Theoretical Physics 6a (QFT): SS 2025
Exercise sheet 10

23.06.2025

Exercise 1. (100 points): Feynman integrals

(a)(25 points) The following identity was proven in lectures:

$$\frac{1}{A_1 A_2} = \int_0^1 \frac{1}{\left[x_1 A_1 + (1-x_1)\, A_2\right]^2}\, dx_1.$$

Usually it is more convenient to write this formula as a double integral (symmetric form):

$$\frac{1}{A_1 A_2} = \int_0^1 \int_0^1 \delta\left(x_1 + x_2 - 1\right) \frac{1}{\left[x_1 A_1 + x_2 A_2\right]^2}\, dx_1 dx_2.$$

Differentiate it $n-1$ times to get the following identity:

$$\frac{1}{A_1 A_2^n} = \int_0^1 \int_0^1 \delta\left(x_1 + x_2 - 1\right) \frac{n x_2^{n-1}}{\left[x_1 A_1 + x_2 A_2\right]^{n+1}}\, dx_1 dx_2.$$

(b)(25 points) Using the result from the previous part, prove the general formula:

$$\frac{1}{A_1 ... A_k} = \int_0^1 ... \int_0^1 \delta\left(x_1 + ... + x_k - 1\right) \frac{(k-1)!}{\left[x_1 A_1 + ... x_k A_k\right]^k}\, dx_1 ... dx_k$$

Hint: Use the induction method - you know that formula is valid for a certain $n = 2$, prove it works for $n+1$.

(c)(25 points) Consider the integral:

$$I = \int \frac{1}{\left(p^2 - \Delta + i\varepsilon\right)^n} \frac{d^D p}{(2\pi)^D}$$

In the lectures this integral was calculated using the Wick rotation with the assumption that $\Delta > 0$. Now show that the Wick rotation method still works for $\Delta < 0$.
Hint: Note that poles in this case remain in the same quarters of the integration plane.

(d)(25 points) Explain why two following integrals are zero:

$$\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{k^\mu}{(k^2 - \Delta + i\epsilon)^n} = 0$$
$$\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{k^\mu k^\nu k^\sigma}{(k^2 - \Delta + i\epsilon)^n} = 0$$

And prove the following identities:

$$\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{k^\mu k^\nu}{(k^2 - \Delta + i\epsilon)^n} = -\frac{i g^{\mu\nu}}{(4\pi)^{D/2}} \frac{\Gamma\left(n - 1 - \frac{D}{2}\right)}{2\Gamma(n)} \frac{(-1)^n}{(\Delta - i\varepsilon)^{n-1-D/2}}$$
$$\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{k^\mu k^\nu k^\sigma k^\rho}{(k^2 - \Delta + i\epsilon)^n} = \frac{i}{(4\pi)^{D/2}} \frac{\Gamma\left(n - 2 - \frac{D}{2}\right)}{\Gamma(n)} \frac{(-1)^n}{(\Delta - i\varepsilon)^{n-2-D/2}} \frac{(g^{\mu\nu} g^{\sigma\rho} + g^{\mu\sigma} g^{\nu\rho} + g^{\mu\rho} g^{\sigma\nu})}{4}$$