Practice Exam
Theoretical Physics 6a (QFT): SS 2025

14.07.2025

Exercise 1. (20+10 Bonus points): Dilation current for scalars

Consider the following Lagrangian for real scalar field:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \alpha\phi^2 - \beta\phi^3 - \gamma\phi^4$$

(a)(10 points) Prove that in case of $\alpha = 0$ and $\beta = 0$ the action $\mathcal{S} = \int \mathcal{L}\, d^4x$ has a dilation symmetry, i.e. it is invariant under the transformation:

$$x^\mu \to x^\mu e^\lambda$$
$$\phi \to \phi e^{-\lambda}$$

(b)(10 points) Derive the corresponding Noether current:

$$j^\mu = \phi\partial^\mu\phi + x_\nu T^{\mu\nu},$$

where $T^{\mu\nu}$ is the energy-momentum tensor.

(Bonus)(10 points) Prove the conservation of this current by direct calculation. *Hint*: use the equations of motion to get rid of γ in $T^{\mu\nu}$.

Exercise 2. (15 points): Discrete symmetries of the vector bilinear

A spinor field $\psi(t, \vec{x})$ transforms under parity (P), time-reversal (T) and charge conjugation (C) according to:

$$\psi(t,\vec{x}) \xrightarrow{P} \gamma^0\psi(t,-\vec{x})$$
$$\psi(t,\vec{x}) \xrightarrow{T} \gamma^1\gamma^3\psi(-t,\vec{x})$$
$$\psi(t,\vec{x}) \xrightarrow{C} \left(\overline{\psi}(t,\vec{x})i\gamma^0\gamma^2\right)^T.$$

Determine the transformation of the vector bilinear $\overline{\psi}(t,\vec{x})\gamma^\mu\psi(t,\vec{x})$ under:
(a) (4 points) a parity transformation,
(b) (4 points) a time reversal transformation,
(c) (4 points) a charge conjugation transformation,
(d) (3 points) a CPT transformation.

Exercise 3. (30 points): Bhabha scattering in QED

The Lagrangian of Quantum Electrodynamics is given in terms of a spinor field ψ and the photon field A^μ as:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi - e\overline{\psi}\gamma^\mu A_\mu\psi,$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$.

(a)(5 points) Derive the tree-level matrix element for the process $e^-e^+ \to e^-e^+$ (Bhabha scattering) and draw the two contributing diagrams.
Hint: Is there a relative sign between the two contributing diagrams?

(b) (15 points) Considering the electron to be massless, show that the spin-averaged squared matrix element is given by:

$$|\mathcal{M}|^2_{\text{spin-averaged}} = 2e^4\left[\frac{u^2+s^2}{t^2} + \frac{2u^2}{st} + \frac{u^2+t^2}{s^2}\right],$$

where $s,\ t,\ u$ are the Mandelstam variables for the process $p_1 + p_2 \to p_3 + p_4$:

$$s = (p_1+p_2)^2, \quad t = (p_1-p_3)^2, \quad u = (p_1-p_4)^2.$$

(c) (10 points) Show that the differential cross section in the center-of-momentum frame can be written as:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{2s}\left[u^2\left(\frac{1}{s}+\frac{1}{t}\right)^2 + \left(\frac{t}{s}\right)^2 + \left(\frac{s}{t}\right)^2\right]$$

Exercise 4. 1-loop correction to the propagator in Yukawa theory (35 + 10 Bonus points)

Consider the interaction between a scalar field ϕ (with mass M) and a spin 1/2 field ψ (with mass m) described by the Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}M^2\phi^2 + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \lambda\bar{\psi}\psi\phi, \tag{1}$$

where λ is a coupling constant.

(a)(5 points) Derive the Feynman rule corresponding to the interaction term.

(b)(10 points) The 1-loop correction to the scalar propagator induced by a fermion loop is presented in Fig. 3. Use the Feynman rules to derive the invariant amplitude $\mathcal{M}$ for this diagram, using the momentum labels as indicated on the figure.

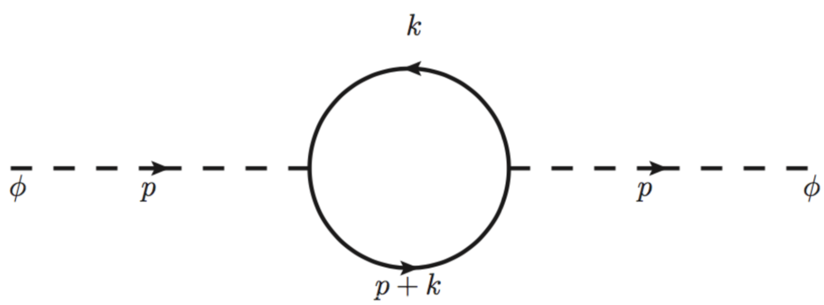

Figure 1: One-loop correction to the scalar propagator in Yukawa theory, given by the Lagrangian in Eq. (1).

(c)(10 points) Use the Feynman parameterization, and perform the one-loop integral using dimensional regularization (and using the formulas given at the end). Show that the result can be expressed as:

$$\mathcal{M}^{1-loop} = i\frac{4(d-1)\lambda^2\mu^{4-d}}{(4\pi)^{d/2}}\Gamma(1-d/2)\int_0^1 dx \frac{1}{[m^2 - p^2 x(1-x)]^{1-d/2}}, \tag{2}$$

where d denotes the dimensionality of space-time, and μ is some arbitrary scale to keep the coupling λ dimensionless.

(d)(10 points) The scalar counterterms (CT) that have to be added to the diagram of Fig. 2 correspond with the Feynman rule:

$$\mathcal{M}^{CT} = i\left[p^2\delta_\phi - M^2(\delta_M + \delta_\phi)\right], \tag{3}$$

where δ_ϕ is the counterterm for the field ϕ, and δ_M is the counterterm for the scalar squared mass M^2.
Defining $\varepsilon \equiv 2 - d/2$, expand the above result for the invariant amplitude $\mathcal{M}$ in ε to extract the pole term in $1/\varepsilon$. Use the MS subtraction scheme, i.e. absorb only the divergent parts, and determine the MS expressions for the counterterms δ_ϕ and δ_M.

(Bonus) (10 points) The renormalized propagator of the scalar field is given by

$$\frac{i}{p^2 - M^2 - \Sigma_R(p^2)}, \tag{4}$$

with the renormalized self-energy $\Sigma_R(p^2) = i(\mathcal{M}^{1-loop} + \mathcal{M}^{CT})$. Using the above result for the invariant amplitude, what is the expression for $\Sigma_R(p^2)$ in the $\overline{MS}$ scheme? You do not need to perform the Feynman parameter integral. What is the expression for the difference between the pole value (M_P^2) and the $\overline{MS}$ value ($M^2_{\overline{MS}}$) of the squared scalar mass ?
Note: This difference determines the shift in the Higgs mass (M) due to the heavy (mass m) top-quark loop.