# Theoretical Physics 6a (QFT): SS 2025
# Exercise sheet 12

07.07.2025

**(0)(0 points)** How much time did it take you to solve this exercise sheet?

## Exercise 1. (100 points): Running coupling and mass in $\phi^4$ theory

Consider the massive real scalar theory in the dimensional regularization:

$$\mathcal{L} = \frac{1}{2}\left(\partial\phi\right)^2 - \frac{1}{2}m^2\phi^2 - \frac{\mu^{4-d}\lambda}{4!}\phi^4$$

The factor $\mu$ has a dimension of mass, so that the coupling constant remains dimensionless.
It is advised to write the counterterms in the following form:

$$\Delta\mathcal{L} = \frac{Z_2 - 1}{2}\left(\partial\phi\right)^2 - \frac{Z-1}{2}m^2\phi^2 - \left(Z_4 - 1\right)\frac{\mu^{4-d}\lambda}{4!}\phi^4$$

Which gives the bare Lagrangian:

$$\mathcal{L} + \Delta\mathcal{L} = \frac{1}{2}\left(\partial\phi_0\right)^2 - \frac{1}{2}m_0^2\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4 = \frac{Z_2}{2}\left(\partial\phi\right)^2 - \frac{Z}{2}m^2\phi^2 - Z_4\frac{\mu^{4-d}\lambda}{4!}\phi^4$$

With bare parameters:

$$\phi_0 = \sqrt{Z_2}\phi$$
$$m_0^2 = \frac{Z}{Z_2}m^2$$
$$\lambda_0 = \mu^{4-d}\frac{Z_4}{Z_2^2}\lambda$$

**(a)(60 points)** Calculate all one-loop diagrams and prove that in MS scheme counterterms are the following ($D = 4 - 2\varepsilon$):

$$Z_4 = 1 + \frac{\lambda}{16\pi^2}\frac{3}{2\varepsilon}$$
$$Z = 1 + \frac{\lambda}{16\pi^2}\frac{1}{2\varepsilon}$$
$$Z_2 = 1$$

*Hint*: there are 4 diagrams, but 3 of them have the same structure.

**(b)(40 points)** Prove that the running coupling and mass:

$$\lambda = \frac{1}{\beta_0 \ln\left(\frac{\mu}{\Lambda}\right)}$$
$$m^2(\mu) = m^2(\mu_0)\left(\frac{\mu}{\mu_0}\right)^{\gamma_m}$$

Have the following coefficients:

$$\beta_0 = -3\left(\frac{1}{4\pi}\right)^2$$
$$\gamma_m = \frac{\lambda}{16\pi^2}$$

*Hint*: Begin from the equation $\mu\frac{d\lambda_0}{d\mu} = 0$ and solve it iteratively for $\lambda(\mu)$. As a first step solve for $Z_4 = Z_2 = 1$. Next solve for $Z_2 = 1$ and $Z_4 = 1 + \frac{\lambda}{16\pi^2}\frac{3}{2\varepsilon}$ (while $\lambda/Z_4 \approx \lambda$). Enforcing that $\lambda(\mu = \mu_0) = \lambda_0$ at the scale $\mu_0$ should give you the correct result. Follow the same procedure for $m^2(\mu)$ (while $\lambda/Z \approx \lambda$).