Theoretical Physics 6a (QFT): SS 2025
Exercise sheet 5

12.05.2025

(0 points) How much time did it take you to solve this exercise sheet?

Exercise 1. (50 points): Photon field

(a)(25 points) Consider the Lagrangian:

$$\mathcal{L} = \mathcal{L}_{Em} + \mathcal{L}_{G.F.} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2$$

where ξ is a real constant. Derive the equations of motion and invert the result to obtain the general photon Feynman propagator.
Hint: In momentum space the equations of motion contain no any other vectors or tensors except for $g^{\mu\nu}$ and k^μ. Thus the propagator can be parameterized as:

$$D^{\mu\nu} = A(k^2)g^{\mu\nu} + B(k^2)k^\mu k^\nu$$

Note: Physical results do not depend on the choice of ξ, because this term goes zero anyway after gauge fixing, thus for simplicity it is often set to be 1. Sometimes the choice of ξ is called by abuse of language a choice of gauge, $\xi = 1$ refers to the Feynman "gauge".

(b)(25 points) Consider a state $|\Psi_T\rangle$ which only contains transverse photons. Futhermore, construct a state $|\Psi'_T\rangle$ as:

$$|\Psi'_T\rangle = \left\{1 + \alpha\left[a^\dagger(\vec{k}, 3) - a^\dagger(\vec{k}, 0)\right]\right\}|\Psi_T\rangle,$$

with α a constant. Show that replacing $|\Psi_T\rangle$ by $|\Psi'_T\rangle$ corresponds to a gauge transformation:

$$\langle\Psi'_T|A^\mu(x)|\Psi'_T\rangle = \langle\Psi_T|A^\mu(x) + \partial^\mu\Lambda|\Psi_T\rangle\,,$$

where Λ is given by:

$$\Lambda(x) = \Re\left(i\alpha\frac{\sqrt{2}}{\omega_k^{3/2}}e^{-ik\cdot x}\right).$$

Note: $\partial_\mu\partial^\mu\Lambda = 0$, which means this is a transformation within Lorentz gauge.

Exercise 2 (50 points) : The angular momentum operator

(a)(25 points) Starting from the transformation law for the classical Dirac field under Lorentz transformations, show that the generators of these transformations are given by:

$$M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \frac{1}{2}\sigma_{\mu\nu}.$$

Hint: The Dirac field transforms as

$$\psi(x) \underset{a}{\rightarrow} \psi'(x) = S(a)\psi(a^{-1}x),$$

under the Lorentz transformation $x \underset{a}{\rightarrow} x' = ax$. The definition of the generator of a Lorentz transformation is,

$$\psi'(x) = e^{-\frac{i}{2}M_{\mu\nu}\omega^{\mu\nu}}\psi(x).$$

Explore the infinitesimal limit $\alpha^\mu_\nu = g^\mu_\nu + \omega^\mu_\nu$, with $\omega^\mu_\nu \ll 1$ of the above equations to find the solution.

(b)(25 points) The angular momentum of the Dirac field is given by:

$$M_{\mu\nu} = \int d^3x\, \psi^\dagger_\alpha(x) \left[i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \frac{1}{2}\sigma_{\mu\nu} \right]_{\alpha\beta} \psi_\beta(x).$$

Prove that

$$[M_{\mu\nu}, \psi_\gamma(x)] = -i(x_\mu \partial_\nu - x_\nu \partial_\mu)\psi_\gamma(x) - \frac{1}{2}\sigma_{\mu\nu}\psi_\gamma(x),$$

where α, β and γ in the above equation correspond to **spinor indices, not Lorentz**!

Hint: Under this notation, the equal-time anti-commutation relations have the form

$$[\psi_\beta(x), \psi_\gamma(y)]_+ = 0,$$
$$\left[\psi^\dagger_\alpha(x), \psi_\beta(y)\right]_+ = \delta_{\alpha\beta}\delta^{(3)}(\vec{x} - \vec{y}).$$