

II.

DIRAC FIELD

- 1) DIRAC EQUATION
- 2) QUANTIZATION OF DIRAC FIELD
- 3) FEYNMAN PROPAGATOR OF DIRAC FIELD
- 4) SYMMETRIES OF DIRAC THEORY

# 1) DIRAC EQUATION

## ⇒ DIRAC EQUATION AND SOLUTIONS

↳ DIRAC EQ. : DESCRIBES RELATIVISTIC SPIN  $1/2$  PARTICLE

$$(i \gamma^\mu \partial_\mu - m) \psi(x) = 0$$

↳  $4 \times 1$  DIRAC SPINOR

↳  $\gamma$ -MATRICES : ( $4 \times 4$  MATRICES)

$$\gamma^\mu (\gamma^0, \vec{\gamma})$$

IN DIRAC REPRESENTATION

$$\gamma^0 = \begin{pmatrix} \mathbb{I}_{2 \times 2} & 0 \\ 0 & -\mathbb{I}_{2 \times 2} \end{pmatrix}$$

$$\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

$$\vec{\sigma} (\sigma^1, \sigma^2, \sigma^3)$$

↳ PAULI MATRICES ( $2 \times 2$ )

$\gamma$ -MATRICES SATISFY ANTI-COMMUTATION RELATION:

$$[\gamma^\mu, \gamma^\nu]_+ \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \mathbb{I}_{4 \times 4}$$

↑

ANTI-COMMUTATOR



↳ HERMITICITY CONDITION

$$\gamma^{0\dagger} = \gamma^0$$

$$\gamma^{i\dagger} = -\gamma^i \quad (i=1,2,3)$$

CAN BE COMBINED AS

$$\underline{\underline{\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0}}$$

↳  $\gamma_5$  MATRIX

IN FOLLOWING, WE WILL ALSO ENCOUNTER

A 5-TH GAMMA :

$$\underline{\underline{\gamma_5 \equiv \frac{i}{4!} \epsilon_{\mu\nu\alpha\beta} \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta}}$$

WITH  $\epsilon_{\mu\nu\alpha\beta}$  TOTAL ANTISYMMETRIC  
LEVI-CIVITA TENSOR IN 4-DIM

CONVENTION USED:

$$\epsilon_{0123} = +1$$

NOTE: THIS CONVENTION IS EQUIVALENT WITH

$$\epsilon^{0123} = -1$$

$$\gamma_5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3$$

$$(\gamma_5)^2 = \mathbb{I}_{4 \times 4}$$

IN DIRAC REPRESENTATION

$$\gamma_5 = \begin{pmatrix} 0 & \mathbb{I}_{2 \times 2} \\ \mathbb{I}_{2 \times 2} & 0 \end{pmatrix} \quad \text{CHANGES UPPER \& LOWER COMPONENTS IN DIRAC SPINOR.}$$

↳ ADJOINT DIRAC FIELD  $\bar{\psi}$

$$\underline{\underline{\bar{\psi}(x) \equiv \psi^\dagger(x) \gamma^0}}$$

TAKE HERMITIAN CONJUGATE OF DIRAC EQ.

$$\left( -i (\partial_\mu \psi^\dagger) (\gamma^\mu)^\dagger - m \psi^\dagger \right) = 0$$

MULTIPLY ON RIGHT BY  $\gamma^0$  & USE  $(\gamma^0)^2 = \mathbb{I}$

$$\left( -i (\partial_\mu \psi^\dagger) \gamma^0 \gamma^\mu (\gamma^0)^2 - m \psi^\dagger \gamma^0 \right) = 0$$

$$\boxed{\left( i (\partial_\mu \bar{\psi}) \gamma^\mu + m \bar{\psi} \right) = 0.}$$

## PLANE-WAVE SOLUTIONS

CHARACTERIZED BY MOMENTUM  $\vec{p}$ , ENERGY  $E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$

§ SPIN PROJECTION  $s = \pm \frac{1}{2}$

(ALONG A CHOSEN QUANTIZATION AXIS)

$$\psi_{\vec{p}, s+}(x) = e^{-i p \cdot x} u(\vec{p}, s) \quad \begin{array}{l} \text{POSITIVE ENERGY} \\ \text{SOLUTION} \\ \text{(PARTICLE)} \end{array}$$

$$\psi_{\vec{p}, s-}(x) = e^{+i p \cdot x} v(\vec{p}, s) \quad \begin{array}{l} \text{NEGATIVE ENERGY} \\ \text{SOLUTION} \\ \text{(ANTI-PARTICLE)} \end{array}$$

↓ PLUG INTO DIRAC EQ.

$$\left\{ \begin{array}{l} (\not{p} - m) u(\vec{p}, s) = 0 \\ (\not{p} + m) v(\vec{p}, s) = 0 \end{array} \right.$$

$$\text{WITH } \not{p} \equiv \gamma_{\mu} p^{\mu}$$

EQUIVALENTLY FOR ADJOINT SPINORS

$$\bar{u}(\vec{p}, s) \equiv u^{\dagger}(\vec{p}, s) \gamma^0 \Rightarrow \bar{u}(\vec{p}, s) (\not{p} - m) = 0$$

$$\bar{v}(\vec{p}, s) \equiv v^{\dagger}(\vec{p}, s) \gamma^0 \Rightarrow \bar{v}(\vec{p}, s) (\not{p} + m) = 0$$



↳ FOR SPIN QUANTIZED ALONG AXIS  $\hat{n}$

$$\bar{S} = \frac{\bar{\sigma}}{2} \quad \leadsto \quad \underline{\underline{\left( \frac{\bar{\sigma}}{2} \cdot \hat{n} \right) \chi_s = s \chi_s}}$$

$$s = \pm \frac{1}{2}$$

$\chi_s$  :  $2 \times 1$  PAULI-SPINOR

- $\hat{n} = \bar{e}_z$       SUPERSCRIPT ON  $\chi$  INDICATES QUANTIZATION AXIS

$$\chi_{+\frac{1}{2}}^{(z)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

SPIN UP

$$\chi_{-\frac{1}{2}}^{(z)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

SPIN DOWN

- $\hat{n} = \bar{e}_x$

$$\sigma_x \chi_s^{(x)} = (2s) \chi_s^{(x)}$$

e.g.  $s = +\frac{1}{2}$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow a = b$$

$$\chi_{+\frac{1}{2}}^{(x)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\chi_{-\frac{1}{2}}^{(x)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

↑  
NORMALIZATION  $\chi_s^\dagger \chi_{s'} = \delta_{ss'}$



↳ IN DIRAC REPRESENTATION : SOLUTION U

$$\left( \gamma^0 E_{\vec{p}} - \vec{\gamma} \cdot \vec{p} - m \right) U(\vec{p}, s) = 0$$

$$\downarrow \quad U(\vec{p}, s) = N \begin{pmatrix} \chi_s \\ \varphi_s \end{pmatrix}$$

↑  
NORMALIZATION

$$\begin{pmatrix} E_{\vec{p}} - m & -\vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & E_{\vec{p}} + m \end{pmatrix} \begin{pmatrix} \chi_s \\ \varphi_s \end{pmatrix} = 0$$

$$\hookrightarrow (E_{\vec{p}} + m) \varphi_s = (\vec{\sigma} \cdot \vec{p}) \chi_s$$

$$\therefore \quad \parallel U(\vec{p}, s) = N \begin{pmatrix} \chi_s \\ \frac{\vec{\sigma} \cdot \vec{p}}{E_{\vec{p}} + m} \chi_s \end{pmatrix}$$

WHERE  $\chi_s$  IS A PAULI-SPINOR  $s = \pm \frac{1}{2}$

NORMALIZATION (COVARIANT NORMALIZATION)

$$\parallel \begin{aligned} \bar{U}(\vec{p}, s) U(\vec{p}, s') &= (2m) \delta_{ss'} \\ U^\dagger(\vec{p}, s) U(\vec{p}, s') &= (2E_{\vec{p}}) \delta_{ss'} \end{aligned}$$

$$\bar{U}(\bar{P}, s) U(\bar{P}, s') = N^2 \left( \chi_s^+ - \chi_s^+ \frac{\bar{\sigma} \cdot \bar{P}}{E_{\bar{P}} + m} \right) \begin{pmatrix} \chi_{s'} \\ \frac{\bar{\sigma} \cdot \bar{P}}{E_{\bar{P}} + m} \chi_{s'} \end{pmatrix}$$

$$= N^2 \left\{ \chi_s^+ \chi_{s'} - \chi_s^+ \frac{(\bar{\sigma} \cdot \bar{P})(\bar{\sigma} \cdot \bar{P})}{(E_{\bar{P}} + m)^2} \chi_{s'} \right\}$$

$$\downarrow (\bar{\sigma} \cdot \bar{P})(\bar{\sigma} \cdot \bar{P}) = \bar{P}^2 = E_{\bar{P}}^2 - m^2$$

$$= N^2 \chi_s^+ \chi_{s'} \left\{ 1 - \frac{E_{\bar{P}}^2 - m^2}{(E_{\bar{P}} + m)^2} \right\}$$

$$\downarrow$$

$$= N^2 \delta_{ss'} \frac{(2m)}{E_{\bar{P}} + m}$$

BY REQUIRING  $\bar{U}(\bar{P}, s) U(\bar{P}, s') = (2m) \delta_{ss'}$

$$\underline{\underline{N = \sqrt{E_{\bar{P}} + m}}}$$

↳ WITH THIS CONVENTION ONE SHOWS THAT

$$U^\dagger(\bar{P}, s) U(\bar{P}, s') = 2E_{\bar{P}} \delta_{ss'}$$

↳ IN DIRAC REPRESENTATION: SOLUTION  $\psi$

$$(\gamma^0 E_{\vec{p}} - \vec{\gamma} \cdot \vec{p} + m) \psi(\vec{p}, s) = 0.$$

$$\downarrow \quad \psi(\vec{p}, s) = N \begin{pmatrix} \varphi'_s \\ \chi'_s \end{pmatrix}$$

$$\begin{pmatrix} E_{\vec{p}} + m & -\vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & -(E_{\vec{p}} - m) \end{pmatrix} \begin{pmatrix} \varphi'_s \\ \chi'_s \end{pmatrix} = 0$$

$$\downarrow \quad (E_{\vec{p}} + m) \varphi'_s - \vec{\sigma} \cdot \vec{p} \chi'_s = 0$$

$$\therefore \left\| \psi(\vec{p}, s) = N \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{p}}{E_{\vec{p}} + m} \chi'_s \\ \chi'_s \end{pmatrix} \right.$$

$$\text{WITH } \left\| \begin{aligned} \chi'_{+\frac{1}{2}} &= \chi_{-\frac{1}{2}} \\ \chi'_{-\frac{1}{2}} &= \chi_{+\frac{1}{2}} \end{aligned} \right.$$

DIRAC: ABSENCE OF NEG. ENERGY SOLUTION  
OF SPIN PROJ  $-\frac{1}{2}$   
INTERPRETED AS ANTI-PARTICLE WITH SPIN  
PROJ  $+\frac{1}{2}$



## NORMALIZATION

$$\begin{aligned}
& \bar{u}(\vec{p}, s) u(\vec{p}, s') \\
&= N^2 \left( \chi_s^{'+} \frac{\vec{\sigma} \cdot \vec{p}}{E_{\vec{p}} + m} - \chi_s^{'+} \right) \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{p}}{E_{\vec{p}} + m} \chi_{s'}^{\prime} \\ \chi_{s'}^{\prime} \end{pmatrix} \\
&= N^2 \left\{ \chi_s^{'+} \frac{(\vec{\sigma} \cdot \vec{p})(\vec{\sigma} \cdot \vec{p})}{(E_{\vec{p}} + m)^2} \chi_{s'}^{\prime} - \chi_s^{'+} \chi_{s'}^{\prime} \right\} \\
&= N^2 \chi_s^{'+} \chi_{s'}^{\prime} \left\{ \frac{E_{\vec{p}}^2 - m^2}{(E_{\vec{p}} + m)^2} - 1 \right\} \\
&\quad \downarrow \\
&= N^2 \delta_{ss'} \frac{(-2m)}{E_{\vec{p}} + m} \\
&\quad \downarrow \quad \text{WITH CHOICE } N = \sqrt{E_{\vec{p}} + m} \\
&= -(2m) \delta_{ss'}
\end{aligned}$$

$$\therefore \left\{ \begin{aligned} \bar{u}(\vec{p}, s) u(\vec{p}, s') &= -(2m) \delta_{ss'} \\ u^\dagger(\vec{p}, s) u(\vec{p}, s') &= (2E_{\vec{p}}) \delta_{ss'} \end{aligned} \right.$$

↳ WE CAN ALSO EASILY CHECK THE ORTHOGONALITY OF PARTICLE ( $u$ ) AND ANTI-PARTICLE ( $v$ ) SPINORS:

$$\begin{aligned} \bar{u}(\vec{p}, s) v(\vec{p}, s') &= 0 \\ \bar{v}(\vec{p}, s) u(\vec{p}, s') &= 0. \end{aligned}$$

NOTE: BECAUSE WE WANT TO WORK WITH LORENTZ INVARIANT NORMALIZATION WE NORMALIZE

$$\bar{u}(\vec{p}, s) u(\vec{p}, s') \text{ TO A CONSTANT } (2m) \delta_{ss'},$$

IN CONTRAST:

$$u^\dagger(\vec{p}, s) u(\vec{p}, s') = 2E_{\vec{p}} \delta_{ss'}$$

IS NOT LORENTZ INVARIANT AS IT INVOLVES THE ENERGY  $E_{\vec{p}}$

∴ IN QFT FOR DIRAC PARTICLES:

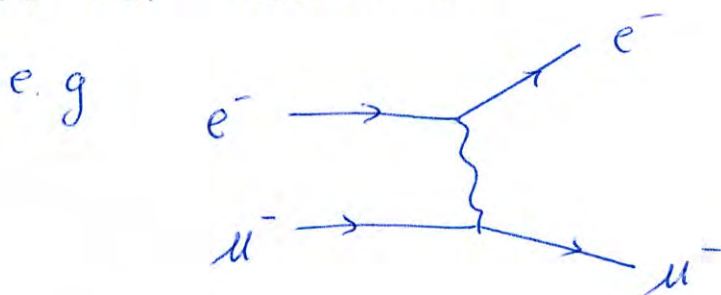
WE WILL WORK WITH  $\psi(x)$  AND  $\bar{\psi}(x)$

( $\bar{\psi}$  INSTEAD OF  $\psi^\dagger$ )



## SPIN SUMS

IN FOLLOWING WE WILL FREQUENTLY  
ENCOUNTER PROCESSES IN WHICH  
THE POLARIZATION OF INITIAL SPIN  $1/2$  BEAM PARTICLE  
IS RANDOMLY ORIENTED,  
AND POLARIZATION OF FINAL SPIN  $1/2$  PARTICLE  
IS NOT OBSERVED



$$\text{INITIAL } e^- : \text{SPIN } s_e \Rightarrow \frac{1}{2} \sum_{s_e}$$

$$\text{FINAL } e^- : \text{SPIN } s'_e \Rightarrow \sum_{s'_e}$$

SUCH UNPOLARIZED PROCESSES WILL LEAD TO  
SPIN SUMS OF FORM

$$\sum_{s=\pm\frac{1}{2}} U(\vec{p}, s) \bar{U}(\vec{p}, s)$$

BY PLUGGING IN A SOLUTION

$$(E_{\vec{p}} + m) \sum_s \begin{pmatrix} \chi_s \\ \frac{\vec{\sigma} \cdot \vec{p}}{E_{\vec{p}} + m} \chi_s \end{pmatrix} \left( \chi_s^\dagger - \chi_s^\dagger \frac{\vec{\sigma} \cdot \vec{p}}{E_{\vec{p}} + m} \right)$$



$$= (E_{\vec{p}} + m) \cdot \left( \begin{array}{cc} \sum_{\lambda} \chi_{\lambda} \chi_{\lambda}^{\dagger} & - \sum_{\lambda} \chi_{\lambda} \chi_{\lambda}^{\dagger} \frac{\vec{\sigma} \cdot \vec{p}}{E_{\vec{p}} + m} \\ \frac{\vec{\sigma} \cdot \vec{p}}{E_{\vec{p}} + m} \sum_{\lambda} \chi_{\lambda} \chi_{\lambda}^{\dagger} & - \frac{\vec{\sigma} \cdot \vec{p}}{E_{\vec{p}} + m} \left( \sum_{\lambda} \chi_{\lambda} \chi_{\lambda}^{\dagger} \right) \frac{\vec{\sigma} \cdot \vec{p}}{E_{\vec{p}} + m} \end{array} \right) \quad \text{DIR 13}$$

↓

USING COMPLETENESS FOR  
PAULI SPINORS

$$\sum_{\lambda = \pm \frac{1}{2}} \chi_{\lambda} \chi_{\lambda}^{\dagger} = \mathbb{I}_{2 \times 2}$$

$$= (E_{\vec{p}} + m) \left( \begin{array}{cc} \mathbb{I} & - \frac{\vec{\sigma} \cdot \vec{p}}{E_{\vec{p}} + m} \\ \frac{\vec{\sigma} \cdot \vec{p}}{E_{\vec{p}} + m} & - \frac{\vec{p}^2}{(E_{\vec{p}} + m)^2} \end{array} \right)$$

$$= \left( \begin{array}{cc} E_{\vec{p}} + m & - \vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & - (E_{\vec{p}} - m) \end{array} \right)$$

∴

$$\sum_{\lambda = \pm \frac{1}{2}} u(\vec{p}, \lambda) \bar{u}(\vec{p}, \lambda) = \not{p} + m.$$

AND ANALOGOUSLY

$$\sum_{\lambda = \pm \frac{1}{2}} v(\vec{p}, \lambda) \bar{v}(\vec{p}, \lambda) = \not{p} - m.$$

## $\Rightarrow$ DIRAC LAGRANGIAN

$\hookrightarrow$  INTERPRETING  $\psi(x)$  AND  $\bar{\psi}(x)$

AS INDEPENDENT FIELDS SATISFYING

$$\left\| \begin{aligned} (i \gamma^\mu \partial_\mu - m) \psi &= 0. \end{aligned} \right.$$

$$\left\| \begin{aligned} (i (\partial_\mu \bar{\psi}) \gamma^\mu + m \bar{\psi}) &= 0. \end{aligned} \right.$$

WE CAN DERIVE DIRAC EQ.

FROM THE DIRAC LAGRANGIAN:

$$\boxed{\mathcal{L}_{\text{DIRAC}} = \bar{\psi}(x) (i \gamma^\mu \partial_\mu - m) \psi(x)}$$

$\hookrightarrow$  EULER LAGRANGE EQS.

$$\left. \begin{aligned} \frac{\partial \mathcal{L}}{\partial \bar{\psi}} &= (i \gamma^\mu \partial_\mu - m) \psi \\ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \bar{\psi})} &= 0 \end{aligned} \right\} \Rightarrow (i \gamma^\mu \partial_\mu - m) \psi = 0$$

$$\left. \begin{aligned} \frac{\partial \mathcal{L}}{\partial \psi} &= \bar{\psi} (-m) \\ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} &= \bar{\psi} i \gamma^\mu \end{aligned} \right\} \Rightarrow i (\partial_\mu \bar{\psi}) \gamma^\mu + m \bar{\psi} = 0$$

# ⇒ DIRAC HAMILTONIAN

↳ CONJUGATE MOMENTA

$$\psi \rightarrow \pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i \bar{\psi} \gamma^0 = i \psi^\dagger$$

$$\bar{\psi} \rightarrow \bar{\pi} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\bar{\psi}}} = 0$$

↳ HAMILTONIAN

$$\mathcal{H} = \pi \dot{\psi} + \cancel{\bar{\psi} \dot{\bar{\pi}}} - \mathcal{L}$$

0

$$= i \bar{\psi} \gamma^0 \partial_0 \psi - \bar{\psi} (i \gamma^\mu \partial_\mu - m) \psi$$

$$= \bar{\psi} (-i \vec{\gamma} \cdot \vec{\nabla} + m) \psi$$

WITH  $\vec{\nabla}^i = \frac{\partial}{\partial x^i} = \partial_i$

$$H = \int d^3 \vec{x} \bar{\psi}(x) (-i \vec{\gamma} \cdot \vec{\nabla} + m) \psi(x)$$



# $\Rightarrow$ ENERGY - MOMENTUM TENSOR

$$\hookrightarrow T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} (\partial^\nu \psi) - g^{\mu\nu} \mathcal{L}$$

$$\partial_\mu T^{\mu\nu} = 0$$

$$T_{\text{DIRAC}}^{\mu\nu} = \bar{\psi} (i \gamma^\mu \partial^\nu \psi - g^{\mu\nu} \bar{\psi} (i \gamma^\alpha \partial_\alpha - m) \psi)$$

$\hookrightarrow$  CONSERVED QUANTITY

4-MOMENTUM  $P^\nu$

$$P^\nu = \int d^3 \vec{x} \quad T^{0\nu}$$

$$= \int d^3 \vec{x} \left( \pi \partial^\nu \psi - g^{0\nu} \bar{\psi} (i \gamma^\alpha \partial_\alpha - m) \psi \right)$$

$$P^0 = H$$

$$P^i = \int d^3 \vec{x} \quad \pi (\partial^i \psi)$$

$(i=1, 2, 3)$

$$= \int d^3 \vec{x} \quad \psi^\dagger (-i \vec{\nabla}^i) \psi \rightarrow \text{3-MOMENTUM OF DIRAC FIELD}$$

## 2) QUANTIZATION OF DIRAC FIELD

⇒ FERMI-DIRAC STATISTICS : ANTI-COMMUTATION RELATIONS

↳ NORMAL MODE EXPANSION FOR  $\Psi$

$$\Psi(x) = \sum_{s=\pm\frac{1}{2}} \int \frac{d^3\vec{p}}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \left\{ a(\vec{p}, s) u(\vec{p}, s) e^{-ip \cdot x} + b^\dagger(\vec{p}, s) v(\vec{p}, s) e^{+ip \cdot x} \right\}$$

$$(i\gamma^\mu \partial_\mu - m) u(\vec{p}, s) e^{-ip \cdot x} = (\not{p} - m) u(\vec{p}, s) e^{-ip \cdot x} = 0$$

$$(i\gamma^\mu \partial_\mu - m) v(\vec{p}, s) e^{+ip \cdot x} = -(\not{p} + m) v(\vec{p}, s) e^{+ip \cdot x} = 0$$

↳ NORMAL MODE EXPANSION FOR ADJOINT FIELD  $\bar{\Psi}$

$$\bar{\Psi}(x) = \sum_{s=\pm\frac{1}{2}} \int \frac{d^3\vec{p}}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \left\{ a^\dagger(\vec{p}, s) \bar{u}(\vec{p}, s) e^{+ip \cdot x} + b(\vec{p}, s) \bar{v}(\vec{p}, s) e^{-ip \cdot x} \right\}$$

↳ SINGLE-PARTICLE STATE ( $e^-$ : ELECTRON OF MOMENTUM  $\vec{p}$ , SPIN PROJ  $s$ )

$$|\vec{p}, s\rangle_{e^-} \equiv \sqrt{2E_{\vec{p}}} a^+(\vec{p}, s) |0\rangle$$

↳ SINGLE ANTI-PARTICLE STATE ( $e^+$ : POSITRON OF MOMENTUM  $\vec{p}$ , SPIN PROJ  $s$ )

$$|\vec{p}, s\rangle_{e^+} \equiv \sqrt{2E_{\vec{p}}} b^+(\vec{p}, s) |0\rangle$$

↳ VACUUM

$$a(\vec{p}, s) |0\rangle = 0$$

$$b(\vec{p}, s) |0\rangle = 0$$

↳ FERMI-DIRAC STATISTICS.

$e^-$  AND  $e^+$  ARE BOTH SPIN  $1/2$  FERMIONS.

↳ PAULI PRINCIPLE:

2 FERMIONS CANNOT OCCUPY THE SAME STATE, STATE IS ANTI-SYMMETRIC UNDER EXCHANGE OF 2 FERMIONS

$$a^+(\vec{p}, s) a^+(\vec{p}, s) |0\rangle = 0$$

$$b^+(\vec{p}, s) b^+(\vec{p}, s) |0\rangle = 0$$

$$a^+(\vec{p}, s) a^+(\vec{p}', s') |0\rangle = -a^+(\vec{p}', s') a^+(\vec{p}, s) |0\rangle$$

$$b^+(\vec{p}, s) b^+(\vec{p}', s') |0\rangle = -b^+(\vec{p}', s') b^+(\vec{p}, s) |0\rangle$$



CAN BE SATISFIED BY

IMPOSING ANTI-COMMUTATION RELATIONS:

$$[a(\vec{p}, s), a(\vec{p}', s')]_+ = 0$$

$$[a^+(\vec{p}, s), a^+(\vec{p}', s')]_+ = 0$$

$$[a(\vec{p}, s), a^+(\vec{p}', s')]_+ = (2\pi)^3 \delta^3(\vec{p} - \vec{p}') \delta_{ss'}$$

$$[b(\vec{p}, s), b(\vec{p}', s')]_+ = 0$$

$$[b^+(\vec{p}, s), b^+(\vec{p}', s')]_+ = 0$$

$$[b(\vec{p}, s), b^+(\vec{p}', s')]_+ = (2\pi)^3 \delta^3(\vec{p} - \vec{p}') \delta_{ss'}$$

§ ALL ANTI-COMMUTATORS BETWEEN  
PARTICLES & ANTI-PARTICLES VANISH

$$[a(\vec{p}, s), b(\vec{p}', s')]_+ = [a(\vec{p}, s), b^+(\vec{p}', s')]_+ = 0$$

$$[a^+(\vec{p}, s), b(\vec{p}', s')]_+ = [a^+(\vec{p}, s), b^+(\vec{p}', s')]_+ = 0$$

WITH ANTI-COMMUTATOR

$$[A, B]_+ \equiv AB + BA$$

OFTEN ONE ALSO USES SYMBOL  $\{A, B\}$  FOR ANTI-COMMUTATOR.

↳ WITH ANTI-COMMUTATION RELATIONS :

SINGLE-PARTICLE STATES ARE NORMALIZED AS :

$$\begin{aligned} {}_e\langle \bar{p}', s' | \bar{p}, s \rangle_e &= \sqrt{2E_{\bar{p}'}} \sqrt{2E_{\bar{p}}} \langle 0 | a(\bar{p}', s') a^\dagger(\bar{p}, s) | 0 \rangle \\ &= \sqrt{2E_{\bar{p}'}} \sqrt{2E_{\bar{p}}} \langle 0 | [a(\bar{p}', s'), a^\dagger(\bar{p}, s)]_+ | 0 \rangle \end{aligned}$$

$$\underline{{}_e\langle \bar{p}', s' | \bar{p}, s \rangle_e = (2\pi)^3 2E_{\bar{p}} \delta^3(\bar{p} - \bar{p}') \delta_{ss'}}$$

AND ANALOGOUSLY

$${}_e\langle \bar{p}', s' | \bar{p}, s \rangle_{e^+} = (2\pi)^3 2E_{\bar{p}} \delta^3(\bar{p} - \bar{p}') \delta_{ss'}$$

## ⇒ ANTI-COMMUTATION RELATIONS FOR FIELDS

$$\hookrightarrow \psi(\bar{x}, t) \rightarrow \pi_\psi(\bar{x}, t) = i \dot{\psi}^\dagger(\bar{x}, t).$$

CONJUGATE MOMENTUM FIELD

EQUAL TIME ANTI-COMMUTATOR

$$\begin{aligned} & [\psi(\bar{x}, t), \psi^\dagger(\bar{x}', t)]_+ \\ &= \sum_s \sum_{s'} \int \frac{d^3 \bar{p}}{(2\pi)^3 \sqrt{2E_{\bar{p}}}} \int \frac{d^3 \bar{p}'}{(2\pi)^3 \sqrt{2E_{\bar{p}'}}} \\ & \left[ a(\bar{p}, s) u(\bar{p}, s) e^{-iE_{\bar{p}}t + i\bar{p}\cdot\bar{x}} + b^\dagger(\bar{p}, s) v(\bar{p}, s) e^{+iE_{\bar{p}}t - i\bar{p}\cdot\bar{x}} \right. \\ & \left. a^\dagger(\bar{p}', s') u^\dagger(\bar{p}', s') e^{+iE_{\bar{p}'}t - i\bar{p}'\cdot\bar{x}'} + b(\bar{p}', s') v^\dagger(\bar{p}', s') e^{-iE_{\bar{p}'}t + i\bar{p}'\cdot\bar{x}'} \right]_+ \end{aligned}$$

$$\begin{aligned}
 & \downarrow \quad [a(\vec{p}, s), a^\dagger(\vec{p}', s')]_+ \\
 & = [b(\vec{p}, s), b^\dagger(\vec{p}', s')]_+ \\
 & = (2\pi)^3 \delta^3(\vec{p} - \vec{p}') \delta_{ss'}
 \end{aligned}$$

$$\begin{aligned}
 & [\psi(\vec{x}, t), \psi^\dagger(\vec{x}', t)]_+ \\
 & = \sum_s \int \frac{d^3 \vec{p}}{(2\pi)^3 2E_{\vec{p}}} \left\{ u(\vec{p}, s) u^\dagger(\vec{p}, s) e^{i\vec{p} \cdot (\vec{x} - \vec{x}')} \right. \\
 & \quad \left. + v(\vec{p}, s) v^\dagger(\vec{p}, s) e^{-i\vec{p} \cdot (\vec{x} - \vec{x}')} \right\}
 \end{aligned}$$

$$\begin{aligned}
 & \downarrow \quad \sum_s u(\vec{p}, s) u^\dagger(\vec{p}, s) = (\not{p} + m) \gamma^0 \\
 & \quad \sum_s v(\vec{p}, s) v^\dagger(\vec{p}, s) = (\not{p} - m) \gamma^0
 \end{aligned}$$

$$\begin{aligned}
 & = \int \frac{d^3 \vec{p}}{(2\pi)^3 2E_{\vec{p}}} \left\{ (\gamma^0 E_{\vec{p}} - \vec{\gamma} \cdot \vec{p} + m) \gamma^0 e^{i\vec{p} \cdot (\vec{x} - \vec{x}')} \right. \\
 & \quad \left. + (\gamma^0 E_{\vec{p}} - \vec{\gamma} \cdot \vec{p} - m) \gamma^0 e^{-i\vec{p} \cdot (\vec{x} - \vec{x}')} \right\}
 \end{aligned}$$

$$\downarrow \quad \text{IN } 2^{\text{O}} \text{ TERM} \quad \vec{p} \rightarrow -\vec{p}$$

$$\begin{aligned}
 & = \int \frac{d^3 \vec{p}}{(2\pi)^3 2E_{\vec{p}}} \left\{ 2E_{\vec{p}} e^{i\vec{p} \cdot (\vec{x} - \vec{x}')} \right\} \mathbb{I}_{4 \times 4} \\
 & \quad \hookrightarrow 4 \times 4 \text{ MATRIX IN DIRAC SPACE} \\
 & = \delta^3(\vec{x} - \vec{x}') \mathbb{I}_{4 \times 4}
 \end{aligned}$$



$$\begin{aligned} \circ \circ \quad & \left[ \psi(\bar{x}, t), \psi^\dagger(\bar{x}', t) \right]_+ = \delta^3(\bar{x} - \bar{x}') \mathbb{1}_{4 \times 4} \\ & \left[ \psi(\bar{x}, t), \psi(\bar{x}', t) \right]_+ = 0 \\ & \left[ \psi^\dagger(\bar{x}, t), \psi^\dagger(\bar{x}', t) \right]_+ = 0. \end{aligned}$$

$\Rightarrow$  HAMILTONIAN IN NORMAL MODE EXPANSION

$$\hookrightarrow H = \int d^3\bar{x} \quad \bar{\psi}(x) \left( -i \vec{\gamma} \cdot \vec{\nabla} + m \right) \psi(x)$$

$\downarrow$  INSERT NORMAL MODE EXPANSIONS  
FOR  $\psi(x)$  &  $\bar{\psi}(x)$

$$\begin{aligned} H = & \int d^3\bar{x} \sum_s \sum_{s'} \int \frac{d^3\bar{p}}{(2\pi)^3 \sqrt{2E_{\bar{p}}}} \int \frac{d^3\bar{p}'}{(2\pi)^3 \sqrt{2E_{\bar{p}'}}} \\ & \cdot \left( a^\dagger(\bar{p}, s) \bar{u}(\bar{p}, s) e^{+i\bar{p} \cdot x} + b(\bar{p}, s) \bar{v}(\bar{p}, s) e^{-i\bar{p} \cdot x} \right) \\ & \cdot \left( -i \vec{\gamma} \cdot \vec{\nabla} + m \right) \left( a(\bar{p}', s') u(\bar{p}', s') e^{-i\bar{p}' \cdot x} + b^\dagger(\bar{p}', s') v(\bar{p}', s') e^{+i\bar{p}' \cdot x} \right) \end{aligned}$$

$$\begin{aligned} H = & \int d^3\bar{x} \sum_s \sum_{s'} \int \frac{d^3\bar{p}}{(2\pi)^3 \sqrt{2E_{\bar{p}}}} \int \frac{d^3\bar{p}'}{(2\pi)^3 \sqrt{2E_{\bar{p}'}}} \\ & \cdot \left( a^\dagger(\bar{p}, s) \bar{u}(\bar{p}, s) e^{+i\bar{p} \cdot x} + b(\bar{p}, s) \bar{v}(\bar{p}, s) e^{-i\bar{p} \cdot x} \right) \\ & \cdot \left( a(\bar{p}', s') (\vec{\gamma} \cdot \vec{p}' + m) u(\bar{p}', s') e^{-i\bar{p}' \cdot x} + b^\dagger(\bar{p}', s') (-\vec{\gamma} \cdot \vec{p}' + m) v(\bar{p}', s') e^{+i\bar{p}' \cdot x} \right) \end{aligned}$$

USING:  $(\not{p}' - m) u(\vec{p}', s') = 0$

$$\Downarrow$$

$$(\gamma^0 E_{\vec{p}'} - \vec{\gamma} \cdot \vec{p}' - m) u(\vec{p}', s') = 0.$$

$$(\not{p}' + m) v(\vec{p}', s') = 0$$

$$\Downarrow$$

$$(\gamma^0 E_{\vec{p}'} - \vec{\gamma} \cdot \vec{p}' + m) v(\vec{p}', s') = 0$$

$$\begin{aligned}
 H = & \sum_s \sum_{s'} \int \frac{d^3 \vec{p}}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \int \frac{d^3 \vec{p}'}{(2\pi)^3 \sqrt{2E_{\vec{p}'}}} \\
 & \cdot \int d^3 \vec{x} \left\{ e^{i(E_{\vec{p}} - E_{\vec{p}'})t - i(\vec{p} - \vec{p}') \cdot \vec{x}} a^+(\vec{p}, s) a(\vec{p}', s') \right. \\
 & \quad \cdot \bar{u}(\vec{p}, s) \gamma^0 E_{\vec{p}'} u(\vec{p}', s') \\
 & - e^{i(E_{\vec{p}} + E_{\vec{p}'})t - i(\vec{p} + \vec{p}') \cdot \vec{x}} a^+(\vec{p}, s) b^+(\vec{p}', s') \\
 & \quad \cdot \bar{u}(\vec{p}, s) \gamma^0 E_{\vec{p}'} v(\vec{p}', s') \\
 & + e^{-i(E_{\vec{p}} + E_{\vec{p}'})t - i(\vec{p} + \vec{p}') \cdot \vec{x}} b(\vec{p}, s) a(\vec{p}', s') \\
 & \quad \cdot \bar{v}(\vec{p}, s) \gamma^0 E_{\vec{p}'} u(\vec{p}', s') \\
 & - e^{-i(E_{\vec{p}} - E_{\vec{p}'})t - i(\vec{p} - \vec{p}') \cdot \vec{x}} b(\vec{p}, s) b^+(\vec{p}', s') \\
 & \quad \cdot \bar{v}(\vec{p}, s) \gamma^0 E_{\vec{p}'} v(\vec{p}', s') \left. \right\}
 \end{aligned}$$

$$\downarrow \int d^3 \bar{x} e^{-i(\bar{p} - \bar{p}') \cdot \bar{x}} = (2\pi)^3 \delta^3(\bar{p} - \bar{p}')$$

$$H = \sum_s \sum_{s'} \int \frac{d^3 \bar{p}}{(2\pi)^3 2E_{\bar{p}}} \cdot E_{\bar{p}} \cdot \left\{ a^\dagger(\bar{p}, s) a(\bar{p}, s') \quad u^\dagger(\bar{p}, s) u(\bar{p}, s') \right. \\ - e^{2iE_{\bar{p}}t} a^\dagger(\bar{p}, s) b^\dagger(-\bar{p}, s') \quad u^\dagger(\bar{p}, s) v(-\bar{p}, s') \\ + e^{-2iE_{\bar{p}}t} b(\bar{p}, s) a(-\bar{p}, s') \quad v^\dagger(\bar{p}, s) u(-\bar{p}, s') \\ \left. - b(\bar{p}, s) b^\dagger(-\bar{p}, s') \quad v^\dagger(\bar{p}, s) v(-\bar{p}, s') \right\}$$

ORTHOGONALITY

$$\downarrow u^\dagger(\bar{p}, s) u(\bar{p}, s') = \delta_{ss'} (2E_{\bar{p}})$$

$$v^\dagger(\bar{p}, s) v(\bar{p}, s') = \delta_{ss'} (2E_{\bar{p}})$$

$$u^\dagger(\bar{p}, s) v(-\bar{p}, s') = 0$$

$$v^\dagger(\bar{p}, s) u(-\bar{p}, s') = 0$$

$$\parallel H = \sum_s \int \frac{d^3 \bar{p}}{(2\pi)^3} E_{\bar{p}} \left\{ a^\dagger(\bar{p}, s) a(\bar{p}, s) \right. \\ \left. - b(\bar{p}, s) b^\dagger(-\bar{p}, s) \right\}$$



USING  $[b(\vec{p}, s), b^\dagger(\vec{p}, s)]_+ = (2\pi)^3 \delta^3(0)$

$$H = \sum_s \int \frac{d^3 \vec{p}}{(2\pi)^3} E_{\vec{p}} \left\{ a^\dagger(\vec{p}, s) a(\vec{p}, s) + b^\dagger(\vec{p}, s) b(\vec{p}, s) + \text{CONSTANT} \right\}$$

↳ CONSTANT TERM CAN BE DROPPED  
IF WE DEFINE VACUUM STATE AS  
STATE OF ZERO ENERGY  
(i.e. EXPRESS ENERGY RELATIVE TO VACUUM STATE)

$$\langle 0 | H | 0 \rangle = \sum_s \int \frac{d^3 \vec{p}}{(2\pi)^3} E_{\vec{p}} \cdot \{ \text{CONSTANT} \}$$

BECAUSE  $\langle 0 | a^\dagger(\vec{p}, s) a(\vec{p}, s) | 0 \rangle = 0$

$$\langle 0 | b^\dagger(\vec{p}, s) b(\vec{p}, s) | 0 \rangle = 0$$

## ↳ NORMAL ORDERING

PRODUCT OF OPERATORS WHERE CREATION OPERATOR STANDS ON LEFT OF ANNIHILATION OPERATOR

$$N(a^\dagger(\bar{p}, s) a(\bar{p}', s')) = a^\dagger(\bar{p}, s) a(\bar{p}', s')$$

$$N(a(\bar{p}, s) a^\dagger(\bar{p}', s')) = -a^\dagger(\bar{p}', s') a(\bar{p}, s)$$

AND ANALOGOUS FOR  $b$

WHEN INTERCHANGING FERMION OPERATORS :

FOR EVERY INTERCHANGE  $\Rightarrow (-1)$

↳ BY DEFINITION  $\langle 0 | N(A.B) | 0 \rangle$

$$e_0 \quad H = \langle 0 | H | 0 \rangle$$

IS EQUIVALENT TO EXPRESSING  $H$  THROUGH NORMAL ORDERED PRODUCT

$$H = \int d^3x \quad N \quad \bar{\Psi}(x) (-i \vec{\gamma} \cdot \vec{\nabla} + m) \Psi(x).$$

↑

IN FOLLOWING : WE WILL EXPRESS ALL QUANTITIES (ENERGY, MOMENTUM, SPIN, CHARGE, ...) RELATIVE TO VACUUM

⇓

NORMAL ORDERING IS UNDERSTOOD IN EXPRESSIONS

## ↳ ENERGY

$$H = \sum_s \int \frac{d^3 \vec{p}}{(2\pi)^3} E_{\vec{p}} \left\{ a^\dagger(\vec{p}, s) a(\vec{p}, s) + b^\dagger(\vec{p}, s) b(\vec{p}, s) \right\}$$

↑  
# PARTICLES  
OF  $(\vec{p}, s)$

↑  
# ANTI-PARTICLE  
OF  $(\vec{p}, s)$

FOR FERMIONS: # PARTICLES = 0, 1

## ↳ MOMENTUM

ANALOGOUSLY, WE CAN OBTAIN FROM

$$\vec{P} = \int d^3 \vec{x} \, N \, \psi^\dagger(x) (-i \vec{\nabla}) \psi(x)$$

↓

$$\vec{P} = \sum_s \int \frac{d^3 \vec{p}}{(2\pi)^3} \vec{p} \left\{ a^\dagger(\vec{p}, s) a(\vec{p}, s) + b^\dagger(\vec{p}, s) b(\vec{p}, s) \right\}$$

↳ NOTE: IF WE WOULD HAVE QUANTIZED DIRAC FIELD  
 ACCORDING TO BOSE-EINSTEIN STATISTICS  
 (i.e. THROUGH COMMUTATORS AS FOR KLEIN-GORDON  
 FIELD), WE WOULD HAVE OBTAINED

$$H = \sum_s \int \frac{d^3 \vec{p}}{(2\pi)^3} E_{\vec{p}} \left\{ a^\dagger(\vec{p}, s) a(\vec{p}, s) \ominus b^\dagger(\vec{p}, s) b(\vec{p}, s) \right\}$$

DUE TO MINUS SIGN;  $H$  WOULD NOT HAVE  
 A LOWER BOUND  $\Rightarrow$  NO STABLE GROUND STATE !





## CHARGE

$$\rightarrow \mathcal{L}_{\text{DIRAC}} = \bar{\Psi}(x) (i\gamma^\mu \partial_\mu - m) \Psi(x)$$

CONSIDER GLOBAL PHASE TRANSFORMATION

$$\Psi(x) \rightarrow e^{i\alpha} \Psi(x), \text{ INFINITESIMAL } \delta\Psi = i\alpha \Psi$$

LEAVES  $\mathcal{L}$  INVARIANT :  $\mathcal{L} \rightarrow \mathcal{L}$

∴ SYMMETRY TF.



→ CONSERVED CURRENT  $\partial_\mu j^\mu = 0$

$$j^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Psi)} \cdot \delta\Psi$$

$$j^\mu = \bar{\Psi}(x) \gamma^\mu \Psi(x)$$

→ CONSERVED CHARGE

$$\begin{aligned} Q &\equiv \int d^3\vec{x} \bar{\Psi}(x) \gamma^0 \Psi(x) \\ &= \int d^3\vec{x} \Psi^\dagger(x) \Psi(x) \end{aligned}$$



BY INSERTING NORMAL MODE EXPANSIONS.

AND CONSIDERING NORMAL ORDERING (EXERCISE!)

$$Q = \sum_s \int \frac{d^3\vec{p}}{(2\pi)^3} \left\{ a^\dagger(\vec{p}, s) a(\vec{p}, s) - b^\dagger(\vec{p}, s) b(\vec{p}, s) \right\}$$

PARTICLE & ANTI-PARTICLE HAVE OPPOSITE CHARGES!  $\nabla$

$\hookrightarrow$  ELECTROMAGNETIC CURRENT & CHARGE

BY MULTIPLYING ABOVE  $j^\mu$  WITH  
ELECTRIC CHARGE  $q$ , WE OBTAIN  
ELECTROMAGNETIC CURRENT  $j_{EM}^\mu$

$$j_{EM}^\mu = q \bar{\Psi}(x) \gamma^\mu \Psi(x)$$

CONSERVED CHARGE : ELECTRIC CHARGE

$$Q_{EM} = q \int d^3\vec{x} \Psi^\dagger(x) \Psi(x)$$

e.g. FOR THEORY OF  $e^-$  AND  $e^+$

FOR  $e^-$  :  $q = -e$  ( $e > 0$ ),  $\frac{e^2}{4\pi} \approx \frac{1}{137}$

$$Q_{EM} = -e \sum_s \int \frac{d^3\vec{p}}{(2\pi)^3} \left\{ a^\dagger(\vec{p}, s) a(\vec{p}, s) - b^\dagger(\vec{p}, s) b(\vec{p}, s) \right\}$$

RELATIVE MINUS SIGN:  $e^-$  AND  $e^+$  HAVE OPPOSITE

ELECTRIC CHARGES :  $q_{e^-} = -e$ ,  $q_{e^+} = +e$

### 3) FEYNMAN PROPAGATOR OF DIRAC FIELD

⇒ 2-POINT CORRELATION FUNCTIONS.

$$\hookrightarrow \langle 0 | \psi_\alpha(x) \bar{\psi}_\beta(y) | 0 \rangle \equiv i S_{\alpha\beta}^+(x-y)$$

$x, y$  : 2 DIFFERENT SPACE-TIME POINTS

$\alpha, \beta = 1, 2, 3, 4$  : SPINOR INDEX OF DIRAC FIELD

$S_{\alpha\beta}$  :  $4 \times 4$  MATRIX IN DIRAC SPACE

↳ INSERTING NORMAL MODE EXPANSIONS.

$$i S^+(x-y)$$

$$= \sum_{\vec{s}} \sum_{s'} \int \frac{d^3 \vec{p}}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \int \frac{d^3 \vec{p}'}{(2\pi)^3 \sqrt{2E_{\vec{p}'}}}$$

$$\text{AS } b(\vec{p}, s) | 0 \rangle = 0$$

$$\langle 0 | \left( a(\vec{p}, s) u(\vec{p}, s) e^{-ip \cdot x} + \cancel{b^+}(\vec{p}, s) \bar{v}(\vec{p}, s) e^{+ip \cdot x} \right)$$

$$\cdot \left( a^+(\vec{p}', s') \bar{u}(\vec{p}', s') e^{+ip' \cdot y} + \cancel{b}(\vec{p}', s') \bar{v}(\vec{p}', s') e^{-ip' \cdot y} \right) | 0 \rangle$$

$$= \sum_{\vec{s}} \sum_{s'} \int \frac{d^3 \vec{p}}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \int \frac{d^3 \vec{p}'}{(2\pi)^3 \sqrt{2E_{\vec{p}'}}} e^{-ip \cdot x} e^{ip' \cdot y} u(\vec{p}, s) \bar{u}(\vec{p}', s')$$

$$\langle 0 | a(\vec{p}, s) a^+(\vec{p}', s') | 0 \rangle$$



$$\begin{aligned}
 & \langle 0 | a(\vec{p}, s) a^\dagger(\vec{p}', s') | 0 \rangle \\
 &= \langle 0 | [a(\vec{p}, s), a^\dagger(\vec{p}', s')]_+ | 0 \rangle \\
 &= (2\pi)^3 \delta^3(\vec{p} - \vec{p}') \delta_{ss'}
 \end{aligned}$$

$$\begin{aligned}
 iS^+(x-y) &= \int \frac{d^3 \vec{p}}{(2\pi)^3 2E_{\vec{p}}} e^{-ip \cdot (x-y)} \bigg|_{p^0 = E_{\vec{p}}} \\
 &\quad \cdot \underbrace{\sum_s U(\vec{p}, s) \bar{U}(\vec{p}, s)}_{(\not{p} + m)} \quad \text{SPIN SUM}
 \end{aligned}$$

$$\therefore iS^+(x-y) = \int \frac{d^3 \vec{p}}{(2\pi)^3 2E_{\vec{p}}} e^{-ip \cdot (x-y)} (\not{p} + m) \bigg|_{p^0 = E_{\vec{p}}}$$

$$\hookrightarrow iS^+(x) = (i\gamma^\mu \partial_\mu + m) \underbrace{\int \frac{d^3 \vec{p}}{(2\pi)^3 2E_{\vec{p}}} e^{-ip \cdot x}}_{\text{SCALAR CORRELATOR FROM KLEIN-GORDON THEORY } i\Delta(x)} \bigg|_{p^0 = E_{\vec{p}}}$$

SCALAR CORRELATOR  
FROM KLEIN-GORDON  
THEORY  $i\Delta(x)$

$$S^+(x) = (i\gamma^\mu \partial_\mu + m) \Delta(x)$$

↳ ANALOGOUSLY :

$$- \langle 0 | \bar{\psi}_\beta(y) \psi_\alpha(x) | 0 \rangle \equiv i S_{\alpha\beta}^-(x-y)$$


---

$$i S_{\alpha\beta}^-(x-y)$$

$$= \sum_\alpha \sum_{\alpha'} \int \frac{d^3 \vec{p}}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \int \frac{d^3 \vec{p}'}{(2\pi)^3 \sqrt{2E_{\vec{p}'}}} e^{i p \cdot x} e^{-i p' \cdot y}$$

$$\cdot (-1) \bar{u}_\beta(\vec{p}', \alpha') u_\alpha(\vec{p}, \alpha)$$

$$\cdot \langle 0 | b(\vec{p}', \alpha') b^\dagger(\vec{p}, \alpha) | 0 \rangle$$

$$= \int \frac{d^3 \vec{p}}{(2\pi)^3 2E_{\vec{p}}} e^{i p \cdot (x-y)} \cdot (-1) \cdot \underbrace{\sum_\alpha u_\alpha(\vec{p}, \alpha) \bar{u}_\beta(\vec{p}, \alpha)}_{(\not{p} - m)_{\alpha\beta}} \Big|_{p^0 = E_{\vec{p}}}$$

$$i S_{\alpha\beta}^-(x-y) = \int \frac{d^3 \vec{p}}{(2\pi)^3 2E_{\vec{p}}} e^{i p \cdot (x-y)} (-\not{p} + m)_{\alpha\beta} \Big|_{p^0 = E_{\vec{p}}}$$

$$i S_{\alpha\beta}^-(x) = + \underbrace{(i \gamma^\mu \partial_\mu + m)_{\alpha\beta} \int \frac{d^3 \vec{p}}{(2\pi)^3 2E_{\vec{p}}} e^{i p \cdot x}}_{i \Delta(-x)}$$

$$S_{\alpha\beta}^-(x) = (i \gamma^\mu \partial_\mu + m)_{\alpha\beta} \Delta(-x)$$

## ⇒ TIME-ORDERED PRODUCT FOR FERMION FIELDS

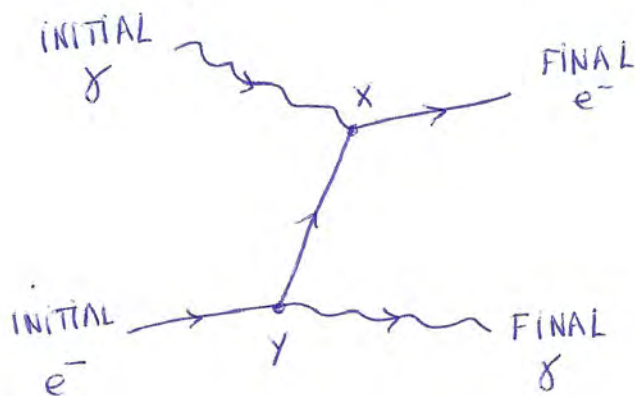
$$T \psi(x) \bar{\psi}(y) \equiv \psi(x) \bar{\psi}(y) \theta(x^0 - y^0) - \bar{\psi}(y) \psi(x) \theta(y^0 - x^0)$$

NOTE! - SIGN BECAUSE OF ANTI-COMMUTATION OF FERMION FIELDS

## ⇒ FEYNMAN PROPAGATOR FOR SPIN 1/2

↳ ENTERS PHYSICAL AMPLITUDES

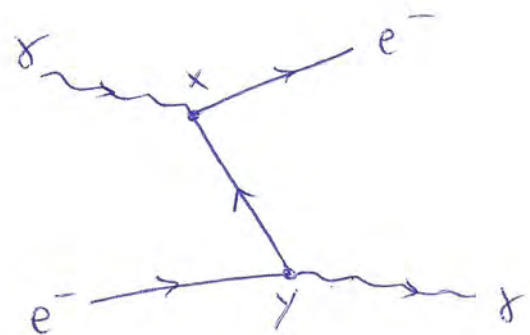
e.g. COMPTON SCATTERING ON  $e^-$  : 2 CONTRIBUTIONS



$$x^0 > y^0$$

PROPAGATION OF PARTICLE FROM  $y \rightarrow x$

$e^-$  PROPAGATES FORWARD IN TIME FROM  $y \rightarrow x$



$$y^0 > x^0$$

PROPAGATION OF ANTI-PARTICLE  $x \rightarrow y$

- AT  $x$  :  $e^- e^+$  PAIR IS CREATED
- $e^+$  PROPAGATES FORWARD IN TIME FROM  $x \rightarrow y$
- AT  $y$  :  $e^+$  ANNIHILATES WITH INITIAL  $e^-$



↳ DEFINITION :

$$\langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle \equiv i S_F(x-y)$$

$$\begin{aligned} \left( i S_F(x-y) \right)_{\alpha\beta} &= \theta(x^0 - y^0) \langle 0 | \psi_\alpha(x) \bar{\psi}_\beta(y) | 0 \rangle \\ &\quad - \theta(y^0 - x^0) \langle 0 | \bar{\psi}_\beta(y) \psi_\alpha(x) | 0 \rangle \end{aligned}$$

$$= \theta(x^0 - y^0) i S_{\alpha\beta}^+(x-y)$$

$$+ \theta(y^0 - x^0) i S_{\beta\alpha}^-(x-y)$$

$$\left( S_F(x) \right)_{\alpha\beta} = \theta(x^0) \cdot \left( i \gamma^\mu \partial_\mu + m \right)_{\alpha\beta} \Delta(x)$$

$$+ \theta(-x^0) \cdot \left( i \gamma^\mu \partial_\mu + m \right)_{\alpha\beta} \Delta(-x)$$

$$S_F(x) = \left( i \gamma^\mu \partial_\mu + m \right) \underbrace{\left( \theta(x^0) \Delta(x) + \theta(-x^0) \Delta(-x) \right)}_{\Delta_F(x)}$$

$$S_F(x) = \left( i \gamma^\mu \partial_\mu + m \right) \Delta_F(x)$$

WITH  $\Delta_F$  : SPIN-0 (KLEIN-GORDON) FEYNMAN PROPAGATOR

↳ FEYNMAN PROPAGATOR IN MOMENTUM SPACE

$$S_F(x) \equiv \int \frac{d^4 p}{(2\pi)^4} e^{-i p \cdot x} \tilde{S}_F(p)$$

WHERE  $\tilde{S}_F(p)$  IS FEYNMAN PROPAGATOR  
FOR SPIN 1/2 PARTICLE  
IN MOMENTUM SPACE

(NOTE: IN FOLLOWING WE WILL FOR SIMPLICITY  
DROP  $\sim$  ON  $\tilde{S}_F(p)$  WHEN IT IS UNDERSTOOD  
THAT WE WORK IN MOMENTUM SPACE)

↳ EXPLICIT FORM

$$\begin{aligned} S_F(x) &= (i \gamma^\mu \partial_\mu + m) \Delta_F(x) \\ &= (i \gamma^\mu \partial_\mu + m) \int \frac{d^4 p}{(2\pi)^4} e^{-i p \cdot x} \frac{1}{p^2 - m^2 + i\epsilon} \\ &= \int \frac{d^4 p}{(2\pi)^4} e^{-i p \cdot x} \frac{(\not{p} + m)}{p^2 - m^2 + i\epsilon} \end{aligned}$$

∴

$$\tilde{S}_F(p) = \frac{(\not{p} + m)}{p^2 - m^2 + i\epsilon}$$

#### 4) SYMMETRIES OF THE DIRAC THEORY

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##### ⇒ LORENTZ INVARIANCE

↳ EQUATIONS OF MOTION SHOULD BE SAME  
IN ANY INERTIAL FRAME

LORENTZ TF:  $x^\mu \rightarrow x'^\mu = a^\mu{}_\nu x^\nu$

$$\text{WITH } \underline{a^\mu{}_\nu a^\lambda{}_\mu = g^\lambda{}_\nu}$$

↓  
ENSURES THAT  $x'^\mu x'_\mu = x^\mu x_\mu$   
SPACE-TIME INTERVAL IS SAME  
FOR ALL INERTIAL OBSERVERS

$$\det(a^\mu{}_\nu) = \pm 1$$

$$\det(a^\mu{}_\nu) = +1$$

PROPER LORENTZ TF  
(3 ROTATIONS,  
3 BOOSTS)

$$\det(a^\mu{}_\nu) = -1$$

IMPROPER LORENTZ TF

↓  
COMBINATIONS OF  
PROPER LORENTZ TF  
WITH

SPACE INVERSION

OR

TIME REVERSAL

$$a^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$a^\mu{}_\nu = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$



↳ TF. FOR SPINOR FIELD

$$x^\mu \xrightarrow{a} x'^\mu = a^\mu_\nu x^\nu$$

$$\underline{\underline{\psi'(x') = S(a) \psi(x)}}$$

LINEAR TF.

IN MATRIX NOTATION

$$x' = a x$$

$$x = a^{-1} x'$$

$$\underline{\underline{\psi(x) \xrightarrow{a} \psi'(x) = S(a) \psi(a^{-1}x)}}$$

INVERSE TF

$$\psi(x) = S^{-1}(a) \psi'(ax)$$

$$= S(a^{-1}) \psi'(ax)$$

$$\underline{\underline{S^{-1}(a) = S(a^{-1})}}$$

DIRAC EQ. IS SAME FOR BOTH INERTIAL OBSERVERS  
(COVARIANCE)

$$\left( i \gamma^\mu \frac{\partial}{\partial x^\mu} - m \right) \psi(x) = 0$$

↓

$$\left( i \gamma^\mu \frac{\partial}{\partial x^\mu} - m \right) S^{-1}(a) \psi'(x') = 0$$

↓ MULTIPLY ON LEFT  
BY  $S(a)$

$$\left( i S(a) \gamma^\mu S^{-1}(a) \frac{\partial}{\partial x^\mu} - m \right) \psi'(x') = 0$$

↓

$$\frac{\partial}{\partial x^\mu} = \frac{\partial x'^\nu}{\partial x^\mu} \frac{\partial}{\partial x'^\nu} = a^\nu_\mu \frac{\partial}{\partial x'^\nu}$$

$$\left( i S(a) \gamma^\mu S^{-1}(a) a^\nu_\mu \frac{\partial}{\partial x'^\nu} - m \right) \psi'(x') = 0$$

LORENTZ COVARIANCE REQUIRES

$$\left( i \gamma^\nu \frac{\partial}{\partial x'^\nu} - m \right) \psi'(x') = 0$$

$\Downarrow$

$$S(a) \gamma^\mu S^{-1}(a) a^\nu_\mu = \gamma^\nu$$

$\Downarrow$

$$\boxed{S^{-1}(a) \gamma^\nu S(a) = a^\nu_\mu \gamma^\mu}$$

$\hookrightarrow$  EXPLICIT CONSTRUCTION OF  $S(a)$  FOR PROPER LORENTZ TF.

- CONSIDER INFINITESIMAL PROPER LORENTZ TF.

$$a^\mu_\nu = g^\mu_\nu + \omega^\mu_\nu$$

$$(a^{-1})^\mu_\nu = g^\mu_\nu - \omega^\mu_\nu$$

FOR LORENTZ TF  $a^\mu_\nu a^\lambda_\mu = g^\lambda_\nu$

$$\left( g^\mu_\nu + \omega^\mu_\nu \right) \left( g^\lambda_\mu + \omega^\lambda_\mu \right) = g^\lambda_\nu$$

$$\omega^\lambda_\nu + \omega_\nu^\lambda = 0 \Rightarrow \underline{\underline{\omega_{\mu\nu} = -\omega_{\nu\mu}}}$$

ANTI - SYMMETRIC

- FOR LINEAR TF :  $S(a)$  CAN BE CONSTRUCTED AS LINEAR IN  $\omega^{\mu\nu}$

$$S(a) = \mathbb{I} - \frac{i}{4} \sigma_{\mu\nu} \omega^{\mu\nu}$$

WITH  $\sigma_{\mu\nu}$  :  $4 \times 4$  ANTI-SYMM MATRICES

$$\sigma_{\mu\nu} = -\sigma_{\nu\mu}$$

6 INDEPENDENT :  $\nearrow$  3 ROTATIONS  
 $\searrow$  3 BOOSTS.

- $\sigma_{\mu\nu}$  DETERMINED FROM

$$S^{-1}(a) \gamma^\mu S(a) = a^\mu{}_\nu \gamma^\nu$$

$$\left[ \mathbb{I} + \frac{i}{4} \sigma_{\alpha\beta} \omega^{\alpha\beta} \right] \gamma^\mu \left[ \mathbb{I} - \frac{i}{4} \sigma_{\alpha\beta} \omega^{\alpha\beta} \right] = \gamma^\mu + \omega^\mu{}_\nu \gamma^\nu$$

$\Downarrow$  KEEPING TERMS LINEAR IN  $\omega$

$$\frac{i}{4} [\sigma_{\alpha\beta}, \gamma^\mu] \omega^{\alpha\beta} = \omega^{\mu\beta} \gamma_\beta$$

$$= \frac{1}{2} \omega^{\alpha\beta} (g^\mu{}_\alpha \gamma_\beta - g^\mu{}_\beta \gamma_\alpha)$$

$\uparrow$   
BECAUSE OF ANTI-SYMM OF  $\omega^{\alpha\beta}$



IN ORDER FOR THIS TO HOLD  $\forall \omega^{\alpha\beta}$  :

$$\underline{\underline{[\sigma_{\alpha\beta}, \gamma^\mu] = -2i (g_\alpha^\mu \gamma_\beta - g_\beta^\mu \gamma_\alpha)}}$$

- THIS EQUATION IS SOLVED BY

$$\sigma_{\alpha\beta} = \frac{i}{2} [\gamma_\alpha, \gamma_\beta] = \frac{i}{2} (\gamma_\alpha \gamma_\beta - \gamma_\beta \gamma_\alpha)$$

PROOF :

$$\begin{aligned} & [\sigma_{\alpha\beta}, \gamma^\mu] \\ &= \sigma_{\alpha\beta} \gamma^\mu - \gamma^\mu \sigma_{\alpha\beta} \\ &= \frac{i}{2} \left\{ (\gamma_\alpha \gamma_\beta - \gamma_\beta \gamma_\alpha) \gamma^\mu - \gamma^\mu (\gamma_\alpha \gamma_\beta - \gamma_\beta \gamma_\alpha) \right\} \\ &= \frac{i}{2} \left\{ (\gamma_\alpha \gamma_\beta - \gamma_\beta \gamma_\alpha) \gamma^\mu - (2g_\alpha^\mu \gamma_\beta - \gamma_\alpha \gamma^\mu) \gamma_\beta \right. \\ &\quad \left. + (2g_\beta^\mu - \gamma_\beta \gamma^\mu) \gamma_\alpha \right\} \\ &= \frac{i}{2} \left\{ (\cancel{\gamma_\alpha \gamma_\beta} - \cancel{\gamma_\beta \gamma_\alpha}) \gamma^\mu - 2g_\alpha^\mu \gamma_\beta + \gamma_\alpha (2g_\beta^\mu - \cancel{\gamma_\beta \gamma^\mu}) \right. \\ &\quad \left. + 2g_\beta^\mu \gamma_\alpha - \gamma_\beta (2g_\alpha^\mu - \cancel{\gamma_\alpha \gamma^\mu}) \right\} \end{aligned}$$

$$\nabla \circ = 2i (g_\beta^\mu \gamma_\alpha - g_\alpha^\mu \gamma_\beta)$$

°° UNDER PROPER LORENTZ TF

$$x \xrightarrow{a} x' = a x$$

$$\psi(x) \xrightarrow{a} \psi'(x') = S(a) \psi(a^{-1}x)$$

↳ INFINITESIMAL  $a^{\mu}_{\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}_{\nu}$

$$S(a) = \mathbb{I} - \frac{i}{4} \sigma_{\mu\nu} \omega^{\mu\nu}$$

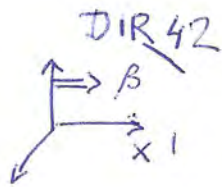
WITH  $\sigma_{\mu\nu} = \frac{i}{2} [\gamma_{\mu}, \gamma_{\nu}]$

↳ FINITE PROPER LORENTZ TF

$$S(a) = \exp \left\{ -\frac{i}{4} \sigma_{\mu\nu} \omega^{\mu\nu} \right\}$$

WITH  $\omega^{\mu\nu}$  FINITE

# ↳ LORENTZ BOOST ALONG X-AXIS



$$\bullet \quad a^{\mu}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\left. \begin{aligned} \gamma &= \cosh w \\ \beta\gamma &= \sinh w \end{aligned} \right\} \beta = \tanh w$$

## • INFINITESIMAL TF

$$\gamma \approx 1, \quad \Delta\beta \ll 1.$$

$$\omega^{\mu}_{\nu} \approx (\Delta\beta) \begin{pmatrix} 0 & -1 & & \\ -1 & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \equiv (\Delta\beta) (\mathcal{I}_1)^{\mu}_{\nu}$$

$$\mathcal{I}_1^2 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

$$\mathcal{I}_1^{2m+1} = \mathcal{I}_1 \quad m = 1, 2, 3, \dots$$

$$\mathcal{I}_1^{2m} = \mathcal{I}_1^2 \quad m = 2, 3, 4, \dots$$

$$(\mathcal{I}_1)^{01} = 1$$

$$(\mathcal{I}_1)^{10} = -1$$



- FINITE LORENTZ BOOST CAN BE OBTAINED FROM SUCCESSION OF INFINITESIMAL BOOSTS  $\Delta\beta = \frac{\omega}{N}$

$$a^\mu{}_\nu = \left( \lim_{N \rightarrow \infty} \left( \mathbb{I} + \frac{\omega}{N} I_1 \right)^N \right)^\mu{}_\nu$$

$$= \left( \exp \left\{ \omega I_1 \right\} \right)^\mu{}_\nu$$

$$\exp \left\{ \omega I_1 \right\} = \left[ \mathbb{I} + \frac{\omega^2}{2!} I_1^2 + \frac{\omega^4}{4!} I_1^4 + \dots \right]$$

$$+ \left[ \omega I_1 + \frac{\omega^3}{3!} I_1^3 + \dots \right]$$

$$= \mathbb{I} - I_1^2 + (\cosh \omega) I_1^2 + (\sinh \omega) I_1$$

$$= \begin{pmatrix} \cosh \omega & -\sinh \omega \\ -\sinh \omega & \cosh \omega \end{pmatrix}$$

- INFINITESIMAL BOOST (ALONG x-AXIS)

$$S(a) = \mathbb{I} - \frac{i}{2} \sigma_{01} (\Delta\beta) (\mathbb{I}_1)^{01}$$

$$\sigma_{01} = -i \begin{pmatrix} 0 & \alpha^1 \\ \sigma^1 & 0 \end{pmatrix} = -i \alpha^1$$

$$\underline{\underline{S(a) = \mathbb{I} - \frac{1}{2} (\Delta\beta) \alpha^1}}$$

- FINITE BOOST (ALONG x-AXIS)

$$\Delta\beta = \frac{\omega}{N}$$

$$S(a) = \lim_{N \rightarrow \infty} \left( \mathbb{I} - \frac{i}{2} \sigma_{01} \frac{\omega}{N} (\mathbb{I}_1)^{01} \right)^N$$

$$= \exp \left\{ - \frac{i}{2} \sigma_{01} \underbrace{\omega (\mathbb{I}_1)^{01}}_{\equiv (\omega)^{01}} \right\}$$

$$\underline{\underline{S(a) = \exp \left\{ - \frac{1}{2} \omega \alpha^1 \right\}}}$$

↳ ROTATION AROUND Z-AXIS OVER ANGLE  $\varphi$

$$\bullet \quad a^u_v = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \varphi & -\sin \varphi & 0 \\ 0 & \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

• INFINITESIMAL TF

$$\Delta \varphi \ll$$

$$\cos \Delta \varphi \approx 1$$

$$\sin \Delta \varphi \approx \Delta \varphi$$

$$\omega^u_v = (\Delta \varphi) i \begin{pmatrix} 0 & & & \\ & 0 & +i & \\ & -i & 0 & \\ & & & 0 \end{pmatrix} \equiv i(\Delta \varphi) (R_3)^u_v$$

$$R_3^{2n} = R_3^2 = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}$$

$$R_3^{2n+1} = R_3$$

$$(R_3)^{12} = -i$$

$$(R_3)^{21} = i$$



- FINITE ROTATION OBTAINED AS SUCCESSION OF INFINITESIMAL ROTATIONS  $\Delta\varphi = \frac{\varphi}{N}$

$$a = \lim_{N \rightarrow \infty} \left( \mathbb{I} + \frac{i\varphi}{N} R_3 \right)^N$$

$$= \exp \{ i\varphi R_3 \}$$

$$= \mathbb{I} - R_3^2 + \sum_{n=0}^{\infty} (-1)^n \frac{\varphi^{2n}}{(2n)!} R_3^{2n}$$

$$+ i \sum_{n=0}^{\infty} (-1)^n \frac{\varphi^{2n+1}}{(2n+1)!} R_3^{2n+1}$$

$$= \mathbb{I} + (\cos \varphi - 1) R_3^2 + i \sin \varphi R_3$$

$$= \begin{pmatrix} 1 & & \\ & \cos \varphi & -\sin \varphi \\ & \sin \varphi & \cos \varphi \\ & & & 1 \end{pmatrix}$$

- INFINITESIMAL ROTATION (AROUND Z-AXIS)

$$S(a) = \mathbb{1} - \frac{i}{2} \sigma_{12} (i \Delta \varphi) (R_3)^{12}$$

$$\begin{aligned} \sigma_{12} &= \frac{i}{2} (\gamma^1 \gamma^2 - \gamma^2 \gamma^1) = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} \\ &\equiv \Sigma^3 \end{aligned}$$

$$\underline{\underline{S(a) = \mathbb{1} - \frac{i}{2} (\Delta \varphi) \Sigma^3}}$$

- FINITE ROTATION (AROUND Z-AXIS)  $\Delta \varphi = \frac{\varphi}{N}$

$$S(a) = \lim_{N \rightarrow \infty} \left( \mathbb{1} - \frac{i}{2} \sigma_{12} i \frac{\varphi}{N} (R_3)^{12} \right)^N$$

$$= \exp \left\{ - \frac{i}{2} \sigma_{12} \underbrace{\varphi (i R_3)^{12}}_{\substack{\text{III} \\ (\omega)^{12}}} \right\}$$

$$\underline{\underline{S(a) = \exp \left\{ - \frac{i}{2} \varphi \Sigma^3 \right\}}}$$

- CONSIDER SPINOR WITH SPIN PROJECTION  $+\frac{1}{2}$  ALONG Z-AXIS

$$\text{i.e. } \sum^3 \psi = + \psi$$



UNDER ROTATION, THIS SPINOR TRANSFORMS AS

$$\psi'(x') = \exp\left(-\frac{i}{2}\varphi\right) \psi(x)$$

IT TAKES A ROTATION OVER  $\underline{4\pi}$   $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  BEFORE SPINOR TURNS INTO ITSELF



PHYSICAL QUANTITIES MUST BE BILINEARS IN  $\psi$   
 ↳ TURN INTO THEMSELVES AFTER ROTATION OVER  $2\pi$

- NOTE: → FOR ROTATIONS

$$\sigma_{ij}^+ = \sigma_{ij} \Rightarrow \underset{\substack{\uparrow \\ \text{ROTATION}}}{S_R^+} = S_R^{-1}$$

→ FOR BOOSTS

$$S_B = \exp\left\{-\frac{1}{2}\omega \alpha^1\right\}$$

$$S_B^+ = S_B$$

FOR PROPER L.T,  $S^{-1} = \gamma_0 S^+ \gamma_0$



↳ TRANSFORMATION PROPERTIES UNDER LORENTZ TF  
OF CURRENT

$$J^\mu(x) = \bar{\psi}(x) \gamma^\mu \psi(x)$$

↓

$$J'^\mu(x') = \bar{\psi}'(x') \gamma^\mu \psi'(x')$$

$$= \psi^\dagger(x) \underbrace{S^\dagger \gamma^0 \gamma^\mu S}_{a^\mu{}_\nu} \psi(x)$$

$$= \psi^\dagger(x) \gamma^0 \underbrace{S^{-1} \gamma^\mu S}_{a^\mu{}_\nu} \psi(x)$$

$$= a^\mu{}_\nu \bar{\psi}(x) \gamma^\nu \psi(x)$$

$$\underline{J'^\mu(x') = a^\mu{}_\nu J^\nu(x)}$$

$J^\mu$  TRANSFORMS AS A 4-VECTOR

## → TRANSFORMATION OF FIELD OPERATORS IN HILBERT SPACE UNDER PROPER L.T. $a$

- STATE  $|\vec{p}, s\rangle = \sqrt{2E_{\vec{p}}} a^\dagger(\vec{p}, s) |0\rangle$  ↗ BOOST AXIS  
↘ ROTATION AXIS  
CONSIDER SPIN QUANTIZATION AXIS EITHER ALONG

$$|\vec{p}, s\rangle \xrightarrow{a} |a\vec{p}, s\rangle = \sqrt{2E_{a\vec{p}}} a^\dagger(a\vec{p}, s) |0\rangle$$

NORM HAS TO BE INVARIANT  $\Rightarrow$  UNITARY T.F.  $U$   
( $U^\dagger = U^{-1}$ )

$$|a\vec{p}, s\rangle = U(a) |\vec{p}, s\rangle$$

$\Downarrow$

$$\sqrt{2E_{a\vec{p}}} a^\dagger(a\vec{p}, s) = U(a) a^\dagger(\vec{p}, s) U^\dagger(a) \sqrt{2E_{\vec{p}}}$$

$\circ_c$

$$\begin{aligned} \left\| \begin{aligned} a(\vec{p}, s) &\xrightarrow{a} U(a) a(\vec{p}, s) U^\dagger(a) \\ &= \sqrt{\frac{E_{a\vec{p}}}{E_{\vec{p}}}} a(a\vec{p}, s) \end{aligned} \right. \end{aligned}$$

- FIELD OPERATOR

$$\psi(x) \xrightarrow{a} U(a) \psi(x) U^\dagger(a)$$

$$= \sum_s \int \frac{d^3\vec{p}}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \left\{ U(a) a(\vec{p}, s) U^\dagger(a) U(\vec{p}, s) e^{-ip \cdot x} + U(a) b^\dagger(\vec{p}, s) U^\dagger(a) v(\vec{p}, s) e^{+ip \cdot x} \right\}$$

$$= \sum_s \int \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} \sqrt{2E_{a\vec{p}}} \left\{ a(a\vec{p}, s) U(\vec{p}, s) e^{-ip \cdot x} + b^\dagger(a\vec{p}, s) v(\vec{p}, s) e^{+ip \cdot x} \right\}$$

CHANGE OF VARIABLE  $\bar{p}' = a \bar{p}$ 

$$\frac{d^3 \bar{p}'}{2E_{\bar{p}'}} = \frac{d^3 \bar{p}}{2E_{\bar{p}}}$$

$$p \cdot x = p' \cdot x' \quad \text{WITH } x' = ax$$

$$U(a) \psi(x) U^\dagger(a)$$

$$= \sum_s \int \frac{d^3 \bar{p}'}{(2\pi)^3 \sqrt{2E_{\bar{p}'}}} \left\{ a(\bar{p}', s) U(a^{-1} \bar{p}', s) e^{-ip' \cdot x'} + b^\dagger(\bar{p}', s) v(a^{-1} \bar{p}', s) e^{+ip' \cdot x'} \right\}$$

$$\downarrow \quad U(a^{-1} \bar{p}', s) = S^{-1}(a) U(\bar{p}', s)$$

(SEE PROOF P. 52-53)

$$= \sum_s \int \frac{d^3 \bar{p}'}{(2\pi)^3 \sqrt{2E_{\bar{p}'}}} \left\{ a(\bar{p}', s) S^{-1}(a) U(\bar{p}', s) e^{-ip' \cdot x'} + b^\dagger(\bar{p}', s) S^{-1}(a) v(\bar{p}', s) e^{+ip' \cdot x'} \right\}$$

$$= S^{-1}(a) \psi(x')$$

$$\therefore \underline{\underline{U(a) \psi(x) U^\dagger(a) = S^{-1}(a) \psi(ax)}}$$

NOTE: THIS IS RELATION FOR AN ACTIVE  
LORENTZ TF  $a$ .



CHECK

$$\underline{U(a\vec{p}, 1) = S(a) U(\vec{p}, 1)}$$

FOR LORENTZ BOOST ALONG X-AXIS  
WITH SPIN QUANTIZATION AXIS CHOSEN ALONG X-AXIS.  
(WE CONSIDER HERE AN ACTIVE TF.)

↳ INVERSE OF PASSIVE TF.  
(COORDINATE TF.)

$$a = \begin{pmatrix} \cosh w & \sinh w & & \\ \sinh w & \cosh w & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

FOR PARTICLE AT REST, WHICH IS BOOSTED  
ALONG X-AXIS

$$\begin{pmatrix} E_{\vec{p}} \\ p_x \end{pmatrix} = \begin{pmatrix} \cosh w & \sinh w \\ \sinh w & \cosh w \end{pmatrix} \begin{pmatrix} m \\ 0 \end{pmatrix}$$

$$\left. \begin{aligned} \cosh w &= \frac{E_{\vec{p}}}{m} \\ \sinh w &= \frac{p_x}{m} = \frac{|\vec{p}|}{m} \end{aligned} \right\} \tanh w = \frac{|\vec{p}|}{E_{\vec{p}}}$$

↓

$$\cosh \frac{w}{2} = \sqrt{\frac{1 + \cosh w}{2}} = \sqrt{\frac{E_{\vec{p}} + m}{2m}}$$

$$\begin{aligned} \sinh \frac{w}{2} &= \frac{\cosh w}{2 \cosh \frac{w}{2}} \tanh w = \frac{|\vec{p}|}{2m} \cdot \sqrt{\frac{2m}{E_{\vec{p}} + m}} \\ &= \frac{|\vec{p}|}{E_{\vec{p}} + m} \sqrt{\frac{E_{\vec{p}} + m}{2m}} \end{aligned}$$

PROOF

$$U(\bar{P}, 1) = S(a) U(\bar{0}, 1)$$

$$\hookrightarrow S(a) = \exp \left\{ \frac{1}{2} \omega \alpha^1 \right\} \quad (\text{ACTIVE TF})$$

$$= \sum_{n=0}^{\infty} \left( \frac{\omega}{2} \right)^{2n} (\alpha^1)^{2n} + \sum_{n=0}^{\infty} \left( \frac{\omega}{2} \right)^{2n+1} (\alpha^1)^{2n+1}$$

$$(\alpha_1)^{2n} = \alpha_1^2 = \mathbb{I}, \quad n=0, 1, \dots$$

$$(\alpha_1)^{2n+1} = \alpha_1, \quad n=0, 1, \dots$$

$$= \cosh \frac{\omega}{2} \mathbb{I} + \sinh \frac{\omega}{2} \alpha^1$$

$$= \left( \begin{array}{c|c} \cosh \frac{\omega}{2} \mathbb{I}_{2 \times 2} & \sinh \frac{\omega}{2} \sigma^1 \\ \hline \sinh \frac{\omega}{2} \sigma^1 & \cosh \frac{\omega}{2} \mathbb{I}_{2 \times 2} \end{array} \right)$$

$$\hookrightarrow S(a) U(\bar{0}, 1)$$

$$= \left( \begin{array}{c|c} \cosh \frac{\omega}{2} \mathbb{I} & \sinh \frac{\omega}{2} \sigma^1 \\ \hline \sinh \frac{\omega}{2} \sigma^1 & \cosh \frac{\omega}{2} \mathbb{I} \end{array} \right) \begin{pmatrix} \chi_s \\ 0 \end{pmatrix} \sqrt{2m}$$

$$= \sqrt{2m} \cosh \frac{\omega}{2} \begin{pmatrix} \chi_s \\ \tanh \frac{\omega}{2} \sigma^1 \chi_s \end{pmatrix}$$

$$= \sqrt{E_{\bar{P}} + m} \begin{pmatrix} \chi_s \\ \frac{\sigma^1 |\bar{P}|}{E_{\bar{P}} + m} \chi_s \end{pmatrix} \stackrel{!}{=} U(\bar{P}, 1)$$

$\hookrightarrow \sigma^1 |\bar{P}| = \vec{\sigma} \cdot \vec{P} \quad (\text{BOOST ALONG } x)$

# $\Rightarrow$ PARITY (SPATIAL INVERSION)

$\hookrightarrow$  SPATIAL INVERSION

$$(t, \vec{x}) \xrightarrow{P} (t, -\vec{x})$$

$$a_P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

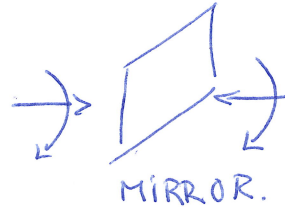
$$\vec{p} \xrightarrow{P} -\vec{p}$$

$$\vec{L} = \vec{r} \times \vec{p} \xrightarrow{P} \vec{L}$$

$$\rightarrow \xrightarrow{P} \rightarrow$$

$$\lambda \xrightarrow{P} -\lambda$$

det  $a_P = -1$   
IMPROPER L.T.



1: SPIN PROJECTION ALONG  
FIXED AXIS IS UNCHANGED

$\lambda$ : HELICITY CHANGES SIGN

$$(\vec{\sigma} \cdot \vec{p}) \chi_\lambda = (2\lambda) \chi_\lambda$$

## $\hookrightarrow$ TRANSFORMATION OF DIRAC SPINOR

$$\psi'(x') = P \psi(x) \quad \text{WITH } x' = (t, -\vec{x})$$

IF WE APPLY TWICE: STATE GOES BACK TO ITSELF

$$P^2 = 1 \Rightarrow P^{-1} = P$$

$$P \gamma^\mu P = (a_P^\mu)_\nu \gamma^\nu$$

$$P \gamma^0 P = \gamma^0$$

$$P \gamma^i P = -\gamma^i \quad (i=1,2,3)$$

$$P = \eta_P \gamma^0$$

$$|\eta_P| = 1$$

PHASE FACTOR

$$\underline{\underline{\psi'(x') = \eta_P \gamma^0 \psi(x)}}$$





## TRANSFORMATION OF FIELD OPERATOR UNDER $\underline{P}$

WE WANT TO DERIVE T.F. LAW FOR FIELD OPERATOR

$\hat{N}(x)$  UNDER  $\underline{P}$

SUCH THAT FOR SINGLE PARTICLE STATE (SPIN PROJ. ALONG FIXED AXIS IN REST FRAME)

WITH  $|\eta_a| = 1$   
PHASE

$$(1) \quad \underline{|\bar{P}, s\rangle} \xrightarrow{\underline{P}} \eta_a^* |-\bar{P}, s\rangle$$

$$\underline{|\bar{P}, s\rangle} \xrightarrow{\underline{P}} U(a_P) |\bar{P}, s\rangle \quad \left\| \begin{array}{l} U(a_P) \text{ UNITARY OPERATOR} \\ \text{SUCH THAT} \\ U^{-1} = U^\dagger \text{ NORM REMAINS SAME} \end{array} \right.$$

$$= U(a_P) \sqrt{2E_P} a^\dagger(\bar{P}, s) |0\rangle$$

$$= \sqrt{2E_P} \underbrace{U(a_P) a^\dagger(\bar{P}, s) U^\dagger(\bar{P}, s)}_{\eta_a^* a^\dagger(-\bar{P}, s)} |0\rangle \quad \left\{ \begin{array}{l} \text{IMPOSE THIS} \\ \text{T.F. LAW} \end{array} \right.$$

$$= \eta_a^* |-\bar{P}, s\rangle$$

$\therefore$  (1) WILL BE SATISFIED BY CHOICE

$$(2) \quad \left\| \begin{array}{l} U(a_P) a(\bar{P}, s) U^\dagger(\bar{P}, s) = \eta_a a(\bar{P}, s) \\ U(a_P) b(\bar{P}, s) U^\dagger(\bar{P}, s) = \eta_b b(-\bar{P}, s) \end{array} \right.$$

$$|\eta_a| = |\eta_b| = 1$$

WITH (2) WE CAN NOW DETERMINE TRANSFORMATION  
LAW OF FIELD OPERATOR  $\hat{\Psi}(x)$  UNDER  $T$

$$\hat{\Psi}(x) = \sum_s \int \frac{d^3 \vec{p}}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \left\{ a(\vec{p}, s) e^{-ip \cdot x} u(\vec{p}, s) + b^\dagger(\vec{p}, s) e^{+ip \cdot x} v(\vec{p}, s) \right\}$$

$$\xrightarrow{T} U(a_P) \hat{\Psi}(x) U^\dagger(a_P)$$

$$= \sum_s \int \frac{d^3 \vec{p}}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \left\{ U(a_P) a(\vec{p}, s) U^\dagger(a_P) e^{-ip \cdot x} u(\vec{p}, s) + U(a_P) b^\dagger(\vec{p}, s) U^\dagger(a_P) e^{+ip \cdot x} v(\vec{p}, s) \right\}$$

$$= \sum_s \int \frac{d^3 \vec{p}}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \left\{ \eta_a a(-\vec{p}, s) e^{-ip \cdot x} u(\vec{p}, s) + \eta_b^* b^\dagger(-\vec{p}, s) e^{+ip \cdot x} v(\vec{p}, s) \right\}$$

$$\downarrow \quad \vec{p}' = -\vec{p} \quad \begin{matrix} p' = (E_{\vec{p}}, -\vec{p}) \\ x' = (t, -\vec{x}) \end{matrix}$$

$$p \cdot x = E_{\vec{p}} t - \vec{p} \cdot \vec{x} = p' \cdot x'$$

$$= \sum_s \int \frac{d^3 \vec{p}'}{(2\pi)^3 \sqrt{2E_{\vec{p}'}}} \left\{ \eta_a a(\vec{p}', s) e^{-ip' \cdot x'} u(-\vec{p}', s) + \eta_b^* b^\dagger(\vec{p}', s) e^{+ip' \cdot x'} v(-\vec{p}', s) \right\}$$

RELATE  $u(-\vec{p}', s)$  TO  $u(\vec{p}', s)$

$\downarrow$   $v(-\vec{p}', s)$  TO  $v(\vec{p}', s)$

$$U(-\vec{p}', s) = N \begin{pmatrix} \chi_s \\ -\frac{\vec{\sigma} \cdot \vec{p}'}{E_{\vec{p}'} + m} \chi_s \end{pmatrix} = \gamma^0 U(\vec{p}', s)$$

$$v(-\vec{p}', s) = N \begin{pmatrix} -\frac{\vec{\sigma} \cdot \vec{p}'}{E_{\vec{p}'} + m} \chi_s' \\ \chi_s' \end{pmatrix} = -\gamma^0 v(\vec{p}', s)$$

$$\hat{\Psi}(t, \vec{x}) \xrightarrow{P} \sum_s \int \frac{d^3 \vec{p}'}{(2\pi)^3 \sqrt{2E_{\vec{p}'}}} \left\{ \eta_a a(\vec{p}', s) \gamma^0 U(\vec{p}', s) e^{-ip' \cdot x'} - \eta_b^* b^\dagger(\vec{p}', s) \gamma^0 v(\vec{p}', s) e^{+ip' \cdot x'} \right\}$$

$$\text{WITH } x' = (t, -\vec{x})$$

$$\text{CHOOSE } \underline{\underline{\eta_b^* = -\eta_a}}$$

(PARTICLE & ANTI-PARTICLE HAVE OPPOSITE INTRINSIC PARITIES)

$$\therefore \boxed{\hat{\Psi}(t, \vec{x}) \xrightarrow{P} \eta_P \gamma^0 \hat{\Psi}(t, -\vec{x})} \quad (\eta_P = \eta_a)$$

$$\hat{\bar{\Psi}}(t, \vec{x}) \xrightarrow{P} \hat{\bar{\Psi}}(t, -\vec{x}) \gamma^0 \eta_P^*$$

$$\text{WITH } |\eta_P|^2 = 1$$

$$\eta_P = \pm 1$$

WE CAN CONVENIENTLY CHOOSE  $\eta_P = +1$

## ↳ TRANSFORMATION OF BILINEARS

### • SCALAR

$$\bar{\Psi}(x) \Psi(x) \xrightarrow{P} |M_a|^2 \bar{\Psi}(x') \gamma^0 \gamma^0 \Psi(x')$$

$$= \bar{\Psi}(x') \Psi(x')$$

$$\text{WITH } x' = (t, -\vec{x})$$

### • PSEUDO-SCALAR

$$\bar{\Psi}(x) \gamma_5 \Psi(x) \xrightarrow{P} |M_a|^2 \bar{\Psi}(x') \gamma^0 \gamma_5 \gamma^0 \Psi(x')$$

$$= - \bar{\Psi}(x') \gamma_5 \Psi(x')$$

↑  
CHANGES SIGN UNDER P

### • VECTOR

$$\bar{\Psi}(x) \gamma^\mu \Psi(x) \xrightarrow{P} |M_a|^2 \bar{\Psi}(x') \gamma^0 \gamma^\mu \gamma^0 \Psi(x')$$

$$= \begin{cases} + \bar{\Psi}(x') \gamma^0 \Psi(x') & , \mu = 0 \\ - \bar{\Psi}(x') \gamma^i \Psi(x') & , \mu = i = 1, 2, 3 \end{cases}$$

TRANSFORMS AS 4-VECTOR UNDER P

↳ TIME COMPONENT  
DOES NOT CHANGE SIGN

↳ SPACE COMPONENTS  
CHANGE SIGN UNDER P



- AXIAL - VECTOR

$$\begin{aligned} \bar{\Psi}(x) \gamma^\mu \gamma_5 \Psi(x) &\xrightarrow{P} |M_a|^2 \bar{\Psi}(x') \gamma^0 \gamma^\mu \gamma_5 \gamma^0 \Psi(x') \\ &= \begin{cases} - \bar{\Psi}(x') \gamma^0 \gamma_5 \Psi(x') , & \mu = 0 \\ + \bar{\Psi}(x') \gamma^i \gamma_5 \Psi(x') , & \mu = i = 1, 2, 3 \end{cases} \end{aligned}$$

↳ TIME COMPONENT CHANGES SIGN

↳ SPACE COMPONENTS DO NOT

CHANGE SIGN UNDER P

(e.g. ANGULAR MOMENTUM  $\vec{L}$ )

- TENSOR

$$\begin{aligned} \bar{\Psi}(x) \sigma^{\mu\nu} \Psi(x) &\xrightarrow{P} \bar{\Psi}(x') \gamma^0 \sigma^{\mu\nu} \gamma^0 \Psi(x') \\ &= \begin{cases} 0 & \mu=0, \nu=0 \\ - \bar{\Psi}(x') \sigma^{\mu\nu} \Psi(x') & \mu=0, \nu=i \\ - \bar{\Psi}(x') \sigma^{\mu\nu} \Psi(x') & \mu=i, \nu=0 \\ \bar{\Psi}(x') \sigma^{\mu\nu} \Psi(x') & \mu=i, \nu=j \\ & i \neq j \end{cases} \\ &= (-1)^{\delta_{\mu i}} (-1)^{\delta_{\nu j}} \bar{\Psi}(x') \sigma^{\mu\nu} \Psi(x') \end{aligned}$$

$\Rightarrow$  TIME REVERSAL. (T)

$$\hookrightarrow (t, \bar{x}) \xrightarrow{T} (-t, \bar{x})$$

$$a_T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$\bar{p} \xrightarrow{T} -\bar{p}$$

$$\vec{L} = \vec{r} \times \vec{p} \xrightarrow{T} -\vec{L}$$

$$\uparrow \xrightarrow{T} -\uparrow$$

$$\begin{array}{c} \rightarrow \bar{p} \quad -\bar{p} \leftarrow \\ \downarrow \quad \quad \uparrow \end{array}$$

SPIN PROJ. ALONG FIXED AXIS  
CHANGES SIGN

$$\lambda \xrightarrow{T} \lambda$$

HELICITY REMAINS UNCHANGED

$\hookrightarrow$  IN QUANTUM MECHANICS

CONSIDER TIME INDEP SCHRÖDINGER EQ.

$$H \psi(t, \bar{x}) = i\hbar \frac{\partial \psi(t, \bar{x})}{\partial t}$$

WITH  $H = -\frac{\hbar^2}{2m} \nabla^2 + V(r)$

e.g. COULOMB  
H-ATOM

$V(r)$  REAL

SOLUTION  $\psi(t, \bar{x})$

TAKE COMPLEX CONJUGATE OF SCHRÖDINGER EQ.

$$H \psi^*(t, \bar{x}) = -i\hbar \frac{\partial \psi^*(t, \bar{x})}{\partial t}$$

$$\downarrow \quad t \rightarrow -t = t'$$

$$H \psi^*(-t', \bar{x}) = i\hbar \frac{\partial \psi^*(-t', \bar{x})}{\partial t'}$$

WHEN  $\psi(t, x)$  IS SOLUTION

$\psi^*(-t, x)$  IS ALSO A SOLUTION

SYMMETRY  $\Rightarrow$  DEGENERACY (KRAMERS)

e.g. H-ATOM

$$\psi(t, x) \sim e^{-iEt} R_{nl}(r) Y_{lm}(\Omega)$$

$$\begin{aligned} \psi^*(-t, x) &\sim e^{-iEt} R_{nl}(r) \underbrace{Y_{lm}^*(\Omega)}_{= (-1)^m Y_{l-m}(\Omega)} \\ &= (-1)^m Y_{l-m}(\Omega) \end{aligned}$$

$|nlm\rangle$  &  $|nl-m\rangle$  ARE DEGENERATE

$\hookrightarrow$  T IN QUANTUM FIELD THEORY (WIGNER)

- WE WANT TO DERIVE TF LAW FOR FIELD OPERATOR  $\hat{\psi}(x)$  UNDER T SUCH THAT FOR SINGLE PARTICLE STATE

$$|\bar{p}, s\rangle \xrightarrow{T} \eta_a^* |-\bar{p}, -s\rangle$$

WITH  $s$ : SPIN PROJ. ALONG FIXED AXIS

$$\begin{aligned} |\bar{p}, s\rangle &\xrightarrow{T} U(a_T) |\bar{p}, s\rangle \\ &= \underbrace{U(a_T) a^\dagger(\bar{p}, s) U^\dagger(a_T)}_{\eta_a^* a^\dagger(-\bar{p}, -s)} \sqrt{2E_{\bar{p}}} |0\rangle \\ &= \eta_a^* |-\bar{p}, -s\rangle \end{aligned}$$

DEFINES  
TF LAW  
FOR STATES

$$|\eta_a| = 1$$

- T ALSO REVERSES INITIAL & FINAL STATES

$$|N_i\rangle \xrightarrow{T} U(a_T) |N_i\rangle \quad U^+ = U^{-1}$$

$$|N_f\rangle \xrightarrow{T} U(a_T) |N_f\rangle$$

$$\& \langle U N_f | U N_i \rangle = \langle N_i | N_f \rangle = \langle N_f | N_i \rangle^*$$

↑  
RELATION FOR  
T-REVERSAL INVARIANCE

THIS REQUIRES AN ANTI-UNITARY OPERATOR

$$\left\| \begin{array}{l} U^+(a_T) = U^{-1}(a_T) \quad \text{AND} \quad U(a_T) f(x) U^+(a_T) = f^*(x) \end{array} \right\|$$

↑  
COMPLEX QUANTITY (e.g. DIRAC SPINOR)

∇ WE NOTICE THAT AS FOR QUANTUM MECHANICS  
0 COMPLEX CONJUGATION IS NEEDED

↳ e.g. FOR TIME EVOLUTION OPERATOR

$$S_{fi} \equiv \langle N_f | e^{-\frac{i}{\hbar} H(t_f - t_i)} | N_i \rangle$$

T-REVERSAL :  
INVARIANCE

$$\underline{S_{fi} = S_{if}}$$

$$e^{-\frac{i}{\hbar} H(t_f - t_i)} \xrightarrow{T} U(a_T) e^{-\frac{i}{\hbar} H(t_f - t_i)} U^+(a_T)$$

$$= e^{-\frac{i}{\hbar} H(t_i - t_f)}$$

REQUIRE T-INV

$$\begin{aligned} S_{fi} &\Downarrow \langle U N_f | U e^{-\frac{i}{\hbar} H(t_f - t_i)} U^+ | U N_i \rangle^* \\ &= \langle U N_f | e^{-\frac{i}{\hbar} H(t_i - t_f)} | U N_i \rangle^* \\ &= \langle U N_i | e^{-\frac{i}{\hbar} H(t_f - t_i)} | U N_f \rangle = S_{if} \end{aligned}$$



• TRANSFORMATION LAW FOR  $\hat{\Psi}(x)$

STARTING FROM

$$U(a_T) a(\bar{p}, s) U^\dagger(a_T) = \eta_a a(\bar{p}, -s)$$

$$U(a_T) b^\dagger(\bar{p}, s) U^\dagger(a_T) = \eta_b^* b^\dagger(-\bar{p}, -s)$$

$$\begin{aligned} \hat{\Psi}(t, \bar{x}) &\xrightarrow{T} U(a_T) \hat{\Psi}(t, \bar{x}) U^\dagger(a_T) \\ &= \sum_s \int \frac{d^3 \bar{p}}{(2\pi)^3 \sqrt{2E_{\bar{p}}}} \left\{ \eta_a a(\bar{p}, s) e^{+ip \cdot x} U^*(\bar{p}, s) \right. \\ &\quad \left. + \eta_b^* b^\dagger(-\bar{p}, -s) e^{-ip \cdot x} v^*(\bar{p}, s) \right\} \end{aligned}$$

$$\begin{aligned} \downarrow \quad p' &= (E_{\bar{p}}, -\bar{p}) & p \cdot x &= -p' \cdot x' \\ x' &= (-t, \bar{x}) \\ s' &= -s \end{aligned}$$

$$= \sum_{s'} \int \frac{d^3 \bar{p}'}{(2\pi)^3 \sqrt{2E_{\bar{p}'}}} \left\{ \eta_a a(\bar{p}', s') e^{-ip' \cdot x'} U^*(-\bar{p}', -s') \right. \\ \left. + \eta_b^* b^\dagger(\bar{p}', s') e^{+ip' \cdot x'} v^*(-\bar{p}', -s') \right\}$$

$$\begin{aligned} U^*(-\bar{p}', -s') &= -\gamma^1 \gamma^3 U(\bar{p}', s') (-1)^{\frac{1}{2}-s'} \\ v^*(-\bar{p}', -s') &= +\gamma^1 \gamma^3 v(\bar{p}', s') (-1)^{\frac{1}{2}-s'} \\ \text{WITH } \gamma^1 \gamma^3 &= i \Sigma^2 = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix} \end{aligned}$$

CHOOSE  $\eta_b^* = -\eta_a$

$$i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\hat{\Psi}(t, \bar{x}) \xrightarrow{T} \eta_T \gamma^1 \gamma^3 \hat{\Psi}(-t, \bar{x})$$

AND CHOOSE  $\eta_a$   
DEPENDENT ON SIGN OF  
 $(-1)^{\frac{1}{2}-s'}$

• PROOF OF RELATIONS FOR SPINORS.

$$\begin{aligned} \left\| \begin{aligned} U^*(-\vec{p}, -s) &= -\gamma^1 \gamma^3 U(\vec{p}, s) (-1)^{\frac{1}{2}-s} \\ U^*(-\vec{p}, -s) &= +\gamma^1 \gamma^3 U(\vec{p}, s) (-1)^{\frac{1}{2}-s} \end{aligned} \right. \end{aligned}$$

FOR SPIN QUANTIZATION AXIS  $\hat{n} = \vec{e}_z$

$$U(\vec{p}, s) = N \begin{pmatrix} \chi_s \\ \frac{1}{E_{\vec{p}} + m} \begin{pmatrix} p_z & p_x - i p_y \\ p_x + i p_y & -p_z \end{pmatrix} \chi_s \end{pmatrix}$$

$$U(\vec{p}, +\frac{1}{2}) = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E_{\vec{p}} + m} \\ \frac{p_x + i p_y}{E_{\vec{p}} + m} \end{pmatrix}$$

$$U(\vec{p}, -\frac{1}{2}) = N \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - i p_y}{E_{\vec{p}} + m} \\ -\frac{p_z}{E_{\vec{p}} + m} \end{pmatrix}$$

$$-\gamma^1 \gamma^3 U(\vec{p}, +\frac{1}{2}) = N \begin{pmatrix} 0 \\ +1 \\ -\frac{p_x + i p_y}{E_{\vec{p}} + m} \\ \frac{p_z}{E_{\vec{p}} + m} \end{pmatrix}$$

$$-\gamma^1 \gamma^3 U(\vec{p}, -\frac{1}{2}) = N \begin{pmatrix} -1 \\ 0 \\ +\frac{p_z}{E_{\vec{p}} + m} \\ +\frac{p_x - i p_y}{E_{\vec{p}} + m} \end{pmatrix}$$

$$= + U^*(-\vec{p}, -\frac{1}{2})$$

$$= - U^*(-\vec{p}, +\frac{1}{2})$$

$$u(\vec{p}, s) = N \begin{pmatrix} \frac{1}{E_{\vec{p}} + m} \begin{pmatrix} p_z & p_x - i p_y \\ p_x + i p_y & -p_z \end{pmatrix} \chi_{-s} \\ \chi_{-s} \end{pmatrix}$$

$$u(\vec{p}, +\frac{1}{2}) = \begin{pmatrix} \frac{p_x - i p_y}{E_{\vec{p}} + m} \\ -\frac{p_z}{E_{\vec{p}} + m} \\ 0 \\ 1 \end{pmatrix}$$

$$u(\vec{p}, -\frac{1}{2}) = \begin{pmatrix} \frac{p_z}{E_{\vec{p}} + m} \\ \frac{p_x + i p_y}{E_{\vec{p}} + m} \\ 1 \\ 0 \end{pmatrix}$$

$$-\gamma^1 \gamma^3 u(\vec{p}, +\frac{1}{2}) = \begin{pmatrix} \frac{p_z}{E_{\vec{p}} + m} \\ \frac{p_x - i p_y}{E_{\vec{p}} + m} \\ -1 \\ 0 \end{pmatrix}$$

$$-\gamma^1 \gamma^3 u(\vec{p}, -\frac{1}{2}) = \begin{pmatrix} -\frac{p_x + i p_y}{E_{\vec{p}} + m} \\ +\frac{p_z}{E_{\vec{p}} + m} \\ 0 \\ 1 \end{pmatrix}$$

$$= -u^*(\vec{p}, -\frac{1}{2})$$

$$= u^*(\vec{p}, +\frac{1}{2})$$



## TRANSFORMATION OF BILINEARS

$$U(a_T) \hat{\psi}(x) U^\dagger(a_T) = \gamma^1 \gamma^3 \hat{\psi}(x') \quad \text{WITH } x' = (-t, \vec{x})$$

$$U(a_T) \hat{\bar{\psi}}(x) U^\dagger(a_T) = + \hat{\bar{\psi}}(x') \gamma^3 \gamma^1$$

- SCALAR

$$\begin{aligned} \bar{\psi}(x) \psi(x) &\xrightarrow{T} \bar{\psi}(x') \gamma^3 \gamma^1 \gamma^1 \gamma^3 \psi(x') \\ &= \bar{\psi}(x') \psi(x') \end{aligned}$$

- PSEUDO SCALAR (NOTE:  $i$  INCLUDED TO MAKE IT HERMITIAN!)

$$\begin{aligned} \bar{\psi}(x) i \gamma_5 \psi(x) &\xrightarrow{T} \bar{\psi}(x') \gamma^3 \gamma^1 (-i \gamma_5) \gamma^1 \gamma^3 \psi(x') \\ &= -\bar{\psi}(x') i \gamma_5 \psi(x') \end{aligned}$$

- VECTOR

$$\begin{aligned} \bar{\psi}(x) \gamma^\mu \psi(x) &\xrightarrow{T} \bar{\psi}(x') \gamma^3 \gamma^1 \gamma^{\mu*} \gamma^1 \gamma^3 \psi(x') \\ &= \begin{cases} \bar{\psi}(x') \gamma^\mu \psi(x') , & \mu = 0. \\ -\bar{\psi}(x') \gamma^\mu \psi(x') , & \mu = 1, 2, 3 \end{cases} \end{aligned}$$

- AXIAL-VECTOR

$$\begin{aligned} \bar{\psi}(x) \gamma^\mu \gamma_5 \psi(x) &\xrightarrow{T} \bar{\psi}(x') \gamma^3 \gamma^1 \gamma^{\mu*} \gamma_5 \gamma^1 \gamma^3 \psi(x') \\ &= \begin{cases} \bar{\psi}(x') \gamma^\mu \gamma_5 \psi(x') , & \mu = 0 \\ -\bar{\psi}(x') \gamma^\mu \gamma_5 \psi(x') , & \mu = 1, 2, 3 \end{cases} \end{aligned}$$



## • TENSOR

$$\sigma^{\mu\nu} = \frac{i}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)$$

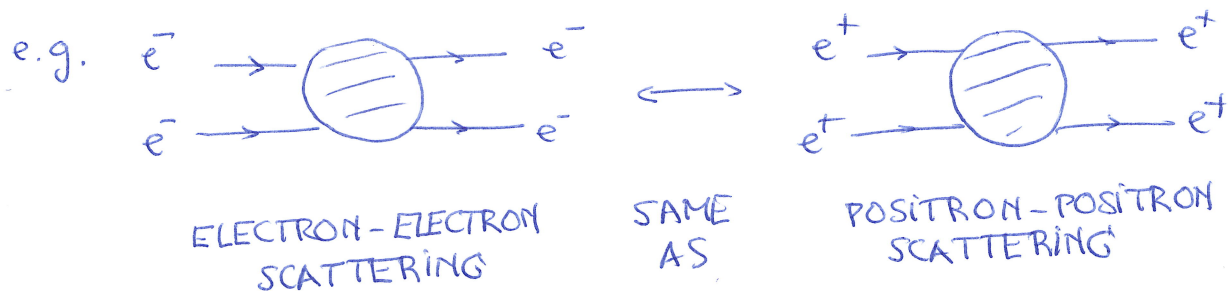
$$\bar{\Psi}(x) \sigma^{\mu\nu} \Psi(x) \xrightarrow{T} \bar{\Psi}(x') \gamma^3 \gamma^1 (\sigma^{\mu\nu})^* \gamma^1 \gamma^3 \Psi(x')$$

$$= \begin{cases} 0 & \mu=0, \nu=0 \\ + \bar{\Psi}(x') \sigma^{\mu\nu} \Psi(x') & \mu=0, \nu=i \\ + \bar{\Psi}(x') \sigma^{\mu\nu} \Psi(x') & \mu=i, \nu=0 \\ - \bar{\Psi}(x') \sigma^{\mu\nu} \Psi(x') & \mu=i, \nu=j \\ & i \neq j \end{cases}$$

$$= - (-1)^{\delta_{\mu i}} (-1)^{\delta_{\nu j}} \bar{\Psi}(x') \sigma^{\mu\nu} \Psi(x')$$

## ⇒ CHARGE CONJUGATION (C)

↳ SYMMETRY UNDER PARTICLE ↔ ANTI-PARTICLE REVERSAL



## ↳ TRANSFORMATION OF SPINORS

- CONSIDER DIRAC EQUATION FOR  $e^-$  IN EXTERNAL ELECTROMAGNETIC FIELD

$$\partial^\mu \rightarrow \partial^\mu + iq A^\mu$$

MINIMAL SUBSTITUTION  
 $q$  : ELECTRIC CHARGE

$$(i \gamma^\mu \partial_\mu + e \gamma^\mu A_\mu - m) \psi(x) = 0$$

FOR  $e^-$  :  $q = -e < 0$   
 FOR  $e^+$  :  $q = +e > 0$

- POSITRON IN EXTERNAL ELECTROMAGNETIC FIELD DESCRIBED BY  $e \rightarrow -e$

$$(i \gamma^\mu \partial_\mu - e \gamma^\mu A_\mu - m) \psi_c(x) = 0$$

WITH  $\psi_c(x)$  : CHARGE CONJUGATED SPINOR

?  $\psi$  &  $\psi_c$  ARE NOT INDEPENDENT  
 HOW TO RELATE  $\psi_c(x)$  TO  $\psi(x)$

- HOW TO GET FROM FIRST DIRAC EQ.  
THE SECOND DIRAC EQ. ?

TO CHANGE ~~SIGN~~ RELATIVE SIGN BETWEEN  
 $i \gamma^\mu \partial_\mu$  &  $e \gamma^\mu A_\mu$  TERMS  $\Rightarrow$  COMPLEX CONJUGATE

$$(i \gamma^\mu \partial_\mu + e \gamma^\mu A_\mu - m) \psi(x) = 0$$

$$\downarrow$$

$$(-i \gamma^{\mu*} \partial_\mu + e \gamma^{\mu*} A_\mu - m) \psi^*(x) = 0$$

$$\downarrow \quad \gamma^{\mu*} = (\gamma^{\mu\dagger})^T = (\gamma^0 \gamma^\mu \gamma^0)^T$$

$$= \gamma^0 \gamma^{\mu T} \gamma^0$$

$$(-i \gamma^0 \gamma^{\mu T} \gamma^0 \partial_\mu + e \gamma^0 \gamma^{\mu T} \gamma^0 A_\mu - m) \psi^*(x) = 0$$

$$\downarrow \quad \text{MULTIPLY ON LEFT BY } \gamma^0$$

$$(-i \gamma^{\mu T} \partial_\mu + e \gamma^{\mu T} A_\mu - m) \gamma^0 \psi^*(x) = 0$$

$$\downarrow \quad \text{INTRODUCE UNITARY OPERATOR } C$$

$$C^\dagger = C^{-1}$$

$$(-i C \gamma^{\mu T} C^{-1} \partial_\mu + e C \gamma^{\mu T} C^{-1} A_\mu - m) C \gamma^0 \psi^*(x) = 0$$

$$\text{REQUIRE } \underline{\underline{C \gamma^{\mu T} C^{-1} = -\gamma^\mu}}$$

$$(i \gamma^\mu \partial_\mu - e \gamma^\mu A_\mu - m) C \gamma^0 \psi^*(x) = 0$$

$$\therefore \text{CHARGE CONJUGATED SPINOR } \underline{\underline{\psi_c(x) = C \gamma^0 \psi^*(x)}}$$

- CONSTRUCTION OF C SUCH THAT  $C \gamma^{\mu T} C^{-1} = -\gamma^{\mu}$  DIR 70

$$\underline{\underline{C = i \gamma^2 \gamma^0}} \Rightarrow C^T = i \gamma^0 \gamma^2 = C^{-1} = -C$$

CHECK

$$\mu=0 : C \gamma^{0T} C^{-1} = -\gamma^0$$

$$\mu=1,3 \quad C \gamma^{1T} C^{-1} = C \gamma^{1+} C^{-1} = C \gamma^0 \gamma^1 \gamma^0 C^{-1} = i \gamma^2 \gamma^1 i \gamma^2 = -\gamma^1$$

ANALOGOUS FOR  $\mu=3$

$$\mu=2 \quad C \gamma^{2T} C^{-1} = -C \gamma^{2+} C^{-1} = -C \gamma^0 \gamma^2 \gamma^0 C^{-1} = -i \gamma^2 \gamma^2 i \gamma^2 = -\gamma^2$$

- CONSTRUCTION OF CHARGE CONJUGATED SPINOR

PROOF GIVEN FOR HELICITY SPINORS

$$\underbrace{C \gamma^0}_{i \gamma^2} (U(\vec{p}, \lambda))^* = N \begin{pmatrix} i \sigma^2 \frac{\vec{\sigma} \cdot \vec{p}}{E_{\vec{p}} + m} \chi_{\lambda}^* \\ -i \sigma^2 \chi_{\lambda}^* \end{pmatrix}$$

$$= N \begin{pmatrix} \frac{|\vec{p}| 2\lambda}{E_{\vec{p}} + m} i \sigma^2 \chi_{\lambda}^* \\ -i \sigma^2 \chi_{\lambda}^* \end{pmatrix}$$

$$\chi_{\lambda=+\frac{1}{2}} = \begin{pmatrix} \cos \theta/2 \\ e^{i\phi} \sin \theta/2 \end{pmatrix} \downarrow$$

$$\chi_{\lambda=-\frac{1}{2}} = \begin{pmatrix} -e^{-i\phi} \sin \theta/2 \\ \cos \theta/2 \end{pmatrix}$$

$$-i \sigma^2 \chi_{\lambda=+\frac{1}{2}}^* = \begin{pmatrix} -e^{-i\phi} \sin \theta/2 \\ \cos \theta/2 \end{pmatrix}$$

$$-i \sigma^2 \chi_{\lambda=-\frac{1}{2}}^* = \begin{pmatrix} -\cos \theta/2 \\ -e^{i\phi} \sin \theta/2 \end{pmatrix}$$

$$= + \chi_{\lambda=-\frac{1}{2}}$$

$$= - \chi_{\lambda=+\frac{1}{2}}$$



$$\therefore \underline{\underline{-i\sigma^2 \chi_\lambda^* = (2\lambda) \chi_{-\lambda}}}$$

$$C \gamma^0 (U(\bar{p}, \lambda))^* = N \begin{pmatrix} |\bar{p}| (-2\lambda) \chi_{-\lambda} \\ \chi_{-\lambda} \end{pmatrix} \cdot (2\lambda)$$

$$= U(\bar{p}, \lambda)$$

NOTE :  $(2\lambda)$  FACTOR IN  $U$  IS EQUIVALENT  
TO PHASE  $\underbrace{(-1)^{\frac{1}{2}-\lambda}}$

ANALOGOUSLY ONE CAN SHOW

$$C \gamma^0 (U(\bar{p}, \lambda))^* = U(\bar{p}, \lambda)$$

$$\therefore \left\{ \begin{aligned} C \gamma^0 U^*(\bar{p}, \lambda) &= C \bar{U}^T(\bar{p}, \lambda) = U(\bar{p}, \lambda) \\ \bar{U}^T &\stackrel{\uparrow}{=} (U^\dagger \gamma^0)^T = \gamma^0 U^* \\ C \gamma^0 U^*(\bar{p}, \lambda) &= C \bar{U}^T(\bar{p}, \lambda) = U(\bar{p}, \lambda) \end{aligned} \right.$$

# ↳ TRANSFORMATION OF FIELDS : UNITARY TF $U(C)$

PARTICLE  $a(\vec{p}, \lambda) \xrightarrow{C} U(C) a(\vec{p}, \lambda) U^\dagger(C) = b(\vec{p}, \lambda)$

ANTI-PARTICLE  $b(\vec{p}, \lambda) \xrightarrow{C} U(C) b(\vec{p}, \lambda) U^\dagger(C) = a(\vec{p}, \lambda)$

PARTICLE & ANTI-PARTICLE GET INTERCHANGED

$$\hat{\Psi}(x) \xrightarrow{C} U(C) \hat{\Psi}(x) U^\dagger(C)$$

$$= \sum_{\lambda} \int \frac{d^3 \vec{p}}{(2\pi)^3 \sqrt{2E_p}} \left\{ b(\vec{p}, \lambda) U(\vec{p}, \lambda) e^{-i p \cdot x} + a^\dagger(\vec{p}, \lambda) v(\vec{p}, \lambda) e^{+i p \cdot x} \right\}$$

USE  $U = C \bar{v}^T$   
 $v = C \bar{u}^T$

$$= \sum_{\lambda} \int \frac{d^3 \vec{p}}{(2\pi)^3 \sqrt{2E_p}} \left\{ b(\vec{p}, \lambda) C \bar{v}^T(\vec{p}, \lambda) e^{-i p \cdot x} + a^\dagger(\vec{p}, \lambda) C \bar{u}^T(\vec{p}, \lambda) e^{+i p \cdot x} \right\}$$

$$= C \hat{\bar{\Psi}}^T(x) = (\hat{\Psi} C^{-1})^T$$

ANALOGOUSLY

$$\hat{\bar{\Psi}}(x) \xrightarrow{C} \hat{\Psi}^T(x) C = (C^{-1} \hat{\Psi}(x))^T$$

## ↳ TRANSFORMATION OF BILINEARS

### • SCALAR

$$\begin{aligned}
 \bar{\psi}(x) \psi(x) &\xrightarrow{C} (i\gamma^0\gamma^2\psi)^T (\bar{\psi} i\gamma^0\gamma^2)^T \\
 &\quad \swarrow \quad \searrow \\
 &= - (\bar{\psi} i\gamma^0\gamma^2) (i\gamma^0\gamma^2\psi) \\
 &\quad \uparrow \\
 &\quad \text{MINUS SIGN DUE TO} \\
 &\quad \text{FERMION ANTI-COMMUTATION} \\
 &= + \bar{\psi}(x) \psi(x)
 \end{aligned}$$

### • PSEUDO-SCALAR

$$\begin{aligned}
 \bar{\psi}(x) i\gamma_5 \psi(x) &\xrightarrow{C} (i\gamma^0\gamma^2\psi)^T i\gamma_5 (\bar{\psi} i\gamma^0\gamma^2)^T \\
 &= - \bar{\psi} i\gamma^0\gamma^2 (i\gamma_5)^T i\gamma^0\gamma^2\psi \\
 &= \bar{\psi} i\gamma_5 \psi
 \end{aligned}$$

### • VECTOR

$$\begin{aligned}
 \bar{\psi}(x) \gamma^\mu \psi(x) &\xrightarrow{C} (i\gamma^0\gamma^2\psi)^T \gamma^\mu (\bar{\psi} i\gamma^0\gamma^2)^T \\
 &= - \bar{\psi} i\gamma^0\gamma^2 \gamma^{\mu T} i\gamma^0\gamma^2\psi \\
 &\quad \downarrow \quad \gamma^{0T} = \gamma^0, \gamma^{2T} = \gamma^2, \gamma^{1T} = -\gamma^1, \gamma^{3T} = -\gamma^3 \\
 &= - \bar{\psi}(x) \gamma^\mu \psi(x)
 \end{aligned}$$

- AXIAL - VECTOR

$$\begin{aligned}
 \bar{\Psi}(x) \gamma^\mu \gamma_5 \Psi(x) &\xrightarrow{C} (i \gamma^0 \gamma^2 \Psi)^T \gamma^\mu \gamma_5 (\bar{\Psi} i \gamma^0 \gamma^2)^T \\
 &= - \bar{\Psi} i \gamma^0 \gamma^2 \gamma_5 \gamma^{\mu T} i \gamma^0 \gamma^2 \Psi \\
 &= - \bar{\Psi} \gamma^0 \gamma^2 \gamma^{\mu T} \gamma^0 \gamma^2 \gamma_5 \Psi \\
 &= + \bar{\Psi}(x) \gamma^\mu \gamma_5 \Psi(x)
 \end{aligned}$$

### 2.5.2.2 TENSOR

- TENSOR

$$\begin{aligned}
 \bar{\Psi} \sigma^{\mu\nu} \Psi &\xrightarrow{C} (i \gamma^0 \gamma^2 \Psi)^T \sigma^{\mu\nu} (\bar{\Psi} i \gamma^0 \gamma^2)^T \\
 &= - \bar{\Psi} i \gamma^0 \gamma^2 (\sigma^{\mu\nu})^T i \gamma^0 \gamma^2 \Psi \\
 &= - \bar{\Psi}(x) \sigma^{\mu\nu} \Psi(x)
 \end{aligned}$$



# ⇒ CPT SYMMETRY

|     | $\bar{\psi}\psi$ | $\bar{\psi}i\gamma_5\psi$ | $\bar{\psi}\gamma^\mu\psi$ | $\bar{\psi}\gamma^4\gamma_5\psi$ | $\bar{\psi}\sigma^{\mu\nu}\psi$               |
|-----|------------------|---------------------------|----------------------------|----------------------------------|-----------------------------------------------|
| P   | +1               | -1                        | $(-1)^{\delta_{\mu i}}$    | $-(-1)^{\delta_{\mu i}}$         | $(-1)^{\delta_{\mu i}}(-1)^{\delta_{\nu j}}$  |
| T   | +1               | -1                        | $(-1)^{\delta_{\mu i}}$    | $(-1)^{\delta_{\mu i}}$          | $-(-1)^{\delta_{\mu i}}(-1)^{\delta_{\nu j}}$ |
| C   | +1               | +1                        | -1                         | +1                               | -1                                            |
| CPT | +1               | +1                        | -1                         | -1                               | +1                                            |

NOTE

$$\begin{aligned} \partial_\mu &\xrightarrow{P} (-1)^{\delta_{\mu i}} \partial_\mu \\ \partial_\mu &\xrightarrow{T} -(-1)^{\delta_{\mu i}} \partial_\mu \\ \partial_\mu &\xrightarrow{C} \partial_\mu \\ \partial_\mu &\xrightarrow{CPT} -\partial_\mu \end{aligned}$$

$\mathcal{L}_{\text{DIRAC}} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$  IS INVARIANT UNDER P, T, C

• GENERAL  $\mathcal{L}$  : LORENTZ SCALAR e.g.  $\bar{\psi}\gamma^\mu\psi A_\mu$

CPT THEOREM : LORENTZ INVARIANT  $\mathcal{L}$  + UNITARITY  
INVARIANT UNDER CPT

⇒ PROOF OF INVARIANCE OF  $\mathcal{L}_{\text{DIRAC}} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$   
 UNDER C, P, AND T SEPARATELY

• PARITY P

$$\bar{\psi}\psi \xrightarrow{P} \bar{\psi}\psi \quad \checkmark$$

$$\partial_\mu = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right) \xrightarrow{P} \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right) = (-1)^\mu \partial_\mu$$

$$\bar{\psi}\gamma^\mu \partial_\mu \psi \xrightarrow{P} (-1)^\mu (-1)^\mu \bar{\psi}\gamma^\mu \partial_\mu \psi = \bar{\psi}\gamma^\mu \partial_\mu \psi \quad \checkmark$$

• TIME - REVERSAL T

$$\bar{\psi}\psi \xrightarrow{T} \bar{\psi}\psi \quad \checkmark$$

$$\partial_\mu = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right) \xrightarrow{T} \left(-\frac{\partial}{\partial t}, \vec{\nabla}\right) = -(-1)^\mu \partial_\mu$$

$$\begin{aligned} \bar{\psi} i\gamma^\mu \partial_\mu \psi &\xrightarrow{T} \bar{\psi} \gamma^3 \gamma^1 (-i) \gamma^{\mu*} \gamma^1 \gamma^3 (\partial_\mu \psi) (-1)(-1)^\mu \\ &= i(-1)^\mu \underbrace{\bar{\psi} \gamma^3 \gamma^1 \gamma^{\mu*} \gamma^1 \gamma^3 (\partial_\mu \psi)}_{(-1)^\mu \bar{\psi} \gamma^\mu \psi} \\ &= \bar{\psi} i\gamma^\mu \partial_\mu \psi \quad \checkmark \end{aligned}$$

# CHARGE CONJUGATION C

$$\bar{\psi} \psi \xrightarrow{C} \bar{\psi} \psi \quad \checkmark \quad \partial_\mu \xrightarrow{C} \partial_\mu$$

$$\bar{\psi} i \gamma^\mu \partial_\mu \psi \xrightarrow{C} (i \gamma^0 \gamma^2 \psi)^T i \gamma^\mu \partial_\mu (\bar{\psi} i \gamma^0 \gamma^2)^T$$

$$= (-1) \left( \partial_\mu \bar{\psi} \right) i \gamma^0 \gamma^2 i \gamma^{\mu T} i \gamma^0 \gamma^2 \psi$$

↑  
FERMION  
ANTI-  
COMMUTATION

$$= i (\partial_\mu \bar{\psi}) \gamma^0 \gamma^2 \gamma^{\mu T} \gamma^0 \gamma^2 \psi$$

↓

$$\gamma^{0T} = \gamma^0, \quad \gamma^{2T} = \gamma^2 \\ \gamma^{1T} = -\gamma^1, \quad \gamma^{3T} = -\gamma^3$$

$$= -i (\partial_\mu \bar{\psi}) \gamma^\mu \psi$$

$$= i \bar{\psi} \gamma^\mu (\partial_\mu \psi) - i \underbrace{\partial_\mu (\bar{\psi} \gamma^\mu \psi)}_{\text{TOTAL DERIVATIVE}}$$

✓

TOTAL  
DERIVATIVE