Theoretical Physics 6a (QFT): SS 2025
Exercise sheet 4

05.05.2025

(0 points) How much time did it take you to solve this exercise sheet?

Exercise 1. (65 points): Dirac field

(a)(25 points) For a Dirac field, the transformations:

$$\psi(x) \to \psi'(x) = e^{i\alpha\gamma_5}\psi(x), \qquad \psi^\dagger(x) \to \psi^{\dagger'}(x) = \psi^\dagger(x)e^{-i\alpha\gamma_5},$$

where α is an arbitrary real parameter, are called chiral phase transformations. Show that the Dirac Lagrangian density $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$ is invariant under chiral phase transformations in the zero-mass limit $m = 0$ only, and that the corresponding conserved current in this limit is the axial vector current $J^\mu_A \equiv \bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)$.
(b)(25 points) Deduce the equations of motion for the fields

$$\psi_L(x) \equiv \frac{1}{2}(\mathbb{1} - \gamma_5)\psi(x), \qquad \psi_R(x) \equiv \frac{1}{2}(\mathbb{1} + \gamma_5)\psi(x),$$

for non-vanishing mass, and show that they decouple in the limit $m = 0$.

Hence, the Lagrangian density $\mathcal{L} = i\bar{\psi}_L\partial\!\!\!/\psi_L$ describes massless fermions with negative helicity and massless anti-fermions with positive helicity only. This field is called the Weyl field and can be used to describe neutrinos as far as the latter can be considered as massless.
Hint: use Weyl representation for gamma matrices:

$$\gamma^\mu_W = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix},$$

where $\sigma^\mu = (1, \vec{\sigma})$ and $\bar{\sigma}^\mu = (1, -\vec{\sigma})$.
(c)(10 points) In the above calculation, you used the Weyl representation, which differs from the one used during the lectures. The standard Dirac representation has the form:

$$\gamma^0_s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \vec{\gamma}_s = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}.$$

Write down a unitary matrix S connecting both representations $\gamma^\mu_s = S\gamma^\mu_W S^{-1}$.
(d)(5 points) Write the γ_5 matrix in both representations.

Exercise 2 (35 points + 10 bonus): Majorana Particles

The Majorana equation

$$i\bar{\sigma}^\mu \partial_\mu \chi - im\sigma^2\chi^2 = 0$$

describes a massless 2-component fermion field ($\chi_a, a = 1,2$) that transforms as the upper two components of a Dirac spinor.

(a)(20 points) Show that the action

$$S = \int \mathrm{d}^4x \left[\chi^\dagger i\bar{\sigma}^\mu \partial_\mu \chi + \frac{im}{2}\left(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*\right)\right]$$

is real and that varying S with respect to χ^* yields the Majorana equation. *Hint*: The field takes Grassmann numbers as values. The complex conjugate of a product of Grassmann numbers is calculated as

$$(\alpha\beta)^* = \beta^*\alpha^*.$$

(b)(15 points) Calculate the divergencies of the currents

$$J_a^\mu = \chi^\dagger\bar{\sigma}^\mu\chi, \quad J_b^\mu = \chi_1^\dagger\bar{\sigma}^\mu\chi_1 - \chi_2^\dagger\bar{\sigma}^\mu\chi_2,$$

where $\chi_{1,2}$ are the components of a Dirac spinor

$$\psi(x) = \begin{pmatrix} \chi_1(x) \\ i\sigma^2\chi_2^*(x) \end{pmatrix}.$$

Express your result in a way that the only Pauli matrix it contains is σ^2.
Hint: Use the Weyl basis for this problem and don't forget that Grassmann numbers anti-commute,

$$\alpha\beta = -\beta\alpha.$$

The relation $\sigma^2\sigma^{\mu *}\sigma^2 = \bar{\sigma}^\mu$ might be useful as well.
(c)(10 bonus points) A useful equation involving the positive energy Dirac spinors $u(\vec{p}, s)$, is the well-known Gordon Identity:

$$\overline{u}(\vec{p}', s)\gamma^\mu u(\vec{p}, s) = \overline{u}(\vec{p}', s)\frac{(p+p')^\mu + i\sigma^{\mu\nu}(p'-p)_\nu}{2m}u(\vec{p}, s),$$

where m is the mass of the particle and $\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right]$. Using the Dirac equation, prove it.