Theoretical Physics 6a (QFT): SS 2025
Exercise sheet 6

19.05.2025

(0 points) How much time did it take you to solve this exercise sheet?

Exercise 1. (50 points) : Wick's Theorem

During the lectures you discussed the proof of Wick's theorem. In the first step of induction, starting from the case of $N = 2$ fields, you showed that

$$T(\phi_1\phi_2) = N(\phi_1\phi_2) + N(\underbracket{\phi_1\phi_2}),$$

where we have defined $\phi_i \equiv \phi(x_i)$. Now it is time for you to do the same for the case of $N = 3$ fields. Prove that:

$$T(\phi_1\phi_2\phi_3) = N(\phi_1\phi_2\phi_3) + N(\underbracket{\phi_1\phi_2}\phi_3) + N(\phi_1\underbracket{\phi_2\phi_3}) + N(\phi_2\underbracket{\phi_1\phi_3}).$$

Hint: Start by explicitly splitting the fields into positive and negative frequency parts.

Exercise 2. (30 points) : The Heisenberg Picture

The Schrödinger picture that you are familiar with assumes that all the time-dependence is carried by the states. The interaction picture on the other hand assumes that the unperturbed Hamiltonian defines the time dependence for the operators and the perturbation for the states. If the perturbed Hamiltonian is absent, then the interaction picture coincides with the Heisenberg picture, i.e. only the operators carry time dependence. In that limit, the operators obey canonical **equal-time** commutation relations,

$$[\hat{x}_i(t), \hat{p}_j(t)] = i\delta_{ij}$$

and the time-dependence for an operator $\hat{O}(t)$ is determined through the Heisenberg equation,

$$\frac{d\hat{O}(t)}{dt} = i[\hat{H}(t), \hat{O}(t)].$$

Consider a non-relativistic particle of mass m and electric charge q subject to a homogeneous constant electric field $\vec{E}$, whose Hamiltonian is

$$\hat{H}(t) = \frac{\hat{p}^2(t)}{2m} - q\vec{E} \cdot \hat{\vec{x}}(t).$$

(a)(20 points) Solve Heisenberg's equations for the position and momentum operators.

(b)(10 points) Calculate the commutators

$$[\hat{x}_i(t), \hat{x}_j(0)] \qquad [\hat{p}_i(t), \hat{p}_j(0)] \qquad [\hat{x}_i(t), \hat{p}_j(0)].$$

Exercise 3. (20 points) : Positronium

Positronium is a bound state of an e^- and an e^+. Their spins can combine into either total spin $S = 0$ or 1.

(a)(5 points) Show that the corresponding spin wavefunctions are either odd ($S = 0$) or even ($S = 1$) under exchange of the spins.

(b)(5 points) Write down all allowed total angular momentum values for a positronium state which has orbital angular momentum $L = 0$ or $L = 1$, using the notation ${}^{2S+1}L_J$ to identify each state.

(c)(5 points) Show that for a state of total spin S and orbital angular momentum L, the parity P of the state (eigenvalue of the P operation) is given by $P = (-1)^{L+1}$.

(d)(5 points) Show that for a state of total spin S and orbital angular momentum L, the C-parity C of the state (eigenvalue of the charge conjugation C operation) is given by $C = (-1)^{L+S}$.