

VI

RENORMALIZATION OF QED

- 1) VACUUM POLARIZATION IN QED
- 2) RUNNING COUPLING IN QED
- 3) ANOMALOUS MAGNETIC MOMENT IN QED
- 4) ELECTRON SELF-ENERGY IN QED
- 5) RENORMALIZED PERTURBATION THEORY (QED)
- 6) RENORMALIZABILITY : GENERAL

1) VACUUM POLARIZATION IN QED

⇒ SCALAR QED

↳ FEYNMAN RULES IN SCALAR QED

↳ SPIN-0 MATTER PARTICLES
e.g. π^+

$$\mathcal{L}_{KG} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - m^2 \phi^\dagger \phi$$

BY REQUIRING THAT $\mathcal{L}$ IS INVARIANT
UNDER $U(1)$ LOCAL PHASE TRANSFORMATION

$$\phi(x) \xrightarrow{U(1)} e^{i\chi(x)} \phi(x)$$

WE NEED TO REPLACE

$$\partial^\mu \rightarrow \mathcal{D}^\mu = \partial^\mu + ie A^\mu$$

WITH

$$A^\mu(x) \xrightarrow{U(1)} A^\mu(x) - \frac{1}{e} \partial^\mu \chi$$

$$\therefore \mathcal{L} = (\mathcal{D}_\mu \phi)^\dagger (\mathcal{D}^\mu \phi) - m^2 \phi^\dagger \phi$$

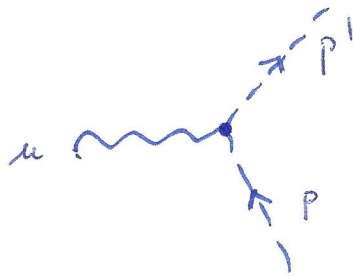
$$= (\partial_\mu \phi^\dagger - ie A_\mu \phi^\dagger) (\partial^\mu \phi + ie A^\mu \phi) - m^2 \phi^\dagger \phi$$

$$= \mathcal{L}_{KG} + \mathcal{L}_{INT}$$

$$\mathcal{L}_{\text{INT}} = -ie \left(\phi^\dagger (\partial^\mu \phi) - (\partial^\mu \phi^\dagger) \phi \right) A^\mu + e^2 A_\mu A^\mu \phi^\dagger \phi$$

2 TYPES OF INTERACTION TERMS FOR SCALAR QED

• LINEAR IN A^μ



$$S = \mathbb{1} + \mathcal{M} (2\pi)^4 \delta(\dots)$$

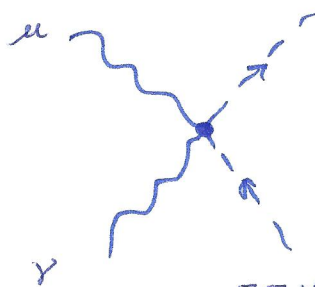
$\mathcal{M}$: INVARIANT AMPLITUDE

$$\mathcal{M}_{fi} = -ie (-ip^\mu - ip'^\mu) \epsilon_\mu$$

FEYNMAN RULE:
FOR VERTEX

$$\boxed{-ie (p + p')^\mu}$$

• QUADRATIC IN A^μ



$\mathcal{M}_{fi} = +2ie^2 g^{\mu\nu} \epsilon_\nu \epsilon_\mu^*$

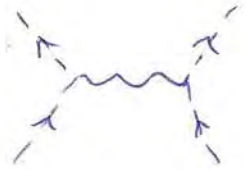
 EACH A_μ FIELD CAN ABSORB INITIAL γ

FEYNMAN RULE:
FOR VERTEX

$$\boxed{+2ie^2 g^{\mu\nu}}$$

↳ 1-LOOP CORRECTION TO PHOTON PROPAGATOR

- LOWEST ORDER E.M. INTERACTION BETWEEN 2 CHARGED PARTICLE PROCEEDS THROUGH PHOTON PROPAGATOR

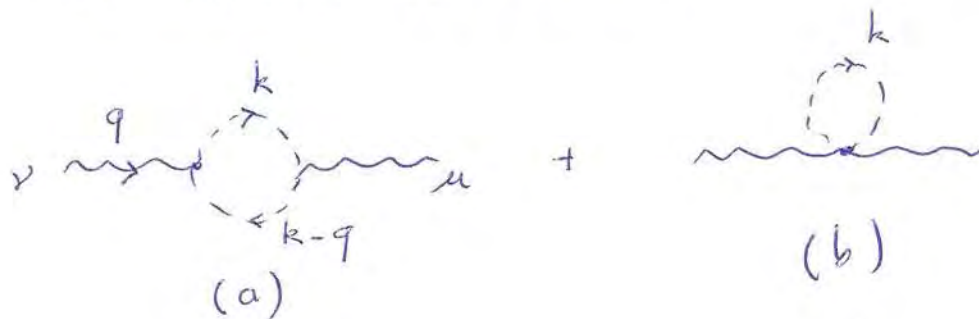


$$i D_0^{\mu\nu}(q) = \frac{-i g^{\mu\nu}}{q^2 + i\epsilon}$$

(FEYNMAN GAUGE)

SUBSCRIPT ZERO DENOTES LOWEST ORDER

- 1st ORDER (in e^2) CORRECTION TO PHOTON PROPAGATOR



$$\begin{aligned} \mathcal{M}_a^{\mu\nu} &= \int \frac{d^4 k}{(2\pi)^4} \frac{i(-ie)(2k-q)^\nu i(-ie)(2k-q)^\mu}{(k^2 - m^2 + i\epsilon)((k-q)^2 - m^2 + i\epsilon)} \\ &= e^2 \int \frac{d^4 k}{(2\pi)^4} \frac{(2k-q)^\nu (2k-q)^\mu}{(k^2 - m^2 + i\epsilon)((k-q)^2 - m^2 + i\epsilon)} \end{aligned}$$

$$\begin{aligned}\mathcal{M}_b^{\mu\nu} &= 2ie^2 g^{\mu\nu} \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \\ &= -2e^2 g^{\mu\nu} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon}\end{aligned}$$

$$\mathcal{M}^{\mu\nu} = \mathcal{M}_a^{\mu\nu} + \mathcal{M}_b^{\mu\nu}$$

$$\begin{aligned}&= -e^2 \int \frac{d^4 k}{(2\pi)^4} \frac{1}{[k^2 - m^2 + i\epsilon][(k-q)^2 - m^2 + i\epsilon]} \\ &\quad \cdot \left\{ -4k^\mu k^\nu + 2k^\mu q^\nu + 2k^\nu q^\mu \right. \\ &\quad \left. - q^\mu q^\nu + 2g^{\mu\nu} [(k-q)^2 - m^2] \right\}\end{aligned}$$

- GAUGE INV. REQUIRES

$$q_\mu \mathcal{M}^{\mu\nu} = 0 \quad , \quad q_\nu \mathcal{M}^{\mu\nu} = 0$$

$$\begin{aligned}q_\mu \mathcal{M}^{\mu\nu} &= -e^2 \int \frac{d^4 k}{(2\pi)^4} \frac{1}{[\dots][\dots]} \\ &\quad \cdot \left\{ -4(k \cdot q) k^\nu + 2(k \cdot q) q^\nu \right. \\ &\quad \left. + 2q^2 k^\nu - q^2 q^\nu \right. \\ &\quad \left. + 2q^\nu [(k-q)^2 - m^2] \right\}\end{aligned}$$

FEYNMAN PARAMETRIZATION

BRING INTEGRAL TO COMMON DENOMINATOR

USING FEYNMAN PARAMETER TRICK

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[A + (B-A)x]^2}$$

PROOF.

$$\int_0^1 dx \frac{1}{[A + (B-A)x]^2}$$

$$= \frac{-1}{(B-A)} \cdot \frac{1}{A + (B-A)x} \Big|_0^1$$

$$= \frac{1}{(A-B)} \cdot \left\{ \frac{1}{B} - \frac{1}{A} \right\}$$

$$= \frac{1}{A \cdot B}$$

• FOR $A = k^2 - m^2 + i\epsilon$

$$B = (k-q)^2 - m^2 + i\epsilon$$

$\Downarrow$

$$B-A = -2k \cdot q + q^2$$

$$\mathcal{M}^{\mu\nu} = -e^2 \int_0^1 dx \int \frac{d^4 k}{(2\pi)^4} \frac{1}{[k^2 + (q^2 - 2k \cdot q)x - m^2 + i\epsilon]^2} \cdot \left\{ -4k^\mu k^\nu + 2k^\mu q^\nu + 2k^\nu q^\mu - q^\mu q^\nu + 2g^{\mu\nu} [(k-q)^2 - m^2] \right\}$$

$$= -e^2 \int_0^1 dx \int \frac{d^4 k}{(2\pi)^4} \frac{1}{[(k-qx)^2 + q^2 x(1-x) - m^2 + i\epsilon]^2} \cdot \{ \dots \}$$

 $\downarrow$

$$k' = k - qx$$

$$= -e^2 \int_0^1 dx \int \frac{d^4 k}{(2\pi)^4} \frac{1}{[k^2 + q^2 x(1-x) - m^2 + i\epsilon]^2} \cdot \left\{ -4(k^\mu + q^\mu x)(k^\nu + q^\nu x) + 2(k^\mu + q^\mu x)q^\nu + 2(k^\nu + q^\nu x)q^\mu - q^\mu q^\nu + 2g^{\mu\nu} [(k - q(1-x))^2 - m^2] \right\}$$

 $\downarrow$

$$\int d^4 k \frac{k^\alpha}{(k^2 + c)^2} = 0.$$

$$\mathcal{M}^{\mu\nu} = -e^2 \int_0^1 dx \int \frac{d^4 k}{(2\pi)^4} \frac{1}{[k^2 + q^2 x(1-x) - m^2 + i\epsilon]^2}$$

$$\cdot \left\{ -4k^\mu k^\nu + q^\mu q^\nu [4x(1-x) - 1] + 2g^{\mu\nu} [k^2 + q^2(1-x)^2 - m^2] \right\}$$

NOTE $\int d^4 k \frac{k^\mu k^\nu}{(k^2 + c)^2}$ IS QUADRATICALLY DIVERGENT ∇_0

• FULL PHOTON PROPAGATOR

$$i D^{\mu\nu}(q) = i D_0^{\mu\nu}(q) + i D_0^{\mu\alpha}(q) \mathcal{M}_{\alpha\beta}(q) i D_0^{\beta\nu}(q)$$

$$\mathcal{M}_{\alpha\beta}(q) \equiv i \Pi_{\alpha\beta}(q)$$

$$D^{\mu\nu}(q) = D_0^{\mu\nu}(q) - D_0^{\mu\alpha}(q) \Pi_{\alpha\beta}(q) D_0^{\beta\nu}(q)$$

$$\boxed{D^{\mu\nu}(q) = \frac{1}{q^2 + i\epsilon} \left\{ -g^{\mu\nu} - \underbrace{\Pi^{\mu\nu}(q)}_{\substack{\text{VACUUM POLARIZATION} \\ \text{CORRECTION}}} \frac{1}{q^2} \right\}}$$

(FEYNMAN GAUGE)

VACUUM POLARIZATION
CORRECTION

WITH

$$\boxed{\begin{aligned} \Pi^{\mu\nu}(q) = & ie^2 \int_0^1 dx \int \frac{d^4 k}{(2\pi)^4} \frac{1}{[k^2 + q^2 x(1-x) - m^2 + i\epsilon]^2} \\ & \cdot \left\{ -4k^\mu k^\nu + q^\mu q^\nu [4x(1-x) - 1] \right. \\ & \left. + 2g^{\mu\nu} [k^2 + q^2 x(1-x) - m^2] \right\} \end{aligned}}$$

BECAUSE k IS INTEGRATED OVER

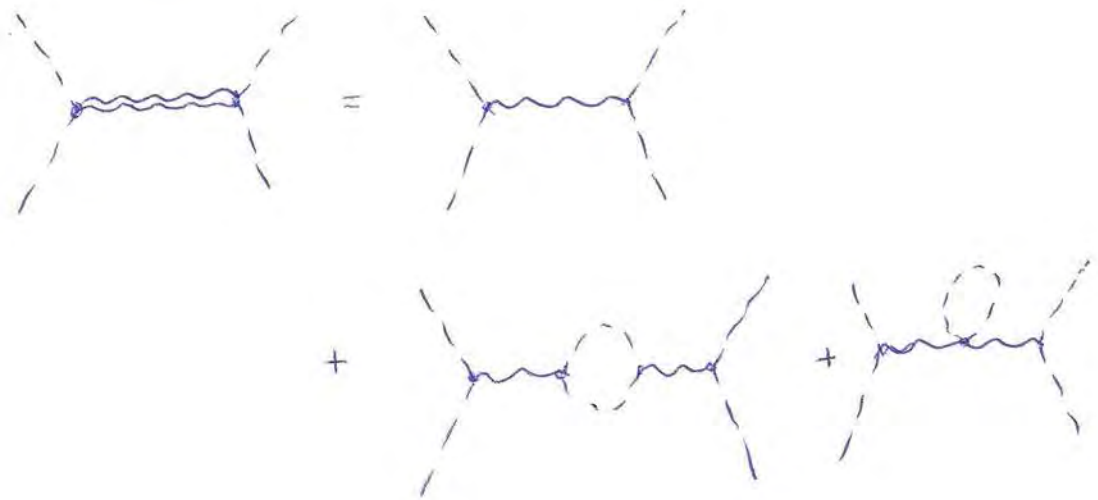
$$\Pi^{\mu\nu}(q) = \Pi_1(q^2) q^2 g^{\mu\nu} + \Pi_2(q^2) q^\mu q^\nu$$

$$\S \text{ GAUGE INVARIANCE} \quad q_\mu \Pi^{\mu\nu} = 0 = q_\nu \Pi^{\mu\nu}$$

$$\boxed{\Pi^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu) \overset{\text{SCALAR FUNCTION}}{\Pi(q^2)}}$$

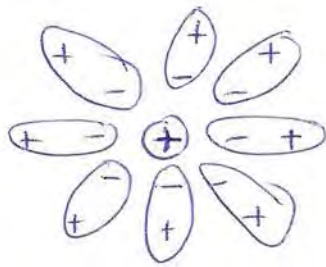
$$\Pi_1(q^2) \equiv \Pi(q^2), \quad \Pi_2(q^2) = -\Pi_1(q^2)$$

• DIVERGENCE OF VACUUM POL.



DUE TO VACUUM POL.

A TEST CHARGE IS POLARIZED IN VACUUM



TEST CHARGE $\rightarrow$ SURROUNDED BY DIPOLES

$\rightarrow$ SCREENED : WE DON'T SEE THE "BARE" CHARGE

$\hookrightarrow$ IN ABSENCE OF VACUUM POL.

RENORMALIZATION: WE WILL ABSORB DIVERGENCE IN REDEFINING UNPHYSICAL "BARE" CHARGE

↳ REGULARIZATION

- CONSIDER INTEGRAL

$$I = \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 - \Lambda + i\epsilon)^2}$$

FOR $k \rightarrow \infty$: I IS LOGARITHMICALLY DIVERGENT
AT HIGH-ENERGY
(UV DIVERGENCE)

IN ORDER TO CALCULATE LOOPS WITH $\mathcal{L}_{\text{INT}}$
WE NEED TO REGULARIZE THEORY
IN INTERMEDIATE STEP

- DIFFERENT REGULARIZATION METHODS :

1) CUT-OFF METHOD

$$\int d^4 k \rightarrow \int_{\Lambda} d^4 k$$

↓
VIOLATES LORENTZ INVARIANCE

2) PAULI-VILLARS REGULARIZATION

$$\frac{1}{k^2 - m^2 + i\epsilon} \rightarrow \frac{1}{k^2 - m^2 + i\epsilon} \cdot \left(\frac{m^2 - \Lambda^2}{k^2 - \Lambda^2 + i\epsilon} \right)$$

EXTRA FACTOR $\left(\frac{m^2 - \Lambda^2}{k^2 - \Lambda^2 + i\epsilon} \right)$ $\xrightarrow{\Lambda \rightarrow \infty}$ 1
IMPROVES CONVERGENCE
BY 2 POWERS

3) DIMENSIONAL REGULARIZATION. ('T HOOFT, VELTMAN)

CALCULATE LOOPS IN D -DIMENSIONS

$$\int d^4 k \rightarrow \int d^D k$$

WITH $D < 4 \Rightarrow$ IMPROVES CONVERGENCE

DIVERGENCE WILL APPEAR AS POLE

$$\text{IN } \underline{\underline{\epsilon \equiv 2 - \frac{D}{2}}} \quad \text{FOR } D < 4, \epsilon > 0$$

$$\frac{1}{\epsilon} \rightarrow \infty \quad \text{IN LIMIT } D \rightarrow 4$$

AT END : WHEN DIVERGENCE IS ABSORBED
BY REDEFINING BARE QUANTITY (CHARGE)
INTO PHYSICAL QUANTITY (OBSERVED)

ONE CAN LET $\epsilon \rightarrow 0$ (ANALYTICAL
CONTINUATION)

↳ EVALUATION OF INTEGRALS : WICK ROTATION

$$I = \int \frac{d^D k}{(2\pi)^D} \cdot \frac{1}{(k^2 - \Delta + i\varepsilon)^n}$$

- INTRODUCE SPHERICAL COORDINATES IN D DIM EUCLIDEAN SPACE

$$\theta_1 \dots \theta_{D-2} : 0 \rightarrow \pi$$

$$\phi : 0 \rightarrow 2\pi$$

$$\begin{aligned} d^D k &= |k|^{D-1} \sin^{D-2} \theta_1 \dots \sin \theta_{D-2} d|k| d\theta_1 \dots d\theta_{D-2} d\phi \\ &= |k|^{D-1} d|k| d\Omega_D \end{aligned}$$

- $d\Omega_D$: SOLID ANGLE ON UNIT SPHERE IN D -DIM EUCLIDEAN SPACE

$$D=2 : \int d\Omega_2 = \int_0^{2\pi} d\phi = 2\pi \quad (\text{CIRCLE})$$

$$D=3 \quad \int d\Omega_3 = \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta = 4\pi \quad (\text{SPHERE})$$

↓
2 DIM SURFACE

$$\begin{aligned} D=4 \quad \int d\Omega_4 &= \int_0^{2\pi} d\phi \int_0^\pi d\theta_1 \sin^2 \theta_1 \int_0^\pi d\theta_2 \sin \theta_2 \\ &= 4\pi \int_0^\pi d\theta_1 \sin^2 \theta_1 \end{aligned}$$

$$\begin{aligned}\int d\Omega_4 &= 4\pi \cdot \int_{-1}^1 dx (1-x^2)^{1/2} \\ &= 4\pi \int_0^1 du u^{-1/2} (1-u)^{1/2} \quad u = x^2\end{aligned}$$

USE BETA-FUNCTION

$$B(m, m) = \int_0^1 dx x^{m-1} (1-x)^{m-1} = \frac{\Gamma(m) \Gamma(m)}{\Gamma(m+m)}$$

$$\int d\Omega_4 = 4\pi B\left(\frac{1}{2}, \frac{3}{2}\right) = 4\pi \frac{\Gamma(\frac{1}{2}) \Gamma(\frac{3}{2})}{\Gamma(2)}$$

$$\Gamma(1) = 1$$

$$\Gamma(n) = (n-1)! \quad \text{FOR INTEGER } n$$

$$\Gamma(x) = (x-1) \Gamma(x-1)$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\int d\Omega_4 = 4\pi \cdot \sqrt{\pi} \cdot \frac{1}{2} \sqrt{\pi} = 2\pi^2$$

(3-SPHERE)

↓
3 DIM SURFACE

ARBITRARY D

$$\int_{(D-1)} d\Omega_D = 2\pi \int_0^\pi d\theta_1 \sin^{\overbrace{D-2}^{D-2}} \theta_1 \cdots \int_0^\pi d\theta_{D-1} \sin \theta_{D-1}$$

$\uparrow$
 $D-1$ DIM SURFACE

$$\text{WITH } \int_0^\pi d\theta \sin^n \theta = \int_{-1}^1 dx (1-x^2)^{\frac{n-1}{2}}$$

$$= \int_0^1 du u^{-\frac{1}{2}} (1-u)^{\frac{n-1}{2}}$$

$$= \frac{\Gamma(\frac{1}{2}) \Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2} + 1)} = \sqrt{\pi} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2} + 1)}$$

$$\therefore \int_{(D-1)} d\Omega_D = 2\pi (\pi)^{\frac{D-2}{2}} \cdot \frac{\Gamma(\frac{D-1}{2})}{\Gamma(\frac{D}{2})} \cdot \frac{\Gamma(\frac{D-2}{2})}{\Gamma(\frac{D-1}{2})} \cdots \frac{\Gamma(1)}{\Gamma(\frac{3}{2})}$$

$$\int_{(D-1)} d\Omega_D = 2 \frac{\pi^{D/2}}{\Gamma(\frac{D}{2})}$$

THROUGH ANALYTICAL CONTINUATION
 RESULT ALSO FOR NON-INTEGER D

- WICK ROTATION

ASSUME $\Delta > 0$ (CAN ALSO BE SHOWN FOR $\Delta < 0$)

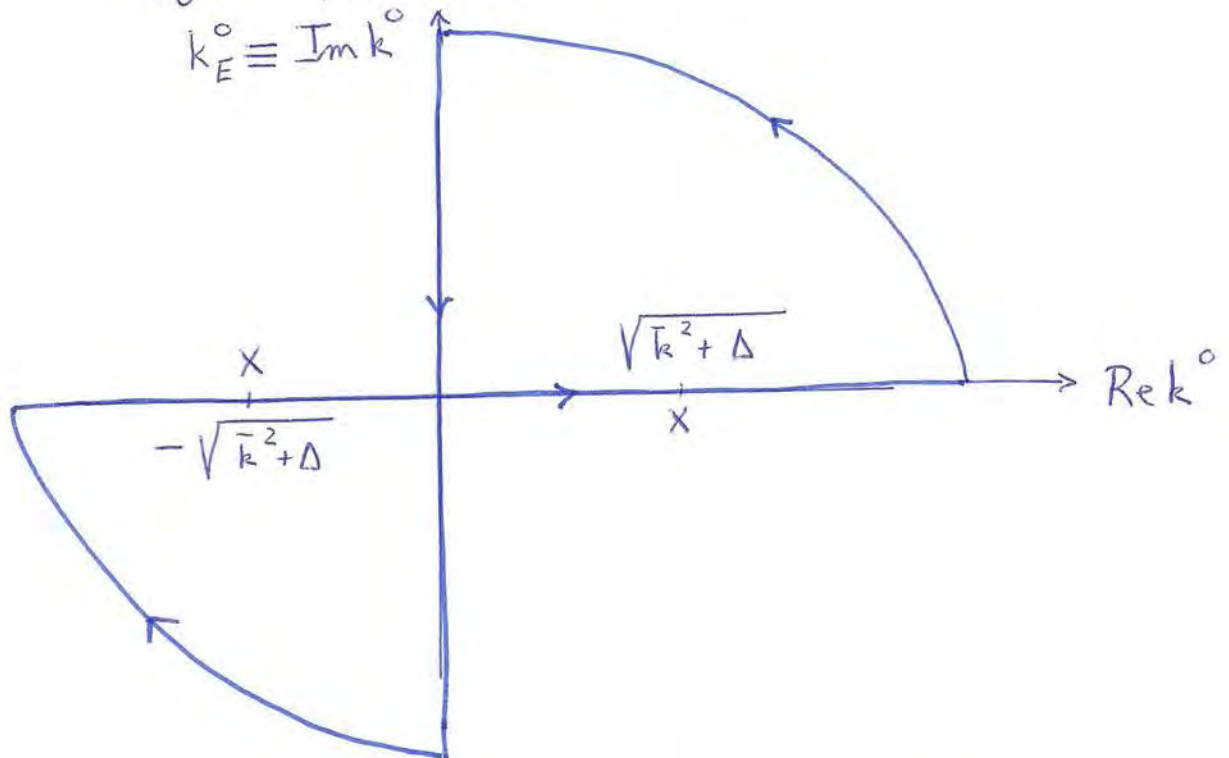
$$\begin{aligned}
 k^2 - \Delta + i\epsilon &= k_0^2 - \vec{k}^2 - \Delta + i\epsilon \\
 &= (k_0 - \sqrt{\vec{k}^2 + \Delta} + i\epsilon) \\
 &\quad \cdot (k_0 + \sqrt{\vec{k}^2 + \Delta} - i\epsilon)
 \end{aligned}$$

INTEGRAL HAS POLES

AT $k_0 = \sqrt{\vec{k}^2 + \Delta} - i\epsilon$

$$k_0 = -\sqrt{\vec{k}^2 + \Delta} + i\epsilon$$

$$k_E^0 \equiv \text{Im } k^0$$



WE CAN REPLACE INTEGRAL ALONG $\text{Re } k_0$

BY INTEGRAL ALONG $\text{Im } k_0$: WICK ROTATION

(BECAUSE ALL POLES ARE OUTSIDE CONTOUR)

$$\underline{k^0 \equiv i k_E^0}$$

k_E^0 : EUCLIDEAN ENERGY

$$k^\mu (k^0, \vec{k})$$

$$k^2 = (k^0)^2 - \vec{k}^2 = -k_E^2$$

$$k_E^\mu (k_E^0, \vec{k})$$

$$k_E^2 \equiv (k_E^0)^2 + \vec{k}^2$$

↳ EUCLIDEAN SPACE
HAS UNIT METRIC

$$\int_{-\infty}^{+\infty} dk^0 \frac{1}{(k^2 - \Delta + i\varepsilon)^m} + i \int_{+\infty}^{-\infty} dk_E^0 \frac{1}{(-k_E^2 - \Delta + i\varepsilon)^m} = 0$$

$$\therefore \int_{-\infty}^{+\infty} dk^0 \frac{1}{(k^2 - \Delta + i\varepsilon)^m} = i \int_{-\infty}^{+\infty} dk_E^0 \frac{1}{(-k_E^2 - \Delta + i\varepsilon)^m}$$

$$I = i \int \frac{d^D k_E}{(2\pi)^D} \cdot \frac{1}{(-k_E^2 - \Delta + i\varepsilon)^m}$$

↓
IN EUCLIDEAN SPACE ;
WE CAN USE SPHERICAL COORDINATES
IN D DIMENSIONS

$$I = i 2 \cdot \frac{\pi^{D/2}}{\Gamma(\frac{D}{2})} \cdot \frac{1}{(2\pi)^D} \int_0^\infty dk_E \frac{k_E^{D-1}}{(-k_E^2 - \Delta + i\varepsilon)^m}$$

$$\downarrow \quad \ell = k_E^2$$

$$I = \frac{i}{(4\pi)^{D/2}} \cdot \frac{(-1)^m}{\Gamma(\frac{D}{2})} \int_0^\infty d\ell \frac{\ell^{\frac{D-2}{2}}}{(\ell + \Delta - i\varepsilon)^m}$$

$$\downarrow \quad x \equiv \frac{\Delta}{\ell + \Delta} \quad dx = -\frac{x^2}{\Delta} d\ell$$

$$1 - x = \frac{\ell}{\ell + \Delta}$$

NOTE Δ STANDS FOR $\Delta - i\varepsilon$

$$I = \frac{i(-1)^m}{(4\pi)^{D/2}} \cdot \frac{1}{\Gamma(\frac{D}{2})} \int_0^1 dx \frac{\Delta}{x^2} (1-x)^{\frac{D}{2}-1} \left(\frac{x}{\Delta}\right)^{m-\frac{D}{2}+1}$$

$$= \frac{i(-1)^m}{(4\pi)^{D/2}} \cdot \frac{1}{\Gamma(\frac{D}{2})} \cdot \frac{1}{\Delta^{m-\frac{D}{2}}} \mathcal{B}\left(m-\frac{D}{2}, \frac{D}{2}\right)$$

$$= \frac{i}{(4\pi)^{D/2}} \cdot \frac{(-1)^m}{\Delta^{m-D/2}} \cdot \frac{1}{\Gamma(\frac{D}{2})} \cdot \frac{\Gamma(m-\frac{D}{2}) \Gamma(\frac{D}{2})}{\Gamma(m)}$$

$$I = \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 - \Delta + i\varepsilon)^m} = \frac{i}{(4\pi)^{D/2}} \cdot \frac{\Gamma(m-\frac{D}{2})}{\Gamma(m)} \cdot \frac{(-1)^m}{(\Delta - i\varepsilon)^{m-\frac{D}{2}}}$$

• EXAMPLE $m = 2$

$$\underline{I} = \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 - \Delta + i\varepsilon)^2} = \frac{i}{(4\pi)^{D/2}} \frac{\Gamma(2 - \frac{D}{2})}{\Gamma(2)} \frac{1}{\Delta^{2-D/2}}$$

$$\downarrow \quad \varepsilon \equiv 2 - \frac{D}{2}$$

$$= \frac{i}{(4\pi)^2} \left(\frac{4\pi}{\Delta} \right)^\varepsilon \Gamma(\varepsilon)$$

USE: LAURENT SERIES EXPANSION OF Γ

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma_E + O(\varepsilon)$$

EULER-MASCHERONI CONSTANT

$$\gamma_E \approx 0.577$$

$$\begin{aligned} \left(\frac{4\pi}{\Delta} \right)^\varepsilon &= \exp \left\{ \varepsilon \ln \left(\frac{4\pi}{\Delta} \right) \right\} \\ &= 1 + \varepsilon \ln \left(\frac{4\pi}{\Delta} \right) + O(\varepsilon^2) \end{aligned}$$

$$\therefore \underline{I} = \frac{i}{(4\pi)^2} \left\{ \frac{1}{\varepsilon} + \ln \left(\frac{4\pi}{\Delta} \right) - \gamma_E + O(\varepsilon) \right\}$$

↑
LOGARITHMIC DIVERGENCE OF $\underline{I}$
IS EXPRESSED AS POLE AT $\varepsilon = 0$
($D=4$)

• FURTHER INTEGRALS

$$\hookrightarrow \int \frac{d^D k}{(2\pi)^D} \cdot \frac{k^\mu}{(k^2 - \Delta + i\varepsilon)^n} = 0$$

ODD FUNCTION
INTEGRATED BETWEEN
SYMM BOUNDARIES

$$\hookrightarrow \int \frac{d^D k}{(2\pi)^D} \cdot \frac{k^\mu k^\nu}{(k^2 - \Delta + i\varepsilon)^n} = A g^{\mu\nu}$$

ONLY RANK-2
SYMM. TENSOR
POSSIBLE

↓

MULTIPLY BY $g_{\mu\nu}$

NOTE: IN D-DIM: $g_{\mu\nu} g^{\mu\nu} = D$

$$A = \frac{1}{D} \int \frac{d^D k}{(2\pi)^D} \frac{k^2}{(k^2 - \Delta)^n}$$

$$= \frac{1}{D} \int \frac{d^D k}{(2\pi)^D} \cdot \left\{ \frac{1}{(k^2 - \Delta)^{n-1}} + \frac{\Delta}{(k^2 - \Delta)^n} \right\}$$

$$= \frac{i}{(4\pi)^{D/2}} \frac{1}{D} \left\{ \frac{\Gamma(n-1-\frac{D}{2})}{\Gamma(n-1)} \frac{(-1)^{n-1}}{\Delta^{n-1-\frac{D}{2}}} + \frac{\Gamma(n-\frac{D}{2})}{\Gamma(n)} \frac{(-1)^n \Delta}{\Delta^{n-\frac{D}{2}}} \right\}$$

$$= \frac{i}{(4\pi)^{D/2}} \frac{1}{D} \frac{(-1)^{n-1}}{\Delta^{n-1-\frac{D}{2}}} \cdot \frac{\Gamma(n-1-\frac{D}{2})}{\Gamma(n)} \underbrace{\left\{ n-1 - (n-1-\frac{D}{2}) \right\}}_{D/2}$$

$$\int \frac{d^D k}{(2\pi)^D} \frac{k^\mu k^\nu}{(k^2 - \Delta + i\varepsilon)^n} = \frac{i g^{\mu\nu}}{(4\pi)^{D/2}} \frac{\Gamma(n-1-\frac{D}{2})}{2\Gamma(n)} \cdot \frac{(-1)^{n-1}}{\Delta^{n-1-D/2}}$$

NOTE

$$\Gamma(-n + \varepsilon) = \frac{(-1)^n}{n!} \left\{ \frac{1}{\varepsilon} + \left(1 + \dots + \frac{1}{n} \right) - \gamma_E + O(\varepsilon) \right\}$$

$n=1, 2, 3, \dots$

→ FIELD DIMENSIONS IN D -DIM SPACE

WHEN GOING TO D -DIM SPACE WE CHANGE DIMENSIONS OF FIELDS

$$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - m^2 \phi^\dagger \phi - ie (\phi^\dagger (\partial^\mu \phi) - (\partial^\mu \phi)^\dagger \phi) A_\mu - e^2 A_\mu A^\mu \phi^\dagger \phi$$

- $D = 4$

$$S^{(1)} = i \int d^4x \mathcal{L}_{\text{INT}}^{(1)}$$

MASS DIMENSION []

$$[S] = 0 \quad S\text{-MATRIX : DIMENSIONLESS}$$

$$[\mathcal{L}] = 4$$

$$[\phi] = 1$$

$$[A^\mu] = 1$$

$$[e] = 0$$

- $D \neq 4$

$$[S] = 0$$

$$[d^Dx] = -D$$

$$[\mathcal{L}] = D$$

$$[\phi] = \frac{D-2}{2}$$

$$[A^\mu] = \frac{D-2}{2}$$

$$[e] = 2 - \frac{D}{2} = \epsilon$$

TO AVOID DIMENSIONAL COUPLING WHEN $D \neq 4$

INTRODUCE AN ARBITRARY MASS SCALE μ
(RENORMALIZATION SCALE)

$$\underline{e \xrightarrow{\text{IN } D\text{-DIM}} \mu^\varepsilon e}$$

AND TREAT e FURTHER AS DIMENSIONLESS.

$$\int \frac{d^4 k}{(2\pi)^4} \frac{e^2}{(k^2 - \Delta + i\varepsilon)^2} \xrightarrow{\text{DIM. REG.}} \mu^{2\varepsilon} e^2 \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 - \Delta^2 + i\varepsilon)^2}$$

NOTE: EVEN THOUGH $\mu^{2\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} 1$

WE NEED TO EXPAND FULL INTEGRAL AROUND $\varepsilon = 0$

$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 - \Delta^2 + i\varepsilon)^2} = \frac{a}{\varepsilon} + b + O(\varepsilon)$$

$$\mu^{2\varepsilon} = 1 + \varepsilon \ln \mu^2 + O(\varepsilon^2)$$

$$\mu^{2\varepsilon} \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 - \Delta^2 + i\varepsilon)^2} = \frac{a}{\varepsilon} + (b + \ln \mu^2) + O(\varepsilon)$$

↑
GIVES A
FINITE
CONTRIBUTION

→ EVALUATION OF VACUUM POLARIZATION 1-LOOP INT.

FROM PAGE 8

$$\begin{aligned}
 \bullet \quad \Pi^{\mu\nu} &= ie^2 \mu^{\frac{2\epsilon}{2}} \int_0^1 dx \\
 &\cdot \left\{ -4 \int \frac{d^D k}{(2\pi)^D} \frac{k^\mu k^\nu}{(k^2 - \Delta)^2} \right. \\
 &+ 2 g^{\mu\nu} \int \frac{d^D k}{(2\pi)^D} \frac{k^2}{(k^2 - \Delta)^2} \\
 &+ \left(q^\mu q^\nu [4x(1-x) - 1] + 2 g^{\mu\nu} [q^2(1-x)^2 - m^2] \right) \\
 &\cdot \left. \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 - \Delta)^2} \right\}
 \end{aligned}$$

WITH $\Delta \equiv m^2 - q^2 x(1-x)$

$$\begin{aligned}
 \bullet \quad \Pi^{\mu\nu} &= ie^2 \mu^{\frac{2\epsilon}{2}} \int_0^1 dx \cdot \frac{i}{(4\pi)^{D/2}} \\
 &\cdot \left\{ + 4 g^{\mu\nu} \frac{\Gamma(-1+\epsilon)}{2} \cdot \frac{1}{\Delta^{-1+\epsilon}} \right. \\
 &- 2 g^{\mu\nu} D \frac{\Gamma(-1+\epsilon)}{2} \frac{1}{\Delta^{-1+\epsilon}} \\
 &+ \frac{\Gamma(\epsilon)}{\Delta^\epsilon} \left(q^\mu q^\nu [4x(1-x) - 1] + 2 g^{\mu\nu} [q^2(1-x)^2 - m^2] \right) \Big\}
 \end{aligned}$$

$$\Pi^{\mu\nu} = - \frac{e^2 \mu^{2\varepsilon}}{(4\pi)^{D/2}} \int_0^1 dx$$

$$\cdot \left\{ 2 g^{\mu\nu} (-1+\varepsilon) \frac{\Gamma(-1+\varepsilon)}{\Delta^{-1+\varepsilon}} \right. \\ \left. + 2 g^{\mu\nu} \frac{\Gamma(\varepsilon)}{\Delta^\varepsilon} [q^2(1-x)^2 - m^2] \right. \\ \left. + q^\mu q^\nu \frac{\Gamma(\varepsilon)}{\Delta^\varepsilon} [4x(1-x) - 1] \right\}$$

$$\downarrow \quad (-1+\varepsilon) \Gamma(-1+\varepsilon) = \Gamma(\varepsilon)$$

$$\Pi^{\mu\nu} = - \frac{e^2 \mu^{2\varepsilon}}{(4\pi)^2} \Gamma(\varepsilon) (4\pi)^\varepsilon$$

$$\cdot \int_0^1 dx \frac{1}{[m^2 - q^2 x(1-x)]^\varepsilon} \left\{ 2 g^{\mu\nu} \left[\cancel{m^2} - q^2 x(1-x) \right. \right. \\ \left. \left. + q^2(1-x)^2 - \cancel{m^2} \right] \right. \\ \left. \underbrace{\hspace{10em}}_{q^2(1-x)(1-2x)} \right. \\ \left. + q^\mu q^\nu \left[-2(1-x)(1-2x) \right. \right. \\ \left. \left. + \cancel{(1-2x)} \right] \right\}$$

 $\downarrow$

NOTE: DUE TO SYMMETRY OF INTEGRAL
(AROUND $x = \frac{1}{2}$)

$$\int_0^1 dx \frac{(1-2x)}{[m^2 - q^2 x(1-x)]^\varepsilon} = 0$$

$$\Pi^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu) \cdot \frac{(-e^2) \mu^{2\varepsilon}}{(4\pi)^2} 2 \Gamma(\varepsilon) (4\pi)^\varepsilon \int_0^1 dx \frac{(\cancel{1-x})(1-2x)}{[m^2 - q^2 x(1-x)]^\varepsilon}$$

NOTE : GAUGE INV. FACTOR $(q^2 g^{\mu\nu} - q^\mu q^\nu)$
COMES OUT EXPLICITLY

- EXPANSION IN ε AROUND $\varepsilon = 0$

$$\begin{aligned} \Pi^{\mu\nu}(q) &= -\frac{e^2}{8\pi^2} (q^2 g^{\mu\nu} - q^\mu q^\nu) \\ &\cdot \left(\frac{1}{\varepsilon} - \gamma_E + \ln(4\pi) + O(\varepsilon) \right) \left(1 + \varepsilon \ln \mu^2 \right) \\ &\cdot \int_0^1 dx \, x(2x-1) \left(1 - \varepsilon \ln [m^2 - q^2 x(1-x)] \right) \end{aligned}$$

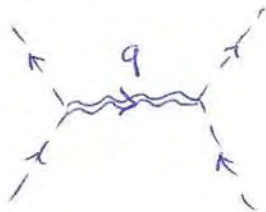
$$\begin{aligned} \Pi^{\mu\nu}(q) &= -\frac{e^2}{8\pi^2} (q^2 g^{\mu\nu} - q^\mu q^\nu) \\ &\cdot \int_0^1 dx \, x(2x-1) \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \left(\frac{4\pi \mu^2}{m^2 - q^2 x(1-x)} \right) \right. \\ &\quad \left. + O(\varepsilon) \right\} \end{aligned}$$

$$\Pi^{\mu\nu}(q) = - \frac{e^2}{48\pi^2} (q^2 g^{\mu\nu} - q^\mu q^\nu)$$

$$\cdot \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln 4\pi + \int_0^1 dx \, x(2x-1) \ln \frac{\mu^2}{m^2 - q^2 x(1-x)} \right\}$$

↓
FOR SPACELIKE PHOTON ($q^2 = -Q^2 < 0$) $Q^2 > 0$

e.g. IN



AND ASSUME $Q^2 \gg m^2$

$$\int_0^1 dx \, x(2x-1) \ln \frac{\mu^2}{Q^2 x(1-x)}$$

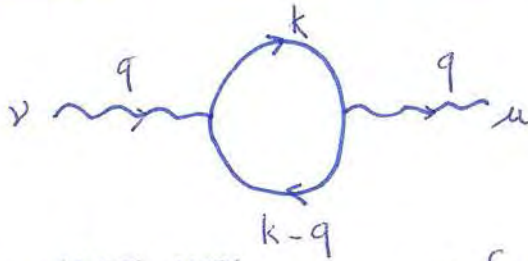
$$= \frac{1}{6} \ln \frac{\mu^2}{Q^2} - \underbrace{\int_0^1 dx \, x(2x-1) \ln x(1-x)}_{-\frac{4}{9}}$$

$$\Pi^{\mu\nu}(q) \stackrel{Q^2 \gg m^2}{\simeq} - \frac{e^2}{48\pi^2} (q^2 g^{\mu\nu} - q^\mu q^\nu)$$

$$\cdot \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln 4\pi + \ln \frac{\mu^2}{Q^2} + \frac{8}{3} \right\}$$

⇒ SPINOR QED

↳ VACUUM POLARIZATION



INVARIANT AMPL.

$$i \Pi^{\mu\nu}(q) = (-1) \int \frac{d^4 k}{(2\pi i)^4} \frac{\text{Tr} \{ (ie\gamma^\mu) i(\not{k} + m) (ie\gamma^\nu) i(\not{k} - q + m) \}}{[k^2 - m^2 + i\epsilon][(k-q)^2 - m^2 + i\epsilon]}$$

↑
CLOSED
FERMION
LOOP

$$\Pi^{\mu\nu}(q) = ie^2 \int \frac{d^4 k}{(2\pi i)^4} \frac{\text{Tr} \{ \gamma^\mu (\not{k} + m) \gamma^\nu (\not{k} - q + m) \}}{[k^2 - m^2 + i\epsilon][(k-q)^2 - m^2 + i\epsilon]}$$

↓ FEYNMAN PARAMETER
 $\int_0^1 dx \quad k \rightarrow k' = k - qx$

$$= ie^2 \int_0^1 dx \int \frac{d^4 k}{(2\pi i)^4} \frac{\text{Tr} \{ \gamma^\mu (\not{k} + q x + m) \gamma^\nu (\not{k} - q(1-x) + m) \}}{[k^2 - (m^2 - q^2 x(1-x)) + i\epsilon]^2}$$

↓

$$\int \frac{d^4 k}{(2\pi i)^4} \frac{k}{[k^2 - \Delta + i\epsilon]^2} = 0$$

TERMS LINEAR IN k IN NUMERATOR
 GIVE 0 UPON INTEGRATION

$$\begin{aligned} \Pi^{\mu\nu}(q) = & i e^2 \int_0^1 dx \int \frac{d^4 k}{(2\pi)^4} \frac{1}{[k^2 - (m^2 - q^2 x(1-x)) + i\epsilon]^2} \\ & \cdot \left\{ \text{Tr} \{ \gamma^\mu \not{k} \gamma^\nu \not{k} \} \right. \\ & \left. - x(1-x) \text{Tr} \{ \gamma^\mu \not{q} \gamma^\nu \not{q} \} + m^2 \text{Tr} \{ \gamma^\mu \gamma^\nu \} \right\} \end{aligned}$$

↳ DIRAC ALGEBRA IN D-DIMENSIONS

WHEN USING DIMENSIONAL REGULARIZATION
WE ALSO NEED TO DEFINE DIRAC γ -MATRICES
IN D DIMENSIONS

- $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu} \mathbb{1}$
↳ UNIT MATRIX IN $f(D)$ DIM
WITH $f(D=4) = 4$
- $\gamma^\mu \gamma_\mu = D \mathbb{1}$
- $\gamma^\alpha \gamma^\mu \gamma_\alpha = \gamma^\alpha (2 g_\alpha^\mu - \gamma_\alpha \gamma^\mu)$
 $= (2 - D) \gamma^\mu$
- $\gamma^\alpha \gamma^\mu \gamma^\nu \gamma_\alpha = \gamma^\alpha \gamma^\mu (2 g_\alpha^\nu - \gamma_\alpha \gamma^\nu)$
 $= 2 \gamma^\nu \gamma^\mu - \gamma^\alpha \gamma^\mu \gamma_\alpha \gamma^\nu$
 $= 4 g^{\mu\nu} - 2 \gamma^\mu \gamma^\nu - (2 - D) \gamma^\mu \gamma^\nu$
 $= 4 g^{\mu\nu} - (4 - D) \gamma^\mu \gamma^\nu$

- γ -MATRICES ARE $f(D) \times f(D)$ MATRICES
WITH $f(D=4) = 4$

$$\begin{aligned} f(D) &= f(4) + f'(4) (D-4) + O(\varepsilon^2) \\ &= 4 - 2\varepsilon f'(4) + O(\varepsilon^2) \end{aligned}$$

WE WILL SEE THAT PHYSICAL OBSERVABLES
WILL NOT DEPEND ON $f'(4)$

↳ CONVENTIONAL CHOICE $f'(4) = 0$

- $\text{Tr } \mathbb{I} = f(D)$
- $\text{Tr } \{ \gamma^\alpha \gamma^\beta \} = f(D) g^{\alpha\beta}$
- $\text{Tr } \{ \gamma^\alpha \gamma^\beta \gamma^\kappa \gamma^\lambda \} = f(D) \{ g^{\alpha\beta} g^{\kappa\lambda} - g^{\alpha\kappa} g^{\beta\lambda} + g^{\alpha\lambda} g^{\beta\kappa} \}$
- $\text{Tr } \{ \text{ODD \# } \gamma\text{-MATRICES} \} = 0.$

$$\begin{aligned} \hookrightarrow \text{Tr} \{ \gamma^\mu k \gamma^\nu k \} &= x(1-x) \text{Tr} \{ \gamma^\mu q \gamma^\nu q \} + m^2 \text{Tr} \{ \gamma^\mu \gamma^\nu \} \\ &= f(D) \left\{ 2k^\mu k^\nu - k^2 g^{\mu\nu} - x(1-x) [2q^\mu q^\nu - q^2 g^{\mu\nu}] + m^2 g^{\mu\nu} \right\} \end{aligned}$$

$\hookrightarrow$ VACUUM POLARIZATION IN DIMENSIONAL REGULARIZATION

$$\begin{aligned} \Pi^{\mu\nu}(q) &= ie^2 \mu^{2\varepsilon} \int_0^1 dx \int \frac{d^D k}{(2\pi)^D} \frac{1}{[k^2 - \Delta + i\varepsilon]^2} \\ &\quad \cdot f(D) \left\{ 2k^\mu k^\nu - k^2 g^{\mu\nu} \right. \\ &\quad \left. + g^{\mu\nu} (m^2 + q^2 x(1-x)) \right. \\ &\quad \left. - x(1-x) 2q^\mu q^\nu \right\} \end{aligned}$$

$$\text{WITH } \Delta = m^2 - q^2 x(1-x)$$

$$\begin{aligned} \Pi^{\mu\nu}(q) &= ie^2 \mu^{2\varepsilon} f(D) \frac{i}{(4\pi)^2} (4\pi)^\varepsilon \\ &\quad \cdot \int_0^1 dx \left\{ -g^{\mu\nu} \frac{(2-D)}{2} \Gamma\left(1 - \frac{D}{2}\right) \cdot \frac{1}{\Delta^{1-D/2}} \right. \\ &\quad \left. + \Gamma(\varepsilon) \frac{1}{\Delta^\varepsilon} \left[g^{\mu\nu} (m^2 + q^2 x(1-x)) \right. \right. \\ &\quad \left. \left. - x(1-x) 2q^\mu q^\nu \right] \right\} \end{aligned}$$

$$\Pi^{\mu\nu}(q) = - \frac{e^2}{(4\pi)^2} (4\pi)^\varepsilon \mu^{2\varepsilon} f(D) \cdot \Gamma(\varepsilon)$$

$$\cdot \int_0^1 dx \Delta^{-\varepsilon} \left\{ - g^{\mu\nu} (\cancel{m^2} - q^2 x(1-x)) \right. \\ \left. + g^{\mu\nu} (\cancel{m^2} + q^2 x(1-x)) \right. \\ \left. - 2q^\mu q^\nu x(1-x) \right\}$$

$$\Pi^{\mu\nu}(q) = - (q^2 g^{\mu\nu} - q^\mu q^\nu) \cdot \frac{e^2}{2\pi^2}$$

$$\cdot \left\{ \frac{1}{\varepsilon} - \gamma_E - \frac{1}{2} f'(4) + \ln(4\pi) + O(\varepsilon) \right\}$$

$$\cdot \int_0^1 dx x(1-x) \left\{ 1 + \varepsilon \ln\left(\frac{\mu^2}{\Delta}\right) + O(\varepsilon^2) \right\}$$

$$\Pi^{\mu\nu}(q) = - (q^2 g^{\mu\nu} - q^\mu q^\nu) \cdot \frac{e^2}{12\pi^2}$$

$$\cdot \left\{ \frac{1}{\varepsilon} - \gamma_E - \frac{1}{2} f'(4) + \ln(4\pi) \right.$$

$$\left. + 6 \int_0^1 dx x(1-x) \ln\left(\frac{\mu^2}{m^2 - q^2 x(1-x)}\right) \right\}$$

$$\Pi^{\mu\nu}(q) \stackrel{Q^2 \gg m^2}{\simeq} - \frac{e^2}{12\pi^2} (q^2 g^{\mu\nu} - q^\mu q^\nu)$$

$$\cdot \left\{ \frac{1}{\varepsilon} - \gamma_E - \frac{1}{2} f'(4) + \ln 4\pi + \ln \frac{\mu^2}{Q^2} + \frac{5}{3} \right\}$$

2) RUNNING COUPLING IN QED

⇒ PHYSICAL INTERPRETATION OF VACUUM POLARIZATION IN QED

QED :

$$\hookrightarrow \Pi^{\mu\nu}(q) = -\frac{e^2}{12\pi^2} (q^2 g^{\mu\nu} - q^\mu q^\nu)$$

$$\left\{ \frac{1}{\epsilon} - \gamma_E - \frac{1}{2} \ln(4) + \ln 4\pi \right. \\ \left. + 6 \int_0^1 dx \, x(1-x) \ln \left(\frac{\mu^2}{m^2 - q^2 x(1-x)} \right) \right\}$$

↳ 'DRESSED' PHOTON PROPAGATOR AT 1-LOOP IN FEYNMAN GAUGE (DYSON EXPANSION)

$$\begin{aligned} \text{wavy line } q \rightarrow \text{wavy line } q &= \text{wavy line } q + \text{wavy line } q \text{ with a fermion loop} \\ &= \text{wavy line } q + \text{wavy line } q \text{ with a fermion loop} + \text{wavy line } q \text{ with two fermion loops} + \dots \end{aligned}$$

$$i D^{\mu\nu}(q) = i D_0^{\mu\nu}(q) + i D_0^{\mu\alpha}(q) i \Pi_{\alpha\beta}(q) i D^{\beta\nu}(q)$$

$$\text{WITH } \Pi^{\alpha\beta}(q) \equiv (q^2 g^{\alpha\beta} - q^\alpha q^\beta) \Pi(q^2)$$

↑
SCALAR
FUNCTION

DYSON SERIES CORRESPONDS WITH
RESUMMATION OF VACUUM POL.

$$\begin{aligned} D^{\mu\nu}(q) = & D_0^{\mu\nu}(q) - D_0^{\mu\alpha}(q) \Pi_{\alpha\beta}(q) D_0^{\beta\nu}(q) \\ & + D_0^{\mu\alpha}(q) \Pi_{\alpha\beta}(q) D_0^{\beta\delta}(q) \Pi_{\delta\epsilon}(q) D_0^{\epsilon\nu}(q) \\ & + \dots \end{aligned}$$

↓ IN FEYNMAN GAUGE $D_0^{\mu\nu} = -\frac{g^{\mu\nu}}{q^2}$

$$\begin{aligned} D^{\mu\nu}(q) = & -\frac{g^{\mu\nu}}{q^2} + \frac{-g^{\mu\alpha}}{q^2} \Delta_\alpha^\nu(q) \Pi(q^2) \\ & + \frac{-g^{\mu\alpha}}{q^2} \Delta_\alpha^\delta(q) \Delta_\delta^\nu(q) \Pi^2(q^2) \\ & + \dots \end{aligned}$$

WHERE WE HAVE DEFINED

$$\Pi_{\alpha\beta}(q) \equiv \Delta_{\alpha\beta}(q) q^2 \Pi(q^2)$$

$$\Delta_{\alpha\beta}(q) \equiv g_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^2}$$

NOTE $\Delta_\alpha^\delta(q) \cdot \Delta_\delta^\nu(q) = \Delta_\alpha^\nu(q)$

$$\begin{aligned}
 \mathcal{D}^{\mu\nu}(q) &= -\frac{g^{\mu\nu}}{q^2} - \frac{1}{q^2} \Delta^{\mu\nu}(q) [\Pi(q^2) + \Pi^2(q^2) + \dots] \\
 &= \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) \frac{1}{q^2} [1 + \Pi(q^2) + \Pi^2(q^2) + \dots] \\
 &\quad - \frac{1}{q^2} \frac{q^\mu q^\nu}{q^2}
 \end{aligned}$$

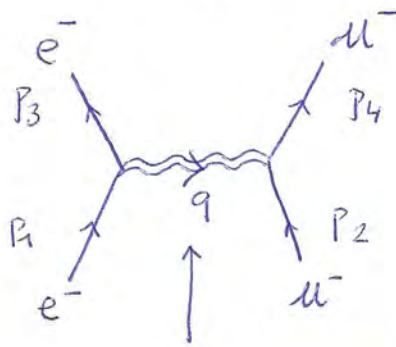
↓ GEOMETRIC SERIES

$$\begin{aligned}
 \mathcal{D}^{\mu\nu}(q) &= \frac{1}{q^2 [1 - \Pi(q^2)]} \cdot \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) \\
 &\quad - \frac{1}{q^2} \frac{q^\mu q^\nu}{q^2}
 \end{aligned}$$

WITH VACUUM POL. TO 1 LOOP

$$\begin{aligned}
 \Pi(q^2) &= -\frac{e^2}{12\pi^2} \left\{ \frac{1}{\epsilon} - \gamma_E - \frac{1}{2} f'(4) + \ln 4\pi \right. \\
 &\quad \left. + 6 \int_0^1 dx \, x(1-x) \ln \left(\frac{\mu^2}{m^2 - q^2 x(1-x)} \right) \right\}
 \end{aligned}$$

↳ PHYSICAL INTERPRETATION (e.g. $e^- \mu^-$ SCATTERING)



FULL PROPAGATOR

$$i D^{\mu\nu}(q) = \frac{-i g^{\mu\nu}}{q^2 [1 - \Pi(q^2)]} + q^\mu q^\nu \text{ TERMS}$$

NOTE : $q^\mu q^\nu$ TERMS GIVE NO CONTRIBUTION TO PHYSICAL AMPLITUDES WHEN COUPLED TO CONSERVED CURRENTS

TRANSITION AMPLITUDE

$$\mathcal{M} = \bar{U}(p_3, s_3) (ie \gamma_\nu) U(p_1, s_1) \cdot i D^{\mu\nu}(q) \cdot \bar{U}(p_4, s_4) (ie \gamma_\mu) U(p_2, s_2)$$

$$\text{AS } q^\nu \bar{U}(3) \gamma_\nu U(1) = \bar{U}(3) (\not{p}_1 - \not{p}_3) U(1) = 0$$

$$\downarrow q^\mu \bar{U}(4) \gamma_\mu U(2) = \bar{U}(4) (\not{p}_2 - \not{p}_4) U(2) = 0$$

$$\boxed{\mathcal{M} = ie^2 \bar{U}(3) \gamma^\mu U(1) \cdot \bar{U}(4) \gamma_\mu U(2) \cdot \frac{1}{q^2 [1 - \Pi(q^2)]}}$$

COULOMB POTENTIAL IS MODIFIED DUE TO VACUUM POL

↳ TO LOWEST ORDER $V_C(q^2) \equiv \frac{e^2}{q^2}$

↳ INCLUDING VACUUM POL $V(q^2) = \frac{e^2}{q^2 [1 - \Pi(q^2)]}$

IF $\Pi(q^2)$ IS A REGULAR AT $q^2=0$

$$\Pi(q^2) = \Pi(0) + q^2 \Pi'(0) + \dots$$

$$V(q^2) = \frac{e^2}{q^2 (1 - \Pi(0))} \left\{ 1 + \frac{\Pi'(0)}{1 - \Pi(0)} q^2 + O(e^4) \right\}$$

∴ FULL PROPAGATOR STILL HAS POLE AT $q^2=0$
(PHOTON REMAINS MASS-LESS)

↳ RESIDUE OF POLE

$$\frac{1}{1 - \Pi(0)} \equiv Z_3$$

$$Z_3 = 1 + \Pi(0) + O(e^4)$$

$$= 1 - \frac{e^2}{12\pi^2} \left\{ \frac{1}{\epsilon} - \gamma_E - \frac{1}{2} f'(4) + \ln \frac{4\pi\mu^2}{m^2} \right\} + O(e^4)$$

⇒ CHARGE RENORMALIZATION IN QED

↳ DEFINE PHYSICAL (RENORMALIZED) CHARGE

AT REFERENCE SCALE (i.e. $q^2 = 0$ CORRESPONDING TO $r = \infty$ IN COORDINATE SPACE)

$$e_R^2 \equiv e^2 Z_3$$

INFINITY IN VACUUM POL. IS ABSORBED IN RE-DEFINITION OF CHARGE (Z_3 : RENORMALIZATION CONSTANT)

↪ e_R^2 IS PHYSICAL (FINITE) CHARGE MEASURED IN EXPERIMENT

↪ e^2 IS CALLED 'BARE' (UNPHYSICAL) CHARGE

↳ POTENTIAL IN TERMS OF e_R^2

$$V(q^2) = \frac{e^2}{q^2 [1 - \Pi(0) - (\Pi(q^2) - \Pi(0))]}$$

$$= \frac{e^2}{q^2 [1 - \Pi(0)]} \left\{ 1 + \frac{\Pi(q^2) - \Pi(0)}{1 - \Pi(0)} + O(e^4) \right\}$$

$$V(q^2) = \frac{e_R^2}{q^2} \left\{ 1 + \frac{e_R^2}{2\pi^2} \int_0^1 dx \, x(1-x) \ln \left[1 - \frac{q^2}{m^2} x(1-x) \right] + O(e_R^4) \right\}$$

→ POTENTIAL IN LIMIT $Q^2 = -q^2 \gg m^2$

FOR SCATTERING PROCESS $q^2 < 0 \Rightarrow Q^2 \equiv -q^2 > 0$

CONSIDER LIMIT $Q^2 \gg m^2$

$$\ln \left[1 + \frac{Q^2}{m^2} x(1-x) \right] = \ln \frac{Q^2}{m^2} + \ln \left[x(1-x) + \frac{m^2}{Q^2} \right]$$

$$\underset{Q^2 \gg m^2}{\approx} \ln \frac{Q^2}{m^2}$$

$$V(q^2) = \frac{e_R^2}{q^2} \left\{ 1 + \frac{e_R^2}{2\pi^2} \int_0^1 dx \, x(1-x) \ln \frac{Q^2}{m^2} + O(e_R^4) \right\}$$

$$V(q^2) \approx \frac{1}{q^2} e_R^2 \left\{ 1 + \frac{e_R^2}{12\pi^2} \ln \frac{Q^2}{m^2} + O(e_R^4) \right\}$$

→ EFFECTIVE CHARGE / RUNNING COUPLING

$$V(q^2) \equiv \frac{e_{\text{eff}}^2(Q^2)}{q^2}$$

AT SCALE $q^2 = -Q^2$

POTENTIAL LOOKS LIKE COULOMB POT. WITH
EFFECTIVE q^2 DEPENDENT CHARGE

$$e_{\text{eff}}^2(Q^2) \equiv e_R^2 \left\{ 1 + \frac{e_R^2}{12\pi^2} \ln \frac{Q^2}{m^2} + O(e_R^4) \right\}$$

NOTE : e_{eff}^2 INCREASES FOR $Q^2 \uparrow$
CORRESPONDING WITH SHORTER
DISTANCE SCALES

(AT LARGE DISTANCES: CHARGE IS MORE AND
MORE SCREENED BY VIRTUAL e^-e^+ DIPOLES
 $\Rightarrow$ DECREASES AT LARGE DISTANCES.

WE CAN TAKE RESUMMATION OF 1-LOOP INTO ACCOUNT AS

$$e_{\text{eff}}^2(Q^2) = \frac{e_R^2}{1 - \frac{e_R^2}{12\pi^2} \ln \frac{Q^2}{m^2}} + O(e_R^4)$$

OR BY DEFINING $\alpha_{\text{em}}(Q^2) \equiv \frac{e_{\text{eff}}^2(Q^2)}{4\pi}$

$$\alpha_{\text{em}}(m^2) \equiv \frac{e_{\text{eff}}^2(m^2)}{4\pi}$$

$$\alpha_{\text{em}}(Q^2) = \frac{\alpha_{\text{em}}(m^2)}{1 - \frac{\alpha_{\text{em}}(m^2)}{3\pi} \ln \frac{Q^2}{m^2}}$$

RUNNING COUPLING IN QED TO 1-LOOP

- NOTE ! • ONE FIXES CHARGE AT LARGE DISTANCE TO EXPERIMENT

$$\alpha_{em}(m^2) \simeq \frac{1}{137} \quad \text{FINE-STRUCTURE CONSTANT}$$

- THEN EFFECTIVE CHARGE AT ANY SCALE CAN BE PREDICTED THROUGH

$$\alpha_{em}(Q^2) = \frac{\alpha_{em}(m^2)}{1 - \frac{\alpha_{em}(m^2)}{3\pi} \ln \frac{Q^2}{m^2}}$$

- EXAMPLE

FOR $Q^2 \simeq (90 \text{ GeV})^2$ NEAR Z^0 POLE

$$\ln\left(\frac{Q^2}{m^2}\right) \simeq 24$$

$$\frac{\alpha_{em}(m^2)}{3\pi} \cdot \ln \frac{Q^2}{m^2} \simeq 0.019$$

($\approx 2\%$ CORRECTION)

- VERY SLOW (LOGARITHMIC RISE) OF COUPLING
- AT VERY HIGH SCALE

$$\frac{\alpha_{em}(m^2)}{3\pi} \ln \frac{Q^2}{m^2} \approx 1 \Rightarrow Q \approx 10^{286} \text{ eV}$$

CORRECTION AS LARGE AS LEADING TERM
(LANDAU POLE) $\rightarrow$ PERTURBATION THEORY BREAKS DOWN

$\Rightarrow$ UEHLING POTENTIAL / LAMB SHIFT

$\hookrightarrow$ CONSIDER EFFECTIVE 1-LOOP POTENTIAL

$$V(q^2) = \frac{e_R^2}{q^2} \left\{ 1 + \frac{e_R^2}{2\pi^2} \int_0^1 dx \, x(1-x) \ln \left[1 - \frac{q^2}{m^2} x(1-x) \right] + O(e_R^4) \right\}$$

IN LARGE DISTANCE LIMIT $q^2 \ll m^2$

$$V(q^2) \underset{q^2 \ll m^2}{\simeq} \frac{e_R^2}{q^2} \left\{ 1 - \frac{e_R^2}{2\pi^2} \frac{q^2}{m^2} \int_0^1 dx \, x^2(1-x)^2 + O(e_R^4) \right\}$$

$$= \frac{e_R^2}{q^2} - \frac{e_R^4}{60\pi^2 m^2} + O(e_R^6)$$

$\hookrightarrow$ POTENTIAL IN COORDINATE SPACE ($q^2 = -\bar{q}^2$)

$$V(r) = \int \frac{d^3 \bar{q}}{(2\pi)^3} e^{i\bar{q} \cdot \vec{r}} V(\bar{q}^2)$$

$$V(r) = \underbrace{\frac{-e_R^2}{4\pi r}}_{\text{COULOMB TERM}} - \underbrace{\frac{e_R^4}{60\pi^2 m^2}}_{\text{UEHLING POTENTIAL}} \delta^3(\vec{r}) + O(e_R^6)$$

COULOMB TERM UEHLING POTENTIAL

$$\frac{e_R^2}{4\pi} = \alpha_{em} \approx \frac{1}{137}$$

FINE-STRUCTURE CONSTANT

$$V(r) \approx -\frac{\alpha_{em}}{r} - \frac{4}{15} \frac{\alpha_{em}^2}{m^2} \delta^3(\vec{r}) + O(\alpha_{em}^3)$$

↳ LAMB SHIFT IN HYDROGEN

- ENERGY LEVELS IN HYDROGEN ATOM

$$E_n = \int d^3\vec{r} \, \psi_n^* \left(-\frac{\alpha_{em}}{r} \right) \psi_n = -\frac{m\alpha_{em}^2}{2} \frac{1}{m^2} \quad (\text{BOHR})$$

- CORRECTION DUE TO UEHLING TERM

$\delta^3(\vec{r}) \rightarrow$ ONLY S-LEVELS ARE AFFECTED

ΔE_{nS}^{VP} : CORRECTION TO nS LEVEL DUE TO VACUUM POLARIZATION

$$\Delta E_{nS}^{VP} = -\frac{4}{15} \frac{\alpha_{em}^2}{m^2} \underbrace{|\psi_{nS}(0)|^2}_{\text{WAVE FUNCTION AT ORIGIN}}$$

$$\downarrow \quad |\psi_{nS}(0)|^2 = \frac{m^3 \alpha_{em}^3}{\pi n^3}$$

$$\Delta E_{nS}^{VP} = -\frac{4}{15\pi} \frac{m \alpha_{em}^5}{n^3}$$

LAMB SHIFT
CONTRIBUTION
DUE TO VP

- FOR $m = 2$

$$\Delta E_{2S} = - \frac{m \alpha_{em}^5}{30 \pi} = - 1.1 \cdot 10^{-7} \text{ eV (H)}$$

- IN DIRAC THEORY

$2S_{1/2}$ & $2P_{1/2}$ ARE DEGENERATE
DUE TO LOOP CORRECTIONS IN QED
DEGENERACY IS LIFTED

DIRAC

$$\overline{2S_{1/2}} \quad \overline{2P_{1/2}}$$

QED

$$\overline{2S_{1/2}} \quad \overline{2P_{1/2}} \quad \downarrow \text{ LAMB SHIFT}$$

VACUUM POL IS ONE CONTRIBUTION TO LAMB SHIFT

$$\begin{aligned} \Delta E^{\text{VAC POL}} &= E(2P_{1/2}) - E(2S_{1/2}) \\ &= \frac{m \alpha_{em}^5}{30 \pi} \\ &\equiv \hbar \omega = h \nu \end{aligned}$$

FREQUENCY

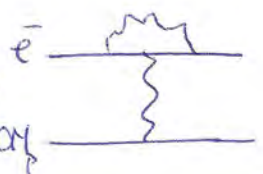
$$\nu_{\text{VAC POL}} = \frac{c}{2\pi \hbar c} \hbar \omega$$

$$= \frac{3 \cdot 10^8 \text{ m/s} \cdot (1.1 \cdot 10^{-7} \text{ eV})}{2\pi \cdot (197) \cdot 10^{-9} \text{ eV m}}$$

$$= \underline{\underline{27 \text{ MHz}}}$$

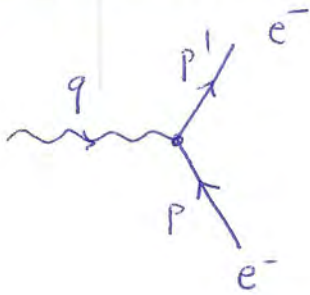
- EXP. LAMB SHIFT (1947)

$$\nu_{LS} \approx 1000 \text{ MHz} : \text{DOMINATED BY VERTEX CORRECTION}$$



3) ANOMALOUS MAGNETIC MOMENT IN QED

⇒ LOWEST ORDER VERTEX

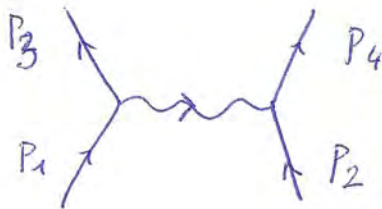


• FEYNMAN RULE

$$\parallel \mathcal{M}_{(0)}^{\mu}(p', p) \equiv \bar{u}(p', s') (ie \gamma^{\mu}) u(p, s)$$

↑
LOWEST ORDER

THIS ENTERS e.g. PHYSICAL PROCESSES AS



$$\mathcal{M}_{(0)} = \mathcal{M}_{(0)}^{\mu}(p_3, p_1) \left(-\frac{ig_{\mu\nu}}{q^2} \right) \mathcal{M}_{(0)}^{\nu}(p_4, p_2)$$

↑
INVARIANT AMPLITUDE

- GORDON DECOMPOSITION

$$\bar{U}(p', s') \gamma^\mu U(p, s)$$

$$= \bar{U}(p', s') \left\{ \frac{(p + p')^\mu}{2m} + i \sigma^{\mu\nu} \frac{q_\nu}{2m} \right\} U(p, s)$$

WITH $q = p' - p$ (MOMENTUM TRANSFER TOWARDS VERTEX)

→ $(p + p')^\mu$ TERM : CONVECTION CURRENT

→ $i \sigma^{\mu\nu} q_\nu$ TERM : MAGNETIZATION CURRENT

RESPONSIBLE FOR $g = 2$

OF DIRAC THEORY

↓

COEFFICIENT IN FRONT OF $i \sigma^{\mu\nu} \frac{q_\nu}{2m}$ TERM

IS $\frac{g}{2} \rightarrow$ IN DIRAC THEORY $\frac{g_{\text{DIRAC}}}{2} = 1$.

$\Rightarrow$ GENERAL VERTEX FOR SPIN $1/2$

- DUE TO LOOP CORRECTIONS,
WE WILL GET A DIFFERENCE FROM
DIRAC RESULT

ANOMALOUS MAGNETIC MOMENT

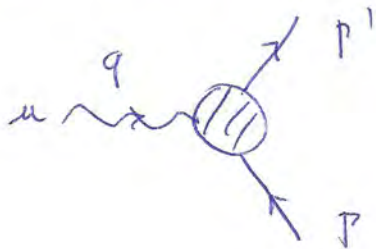
= DEVIATION FROM DIRAC THEORY

$$a \equiv \frac{g - 2}{2}$$

WE WILL CALCULATE



- IN GENERAL



$$\mathcal{M}^\mu(p', p) \equiv \bar{U}(p', s') (ie \Gamma^\mu) U(p, s)$$

$\hookrightarrow \Gamma^\mu$ IS A 4-VECTOR

INDEPENDENT 4-VECTORS $q^\mu, (p + p')^\mu$

$$\bar{U} \gamma^\mu U$$

NOTE : $\bar{U} \sigma^{\mu\nu} q_\nu U$ IS NOT INDEPENDENT

AS IT CAN BE REDUCED TO
ABOVE 3 USING GORDON ID

$$\mathcal{M}^\mu(p', p) = \bar{U}(p', s') \left\{ a \gamma^\mu + b (p + p')^\mu + c q^\mu \right\} U(p, s)$$

↳ GAUGE INVARIANCE

$\mathcal{M}^\mu$ IS A CONSERVED CURRENT

$$q_\mu \mathcal{M}^\mu = 0$$

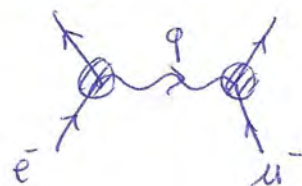
$$\bullet \quad \bar{U}(p') \not{q} U(p) = \bar{U}(p') (\not{p}' - \not{p}) U(p) = \bar{U}(p') (m - m) U(p) = 0$$

$$\begin{aligned} \bullet \quad \bar{U}(p') q \cdot (p + p') U(p) &= \bar{U}(p') (p'^2 - p^2) U(p) \\ &= \bar{U}(p') (m^2 - m^2) U(p) \\ &= 0 \end{aligned}$$

↑
ON-SHELL e^-
INITIAL & FINAL

$$\bullet \quad \bar{U}(p') q \cdot q U(p) = \bar{U}(p') q^2 U(p) \neq 0$$

BECAUSE γ IS IN GENERAL OFF-SHELL
IN A PROCESS AS



$$q^2 < 0$$

TO ENSURE GAUGE INVARIANCE
WE NEED $C = 0$

↳ WE CAN RE-EXPRESS $(p + p')^\mu$ TERM
IN TERMS OF γ^μ AND $\sigma^{\mu\nu} q_\nu$ TERMS
USING GORDON ID

MOST GENERAL VERTEX FOR SPIN $1/2$

$$\mathcal{M}^\mu(p', p) = ie \bar{U}(p', s') \left\{ F_1(q^2) \gamma^\mu + F_2(q^2) i \sigma^{\mu\nu} \frac{q_\nu}{2m} \right\} U(p, s)$$

NOTE: COEFF IN FRONT OF γ^μ & $i \sigma^{\mu\nu} q_\nu$

ARE SCALAR FUNCTIONS

CAN DEPEND ON ALL INDEP. SCALARS

$$q^2, q \cdot p, p^2$$

BUT $p^2 = m^2$ CONSTANT

$$p'^2 = m^2 = (p + q)^2 \Rightarrow 2q \cdot p = -q^2$$

∴ ONLY q^2 IS INDEPENDENT SCALAR

$F_1(q^2)$, $F_2(q^2)$ ARE CALLED 'FORM FACTORS' (FF)

$F_1(q^2)$: DIRAC FF

$F_2(q^2)$: PAULI FF

$$\hookrightarrow F_1(q^2=0) = 1 \quad \text{CHARGE IN UNITS } e$$

$$\hookrightarrow F_2(q^2=0) = a \quad \text{ANOMALOUS MAGNETIC MOMENT}$$

TOTAL MAGNETIC MOMENT

$$\vec{\mu} = \frac{e}{2m} (1 + a) \vec{\sigma}$$

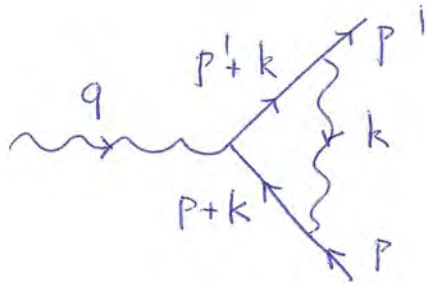
↑
PAULI MATRICES

↑
DIRAC
VALUE
($g=2$)

NOTE : $g = 2(1 + a)$

⇒ ONE-LOOP CORRECTION TO a

WE WILL CALCULATE 1-LOOP CORRECTION
TO ANOMALOUS MAGNETIC MOMENT OF ELECTRON (FERMION)
DUE TO VERTEX CORRECTION



$$\bullet \mathcal{M}_{(1)}^{\mu}(p', p) = ie \bar{U}(p', s') \int \frac{d^4 k}{(2\pi)^4} \frac{(ie\gamma^{\alpha}) i(\not{p}' + \not{k} + m) (-ig_{\alpha\beta}) \gamma^{\mu} i(\not{p} + \not{k} + m) (ie\gamma^{\beta})}{[(p' + k)^2 - m^2 + i\epsilon][(p + k)^2 - m^2 + i\epsilon][k^2 + i\epsilon]}$$

$\uparrow$
1-LOOP

$$= ie \bar{U}(p', s') U(p, s) \quad \left. \begin{array}{l} \text{ } \end{array} \right\} p^2 = p'^2 = m^2$$

$$\bullet (-ie^2) \int \frac{d^4 k}{(2\pi)^4} \frac{(2p'^{\alpha} + \gamma^{\alpha} \not{k}) \gamma^{\mu} (2p_{\alpha} + \not{k} \gamma_{\alpha})}{(k^2 + 2k \cdot p' + i\epsilon)(k^2 + 2k \cdot p + i\epsilon)(k^2 + i\epsilon)} U(p, s)$$

NOTE: WE HAVE USED

$$\bar{U}(p', s') \gamma^{\alpha} (\not{p}' + m) \dots = \bar{U}(p', s') (2p'^{\alpha}) \dots$$

$$\dots (\not{p} + m) \gamma_{\alpha} U(p, s) = \dots (2p_{\alpha}) U(p, s)$$

• FEYNMAN PARAMETERIZATION

$$\frac{1}{ABC} = 2 \int_0^1 dx \int_0^x dy \frac{1}{[A + (B-A)x + (C-B)y]^3}$$

$$A = k^2 - m^2$$

$$B = k^2 + 2k \cdot p$$

$$C = k^2 + 2k \cdot p'$$

$$\mathcal{M}_{(1)}^\mu(p', p) = ie \bar{U}(p', s')$$

$$\cdot (-ie^2) \cdot \int \frac{d^4 k}{(2\pi)^4} 2 \int_0^1 dx \int_0^x dy \frac{(2p'x + \gamma^\alpha k) \gamma^\mu (2p_\alpha + k \gamma_\alpha)}{[k^2 - m^2 + (2k \cdot p + m^2)x + 2k \cdot q y]^3}$$

$$\cdot U(p, s)$$

$$k' = k + p x + q y$$

$$= k + p(x-y) + p' y$$

$$= k + p' x + q(y-x)$$

$$k'^2 + 2k \cdot p x + 2k \cdot q y$$

$$= (k + p x + q y)^2 - (p x + q y)^2$$

$$= k'^2 - m^2 x^2 - q^2 y^2 - \underbrace{2p \cdot q}_{-q^2} x y$$

$$= k'^2 - m^2 x^2 + q^2 y(x-y)$$

$$\mathcal{M}_{(1)}^{\mu}(p', p) = ie \bar{U}(p', s')$$

$$\cdot (-ie^2) 2 \int_0^1 dx \int_0^x dy \int \frac{d^4 k}{(2\pi)^4} \frac{1}{[k^2 - \Delta + i\varepsilon]^3}$$

$$\cdot \left(2p'^{\alpha}(1-x) + \gamma^{\alpha} m x - \gamma^{\alpha} \not{q} (y-x) + \gamma^{\alpha} \not{k} \right) \gamma^{\mu}$$

$$\cdot \left(2p_{\alpha} (1-x) + \gamma_{\alpha} m x - \not{q} \gamma \gamma_{\alpha} + \not{k} \gamma_{\alpha} \right) U(p, s)$$

NOTE 1) BEHAVIOR OF INTEGRAND FOR $k \rightarrow \infty$

$$d^4 k \frac{1}{k^6} \cdot k^2 = \frac{d^4 k}{k^4} \quad \text{LOGARITHMIC DIVERGENCE}$$

↓

WE USE DIM REG FOR DIVERGENT TERM $\sim k^2$

FINITE TERM : IND. OF k IN NUMERATOR
CAN BE EVALUATED IN $D=4$

TERMS LINEAR IN $k \rightarrow$ GIVE 0 UPON $\int d^4 k \dots$

$$2) \quad \boxed{\Delta \equiv m^2 x^2 - q^2 y(x-y)}$$

• NUMERATOR

$$\begin{aligned} \bar{U}(P', s') & \left\{ \left(2P'^{\alpha} (1-x) + \gamma^{\alpha} m_x - \gamma^{\alpha} q (y-x) \right) \gamma^{\mu} \right. \\ & \cdot \left(2P_{\alpha} (1-x) + \gamma_{\alpha} m_x - q_{\gamma} \gamma_{\alpha} \right) \\ & \left. + \gamma^{\alpha} k \gamma^{\mu} k \gamma_{\alpha} \right\} U(P, s) \end{aligned}$$

→ FOR 2^o TERM : RULE IN D DIM

$$\begin{aligned} \gamma^{\alpha} k \gamma^{\mu} k \gamma_{\alpha} &= -2 k \gamma^{\mu} k + 2\epsilon k \gamma^{\mu} k \\ &= -2(1-\epsilon) k \gamma^{\mu} k \end{aligned}$$

→ FOR 1^o TERM : RULES IN 4 DIM

$$\begin{aligned} \gamma^{\alpha} \gamma_{\alpha} &= 4 \\ \gamma^{\alpha} \gamma^{\mu} \gamma_{\alpha} &= -2\gamma^{\mu} \\ \gamma^{\alpha} \gamma^{\mu} \gamma^{\nu} \gamma_{\alpha} &= 4g^{\mu\nu} \\ \gamma^{\alpha} \gamma^k \gamma^{\lambda} \gamma^{\mu} \gamma_{\alpha} &= -2\gamma^{\mu} \gamma^{\lambda} \gamma^k \end{aligned}$$

NUMERATOR

$$\begin{aligned}
 &= \bar{U}(p', s') \left\{ \underbrace{4 p \cdot p' (1-x)^2 \gamma^\mu}_{-2m^2 x^2 \gamma^\mu} + \frac{2 m x (1-x) \overbrace{\gamma^\mu p'}^{2p^\mu - m \gamma^\mu}}{2p^\mu - m \gamma^\mu} \right. \\
 &\quad - \underbrace{2 \gamma (y-x) \not{q} \gamma^\mu \not{q}}_{2q^\mu - \not{q} \gamma^\mu} \\
 &\quad - \underbrace{2 \gamma (1-x) \gamma^\mu \not{q} p'}_{\dots\dots\dots} - \underbrace{2 (y-x)(1-x) \not{p} \not{q} \gamma^\mu}_{\dots\dots\dots} \\
 &\quad - \underbrace{4 m \gamma x q^\mu}_{\dots\dots\dots} - \underbrace{4 m (y-x) x q^\mu}_{\dots\dots\dots} \\
 &\quad \left. - 2(1-\epsilon) \not{k} \gamma^\mu \not{k} \right\} U(p, s)
 \end{aligned}$$

$$\begin{aligned}
 &= \bar{U}(p', s') \left\{ \underbrace{2(2m^2 - q^2)(1-x)^2 \gamma^\mu}_{-4m^2 x(1-x) \gamma^\mu - 2m^2 x^2 \gamma^\mu} + \frac{4 m x (1-x) (p+p')^\mu}{\dots\dots\dots} \right. \\
 &\quad - \underbrace{4 \gamma (y-x) \not{q} \not{q}}_{\dots\dots\dots} + \underbrace{2 \gamma (y-x) q^2 \gamma^\mu}_{\dots\dots\dots} \\
 &\quad - \underbrace{4 m x (2y-x) q^\mu}_{\dots\dots\dots} - 2(1-\epsilon) \not{k} \gamma^\mu \not{k} \\
 &\quad - 2 \gamma (1-x) \left[\underbrace{2m p^\mu}_{\dots\dots\dots} + \underbrace{(q^2 - 2m^2) \gamma^\mu}_{\dots\dots\dots} \right] \\
 &\quad \left. + 2 (y-x)(1-x) \left[\underbrace{2m p^\mu}_{\dots\dots\dots} + \underbrace{(q^2 - 2m^2) \gamma^\mu}_{\dots\dots\dots} \right] \right\} U(p, s)
 \end{aligned}$$

$$\begin{aligned}
 \text{USING: } \bar{U} \not{q} \not{p'} U &= \bar{U} \gamma^\mu (m^2 - p \cdot p') U = \bar{U} \gamma^\mu (m^2 - 2 p \cdot p' + m p') U \\
 &= \bar{U} \left\{ 2m p^\mu + (q^2 - 2m^2) \gamma^\mu \right\} U
 \end{aligned}$$

$$\bar{U} \not{p} \not{q} \gamma^\mu U = \bar{U} \left\{ -2m p^\mu - (q^2 - 2m^2) \gamma^\mu \right\} U$$

$$\downarrow \quad \begin{aligned} 2P^\mu &= (P+P')^\mu - q^\mu \\ 2P'^\mu &= (P+P')^\mu + q^\mu \end{aligned}$$

OUR EXPRESSION HAS TERMS

$$\sim \gamma^\mu, (P+P')^\mu, q^\mu$$

• TERMS IN q^μ :

$$\begin{aligned} & -4m x(2\gamma - x) - 2m \gamma(1-x) - 2m(\gamma-x)(1-x) \\ &= -4m x(2\gamma - x) - 2m(1-x)(2\gamma - x) \\ &= -2m(1+x)(2\gamma - x) \end{aligned}$$

NOTE $\int_0^1 dx \int_0^x dy \frac{(2\gamma - x)}{[\dots q^2 \gamma(x-y)]^3}$

$$= \int_0^1 dx \left\{ \int_0^{x/2} dy + \int_{x/2}^x dy \right\} \frac{(2\gamma - x)}{[\dots q^2 \gamma(x-y)]^3}$$

$$\begin{aligned} & \downarrow \text{in 2nd INT } \gamma' = x - \gamma \quad \frac{(-1)(2\gamma' - x)}{(x - 2\gamma')} \\ & \int_{x/2}^x dy \frac{(2\gamma - x)}{[\dots q^2 \gamma(x-y)]^3} = \int_0^{x/2} dy' \frac{(x - 2\gamma')}{[\dots q^2 (x-y')\gamma']^3} \end{aligned}$$

2 INTEGRALS ARE EXACTLY OPPOSITE

∴ q^μ TERMS CANCEL ∇ AS REQUIRED
BY GAUGE INVARIANCE

• $F_2(q^2)$ TERM

BY USING GORDON ID.,

WE CAN IDENTIFY $F_2(q^2)$ TERM

AS TERM PROPORTIONAL TO $-\frac{(P+P')^u}{2m}$

$$F_2(q^2) = -ie^2 2 \int_0^1 dx \int_0^x dy \int \frac{d^4 k}{(2\pi)^4} \frac{1}{[k^2 - \Delta + i\varepsilon]^3}$$

$$\cdot (-2m) \left\{ 4m x (1-x) - 2m y (1-x) + 2m (y-x)(1-x) \right\}$$

$$= ie^2 8m^2 \int_0^1 dx \int_0^x dy \int \frac{d^4 k}{(2\pi)^4} \frac{x(1-x)}{[k^2 - \Delta + i\varepsilon]^3}$$

$$F_2(q^2) = ie^2 8m^2 \int_0^1 dx x(1-x) \int_0^x dy \frac{i}{(4\pi)^2} \frac{(-1)}{2} \frac{1}{[m^2 x^2 - q^2 y(x-y)]}$$

• $F_2(0) = a$

$$a = F_2(0) = \frac{e^2}{4\pi^2} \cdot m^2 \int_0^1 dx x(1-x) \int_0^x dy \frac{1}{m^2 x^2}$$

$$= \frac{\alpha_{em}}{\pi} \cdot \int_0^1 dx (1-x)$$

$$a = \frac{\alpha_{em}}{2\pi} \quad \text{SCHWINGER (1948)}$$

$\Rightarrow$ STATUS OF a_μ : MUON ($g-2$)

- $$a_\mu^{(1)} \rightarrow 1 \text{ LOOP}$$

$$a_\mu = \frac{\alpha_{em}}{2\pi} \simeq 1.16 \cdot 10^{-3}$$

$$g = 2(1 + a)$$

$$g^{(1)} = 2.0023 \dots$$

- PRESENT THEORETICAL VALUE

QED : CALCULATED UP TO 5 LOOPS !

$\sim 12\,000$ DIAGRAMS

ELECTROWEAK

QCD : LARGEST UNCERTAINTY AT PRESENT !

$$a_\mu^{\text{THEORY}} = (11659\,182.8 \pm 4.9) \cdot 10^{-10}$$

- COMPARISON WITH EXPERIMENT (BROOKHAVEN) 2004

$$a_\mu^{\text{EXP}} = (11\,659\,208.9 \pm 6.3) \cdot 10^{-10}$$

- DIFFERENCE EXP - TH

$$a_\mu^{\text{EXP}} - a_\mu^{\text{TH}} = (26.1 \pm 8.0) \cdot 10^{-10}$$

$\sim 4\sigma$ DEVIATION ! (2017)

SIGNAL NEW PHYSICS ?

NEW EXPERIMENT IS ONGOING AT FERMILAB (USA)

$\hookrightarrow$ WILL IMPROVE EXP. ERROR FROM $6.3 \cdot 10^{-10} \rightarrow 1.6 \cdot 10^{-10}$

4) ELECTRON SELF-ENERGY IN QED

⇒ FERMION PROPAGATOR

↳ FREE FERMION PROPAGATOR

$$\begin{array}{c} \longrightarrow \\ p \end{array} \quad i S_0(p) = \frac{i (\not{p} + m)}{p^2 - m^2 + i\epsilon}$$

↳ FULL FERMION PROPAGATOR

- DUE TO 1-LOOP CORRECTION

$$\begin{array}{c} \longrightarrow \\ p \end{array} \Rightarrow \begin{array}{c} \longrightarrow \\ p \end{array} + \begin{array}{c} \text{wavy line} \\ \longrightarrow \end{array}$$

↑
e⁻ SELF-ENERGY
AT 1-LOOP

- DEFINE e⁻ SELF-ENERGY $-i\Sigma(p)$
AS ALL 1-FERMION IRREDUCIBLE CORRECTION GRAPHS

i.e. DO NOT BECOME DISCONNECTED
BY CUTTING 1 FERMION LINE

$$\begin{array}{c} p \\ \longrightarrow \text{---} \text{---} \text{---} \end{array} = \begin{array}{c} \text{wavy line} \\ \longrightarrow \end{array} + \begin{array}{c} \text{wavy line} \\ \text{---} \end{array} + \begin{array}{c} \text{wavy line} \\ \text{---} \end{array}$$

$-i\Sigma(p)$

- FULL FERMION PROPAGATOR

$$\begin{array}{c} \text{double line} \\ p \end{array} = \begin{array}{c} \longrightarrow \\ p \end{array} + \begin{array}{c} \longrightarrow \text{---} \text{---} \end{array} + \begin{array}{c} \text{---} \text{---} \text{---} \end{array} + \dots$$

$$iS(p) = iS_0(p) + iS_0(-i\Sigma)iS_0 + iS_0(-i\Sigma)iS_0(-i\Sigma)iS_0 + \dots$$

↓ WE CAN RESUM THE 1-FERMION REDUCIBLE GRAPHS IN A DYSON EQUATION

$$\text{Diagram: double line with arrow and } p \text{ below} = \text{Diagram: single line with arrow and } p \text{ below} + \text{Diagram: double line with arrow, a loop with two internal lines, and } p \text{ below}$$

$$iS(p) = iS_0(p) + iS_0(p)(-i\Sigma(p))iS(p)$$

↓ MULTIPLY ON LEFT BY $S_0^{-1}(p) = \not{p} - m$

$$S_0^{-1}(p)S(p) = 1 + \Sigma(p)S(p)$$

$$S(p) = \frac{1}{S_0^{-1}(p) - \Sigma(p)}$$

$$S(p) = \frac{1}{\not{p} - m - \Sigma(p)}$$

- EXPANSION OF SELF-ENERGY (MATRIX IN DIRAC SPACE)

$$\Sigma(p) = \not{p} f_2(p^2) + m f_m(p^2)$$

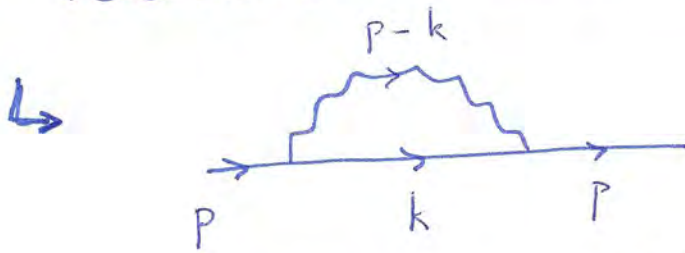
f_2 & f_m ARE SCALAR FUNCTIONS OF p^2

$$S(p) = \frac{1}{(1 - f_2(p^2))\not{p} - m(1 + f_m(p^2))}$$

f_2 : WILL RENORMALIZE FERMION FIELD

f_m : WILL RENORMALIZE FERMION MASS

⇒ ELECTRON SELF-ENERGY IN QED TO 1-LOOP



DENOTE ELECTRON MASS
 AS m_B : 'BARE' MASS,
 DEFINED IN ABSENCE OF
 INTERACTIONS

TO 1-LOOP : $O(e^2)$

$$-i\Sigma(p) = \int \frac{d^4k}{(2\pi)^4} \frac{(ie\gamma^\mu) i(\not{k} + m_B) (ie\gamma^\nu) (-ig_{\mu\nu})}{[k^2 - m_B^2 + i\epsilon][(p-k)^2 + i\epsilon]}$$

↓ FEYNMAN PARAMETERIZATION

$$\Sigma(p) = -ie^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\mu (\not{k} + m_B) \gamma_\mu}{[k^2 - m_B^2 + (p^2 + m_B^2 - 2p \cdot k)x + i\epsilon]^2}$$

↓ $k' = k - px$

$$\Sigma(p) = -ie^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\mu (\not{k} + \not{p}x + m_B) \gamma_\mu}{(k^2 - \Delta + i\epsilon)^2}$$

TERM LINEAR
 IN k
 DOES NOT
 CONTRIBUTE

$$\Delta \equiv m_B^2(1-x) - p^2x(1-x)$$

$$\underline{\underline{\Delta = (m_B^2 - p^2x)(1-x)}}$$

$$\Sigma(p) = -ie^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\mu (\not{p}x + m_B) \gamma_\mu}{(k^2 - \Delta + i\epsilon)^2}$$

↳ FOR LARGE k :

$$\text{INTEGRAND} \sim \frac{d^4 k}{k^4}$$

↓
INTEGRAL IS UV DIVERGENT

↓
WE WILL USE DIMENSIONAL REGULARIZATION
TO REGULARIZE THE INTEGRAL

↳ WE WILL DISCUSS 2 SUBTRACTION (RENORMALIZATION)
SCHEMES TO SUBTRACT THE UV DIV.

1) (MODIFIED) MINIMAL SUBTRACTION ($\overline{\text{MS}}$) $\Rightarrow$ USED IN QCD.

SUBTRACT THE $\frac{1}{\epsilon} - \gamma_E + \ln 4\pi$ TERMS

2) ON-SHELL RENORMALIZATION SCHEME (OS)

↳ CONVENTIONALLY USED IN QED

CAVEAT: } IN ON-SHELL SCHEME
 } ON WILL INTRODUCE IR DIVERGENCES
TO AVOID THEM ONE USUALLY INTRODUCES
A SMALL PHOTON MASS λ
IN PHYSICAL OBSERVABLES $\Rightarrow$ ALL IR DIV CANCEL
AND PHOTON MASS $\lambda \rightarrow 0$.

WHEN WORKING IN ON-SHELL SCHEME
ONE THEREFORE REPLACES

$$\Delta \rightarrow \underline{\underline{\Delta_f = (m_B^2 - p^2 x)(1-x) + \lambda^2 x}}$$

→ DIMENSIONAL REGULARIZATION

$$\Sigma(p) = -ie^2 \int_0^1 dx \mu^{2\varepsilon} \int \frac{d^D k}{(2\pi)^D} \frac{\gamma^\mu (\not{p}x + m_B) \gamma_\mu}{(k^2 - \Delta + i\varepsilon)^2}$$

WITH $\Delta = (m_B^2 - p^2 x)(1-x)$

↓
IN D DIM $\varepsilon \equiv 2 - D/2$

$$\gamma^\mu \not{p} \gamma_\mu = (2-D) \not{p} = -2(1-\varepsilon) \not{p}$$

$$\gamma^\mu \gamma_\mu = D = 4(1 - \frac{\varepsilon}{2})$$

$$\Sigma(p) = -2ie^2 \mu^{2\varepsilon} \int_0^1 dx \left[-(1-\varepsilon) \not{p} x + 2(1 - \frac{\varepsilon}{2}) m_B \right] \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 - \Delta + i\varepsilon)^2}$$

$$= 2 \frac{e^2}{(4\pi)^2} (4\pi\mu^2)^\varepsilon \Gamma(\varepsilon)$$

$$\cdot \int_0^1 dx \left[-(1-\varepsilon) \not{p} x + 2(1 - \frac{\varepsilon}{2}) m_B \right] \cdot \left[1 - \varepsilon \ln \Delta \right]$$

$$= \frac{e^2}{8\pi^2} \int_0^1 dx \left\{ \left[\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi + \ln \frac{\mu^2}{\Delta} \right] \cdot \left[-\not{p}x + 2m_B \right] + \not{p}x - m_B \right\}$$

$$\begin{aligned}
 \Sigma(P) = & \cancel{P} \cdot \left(-\frac{e^2}{16\pi^2} \right) \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln 4\pi - 1 \right. \\
 & \left. + 2 \int_0^1 dx \times \ln \left(\frac{\mu^2}{\Delta} \right) \right\} \\
 & + m_B \cdot \left(\frac{e^2}{4\pi^2} \right) \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln 4\pi - \frac{1}{2} \right. \\
 & \left. + \int_0^1 dx \ln \left(\frac{\mu^2}{\Delta} \right) \right\}
 \end{aligned}$$

NOTE : TO ORDER e^2 WE CAN REPLACE

$$e^2 \rightarrow e_R^2 + O(e_R^4)$$

$$e^2 = Z_3^{-1} e_R^2$$

$$= e_R^2 (1 + O(e_R^2))$$

$$\frac{e^2}{4\pi} = \frac{e_R^2}{4\pi} (1 + O(e_R^2))$$

$$= \alpha_{em} (1 + O(\alpha_{em}))$$

⇒ WAVE FUNCTION & MASS RENORMALIZATION

↳ IN $\Sigma(p)$ BOTH TERMS IN $\not{p}$ AND IN m_B ARE DIVERGENT : WE NEED TO RENORMALIZE 2 QUANTITIES TO ABSORB DIVERGENCES

↳ FERMION WAVE FUNCTION RENORMALIZATION

↳ FERMION MASS RENORMALIZATION

↳ WAVE FUNCTION RENORMALIZATION

FERMION PROPAGATOR

$$i S_F(x-y) = \langle 0 | T \{ \psi(x) \bar{\psi}(y) \} | 0 \rangle$$

- FIELDS WERE DEFINED IN ABSENCE OF INTERACTIONS

↳ "BARE" FIELDS ψ^B

WHAT WE HAVE CALCULATED IS THE CORRECTION TO BARE PROPAGATOR S^B

$$S_0^B(p) \rightarrow S^B(p) = \frac{1}{\not{p} - m_B - \Sigma(p)}$$

- WE INTRODUCE PHYSICAL (RENORMALIZED) FIELDS ψ^R THROUGH A RENORMALIZATION CONSTANT Z_2

$$\boxed{\psi^B = Z_2^{1/2} \psi^R}$$

- PHYSICAL (RENORMALIZED) PROPAGATOR S^R

$$S^B(p) = Z_2 S^R(p)$$

$$\downarrow \quad Z_2 \equiv 1 + \delta_2$$

δ_2 is $O(e^2)$

$\hookrightarrow$ IS CALLED A COUNTERTERM

$$S^R(p) = \frac{1}{(1 + \delta_2)} S^B(p)$$

$\hookrightarrow$ MASS RENORMALIZATION

WE DEFINE PHYSICAL MASS m^R

THROUGH A MASS RENORMALIZATION CONSTANT Z_m

$$m_B = Z_m m_R$$

$$Z_m \equiv (1 + \delta_m) \quad \text{WITH} \quad \delta_m = O(e^2)$$

$\hookrightarrow$ MASS COUNTERTERM

$\hookrightarrow$

$$S^R(p) = \frac{1}{(1 + \delta_2) \not{p} - (1 + \delta_2)(1 + \delta_m)m_R - \Sigma(p) + O(e^4)}$$

$$S^R(p) = \frac{1}{\not{p} - m_R + \delta_2 \not{p} - (\delta_2 + \delta_m)m_R - \Sigma(p) + O(e^4)}$$

$\Rightarrow$ (MODIFIED) MINIMAL SUBTRACTION SCHEME

$\hookrightarrow$ MINIMAL SUBTRACTION SCHEME (MS)

WE CHOOSE RENORMALIZATION CONSTANTS SUCH THAT THEY JUST ABSORB THE DIVERGENT $\frac{1}{\epsilon}$ TERMS IN $\Sigma(p)$

$$\boxed{\delta_2^{\text{MS}} \equiv -\frac{\alpha_{\text{em}}}{4\pi} \frac{1}{\epsilon}} \Rightarrow Z_2^{\text{MS}} = 1 - \frac{\alpha_{\text{em}}}{4\pi} \frac{1}{\epsilon} + O(\alpha_{\text{em}}^2)$$

$$\overset{\text{MS}}{\delta_2} + \overset{\text{MS}}{\delta_m} = -\frac{\alpha_{\text{em}}}{\pi} \frac{1}{\epsilon}$$

$\downarrow$

$$\boxed{\delta_m^{\text{MS}} \equiv -\frac{3\alpha_{\text{em}}}{4\pi} \frac{1}{\epsilon}} \Rightarrow Z_m^{\text{MS}} = 1 - \frac{3\alpha_{\text{em}}}{4\pi} \frac{1}{\epsilon} + O(\alpha_{\text{em}}^2)$$

$\hookrightarrow$ MODIFIED MINIMAL SUBTRACTION SCHEME ($\overline{\text{MS}}$)

AS $\frac{1}{\epsilon}$ ALWAYS COMES IN COMBINATION

$$\frac{1}{\epsilon} - \gamma_E + \ln(4\pi)$$

WE CAN DEFINE RENORM. CONSTANTS BY ABSORBING THIS COMBINATION

$$\boxed{\begin{aligned} Z_2^{\overline{\text{MS}}} &= 1 - \frac{\alpha_{\text{em}}}{4\pi} \left[\frac{1}{\epsilon} - \gamma_E + \ln 4\pi \right] + O(\alpha_{\text{em}}^2) \\ Z_m^{\overline{\text{MS}}} &= 1 - \frac{3\alpha_{\text{em}}}{4\pi} \left[\frac{1}{\epsilon} - \gamma_E + \ln 4\pi \right] + O(\alpha_{\text{em}}^2) \end{aligned}}$$

IN THIS SCHEME m_R IS KNOWN AS $\overline{\text{MS}}$ MASS

$\Rightarrow$ RENORMALIZED PROPAGATOR / POLE MASS

$\hookrightarrow$ IN $\overline{MS}$ SCHEME

$$\left\| [S^R(p)]_{\overline{MS}} \right\| = \frac{1}{\not{p} - m_R - \Sigma_R^{\overline{MS}}(p)}$$

$$\left\| \Sigma_R^{\overline{MS}}(p) \right\| = \not{p} \left(-\frac{\alpha_{em}}{4\pi} \right) \left\{ -1 + 2 \int_0^1 dx \times \ln \left(\frac{\mu^2}{\Delta} \right) \right\} \\ + m_R \left(\frac{\alpha_{em}}{\pi} \right) \left\{ -\frac{1}{2} + \int_0^1 dx \ln \left(\frac{\mu^2}{\Delta} \right) \right\} \\ + O(\alpha_{em}^2)$$

$$\Delta = (m_R^2 - p^2 x)(1-x)$$

$\hookrightarrow$ POLE MASS

S^R HAS POLE FOR $\not{p} = m_P$

$\uparrow$
POLE MASS (PHYSICAL e^- MASS)

$$m_P - m_R = \Sigma_R^{\overline{MS}}(m_P)$$

$$m_R = m_P - \Sigma_R^{\overline{MS}}(m_P)$$

$$= m_P \left[1 - \frac{\alpha_{em}}{4\pi} \left(-1 + 3 \ln \frac{\mu^2}{m_P^2} - 2 \int_0^1 dx (2-x) \ln [(1-x)^2] \right) \right] + O(\alpha_{em}^2)$$

$\overline{MS}$
MASS

$$m_R(\mu) = m_P \left[1 - \frac{\alpha_{em}}{4\pi} \left(4 + 3 \ln \frac{\mu^2}{m_P^2} \right) + O(\alpha_{em}^2) \right]$$

IN $\overline{MS}$ SCHEME

m_R IS FUNCTION OF ARBITRARY SCALE μ

$m_R(\mu)$: RUNNING MASS

FOR $\mu \uparrow \Rightarrow m_R \downarrow$

NOTE : PHYSICAL OBSERVABLES O ARE INDEPENDENT OF μ

$$\frac{dO}{d\mu} = 0 \quad : \quad \text{RENORMALIZATION GROUP EQ.}$$

→ RENORMALIZED PROPAGATOR IN $\overline{MS}$ SCHEME

$$\begin{aligned} [S^R(p)]^{-1} = & \not{p} \left[1 + \frac{\alpha_{em}}{4\pi} \left\{ -1 + \ln \frac{\mu^2}{m_R^2} \right. \right. \\ & \left. \left. - 2 \int_0^1 dx \times \ln \left[\left(1 - \frac{p^2}{m_R^2} x \right) (1-x) \right] \right\} \right] \\ & - m_R \left[1 + \frac{\alpha_{em}}{\pi} \left\{ -\frac{1}{2} + \ln \frac{\mu^2}{m_R^2} \right. \right. \\ & \left. \left. - \int_0^1 dx \ln \left[\left(1 - \frac{p^2}{m_R^2} x \right) (1-x) \right] \right\} \right] \end{aligned}$$

$$\begin{aligned} [S^R(p)]_{\overline{MS}}^{-1} = & \not{p} \left[1 + \frac{\alpha_{em}}{4\pi} \left\{ \ln \frac{\mu^2}{m_R^2} \right. \right. \\ & \left. \left. + \left(1 + \frac{m_R^2}{p^2} \right) \left[1 - \left(1 - \frac{m_R^2}{p^2} \right) \ln \left(1 - \frac{p^2}{m_R^2} \right) \right] \right\} \right] \\ & - m_R \left[1 + \frac{\alpha_{em}}{\pi} \left\{ \ln \frac{\mu^2}{m_R^2} + \frac{3}{2} \right. \right. \\ & \left. \left. - \left(1 - \frac{m_R^2}{p^2} \right) \ln \left(1 - \frac{p^2}{m_R^2} \right) \right\} \right] \\ & + O(\alpha_{em}^2) \end{aligned}$$

⇒ ON-SHELL RENORMALIZATION SCHEME

↳ RENORMALIZED PROPAGATOR

$$S^R(p) = \frac{1}{\not{p} - m_R - \Sigma_R(p)}$$

WITH $\Sigma_R(p) \equiv \Sigma(p) - \delta_2 \not{p} + (\delta_2 + \delta_m) m_R$

↳ FINITE RENORMALIZED SELF-ENERGY

↳ IN QED ONE NORMALLY USES ON-SHELL RENORMALIZATION SCHEME

$m_R = m_p$: POLE MASS OF PROPAGATOR
(e^- : $m_p = m_e = 0.511 \text{ MeV}$)

&
RESIDUE OF POLE $\not{p} = m_p$ IN S^R IS UNITY

↳ ON-SHELL SCHEME

• $m_R = m_p$ IMPLIES

$$\Sigma_R^{\text{OS}}(m_p) = 0$$

• RESIDUE OF POLE $\not{p} = m_p$ IS UNITY IMPLIES:

$$\left. \frac{d}{d\not{p}} \Sigma_R^{\text{OS}}(p) \right|_{\not{p} = m_p} = 0$$

L

$$\delta_m^{OS}$$

$$\sum_R^{OS} (m_p) = 0$$

$$\updownarrow$$

$$\sum (m_p) = -\delta_m^{OS} m_p$$

$$\delta_m^{OS} = \left(-\frac{\alpha_{em}}{4\pi}\right) \left\{ 3 \left[\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi \right] - 1 + 3 \ln \frac{\mu^2}{m_p^2} - 2 \int_0^1 dx (2-x) \ln \left[(1-x)^2 + \frac{\lambda^2}{m_p^2} x \right] \right\}$$

5

CAN BE
NEGLECTED
AS $\int$ IS FINITE

$$\delta_m^{OS} = \left(-\frac{\alpha_{em}}{4\pi}\right) \left\{ 3 \left[\frac{1}{\varepsilon} - \gamma_E + \ln \left(\frac{4\pi\mu^2}{m_p^2} \right) \right] + 4 \right\}$$

NOTE : $\delta_m^{OS} - \delta_m^{\overline{MS}} = \left(-\frac{\alpha_{em}}{4\pi}\right) \left\{ 3 \ln \frac{\mu^2}{m_p^2} + 4 \right\}$

FINITE DIFFERENCE

$$\hookrightarrow \delta_2^{OS}$$

$$\left. \frac{d}{d\bar{p}} \sum_R^{OS}(p) \right|_{\bar{p}=m_P} = 0$$



$$\boxed{\delta_2^{OS} = \left. \frac{d}{d\bar{p}} \sum(p) \right|_{\bar{p}=m_P}}$$

$$\sum(p) = \bar{p} f_2(p^2) + m_P f_m(p^2)$$

$$\downarrow \quad \frac{d}{d\bar{p}} f_2(p^2) = f_2'(p^2) \cdot 2\bar{p}$$

$$\frac{d}{d\bar{p}} \sum(p) = f_2(p^2) + \bar{p} \cdot 2\bar{p} f_2'(p^2) + 2\bar{p} m_P f_m'(p^2)$$

$$\left\| \delta_2^{OS} = f_2(m_P^2) + 2m_P^2 \left[f_2'(m_P^2) + f_m'(m_P^2) \right] \right\|$$

$$\bullet \quad f_2(m_P^2) = -\frac{\alpha_{em}}{4\pi} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln 4\bar{u} + \ln \frac{\mu^2}{m_P^2} \right. \\ \left. - 1 - \underbrace{2 \int_0^1 dx \times \ln \left[(1-x)^2 + \frac{\lambda^2}{m_P^2} x \right]}_3 \right\}$$

$$f_2(m_P^2) = \left(-\frac{\alpha_{em}}{4\pi} \right) \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln 4\bar{u} + \ln \frac{\mu^2}{m_P^2} + 2 \right\}$$

$$\bullet \quad f_2(p^2) + f_m(p^2) = \left(\frac{\alpha_{em}}{4\pi} \right) \left\{ 3 \left[\frac{1}{\epsilon} - \gamma_E + \ln \left(\frac{4\pi \mu^2}{m_p^2} \right) \right] - 1 \right. \\ \left. - 2 \int_0^1 dx (2-x) \ln \left[\left(1 - \frac{p^2}{m_p^2} x \right) (1-x) + \frac{\lambda^2}{m_p^2} x \right] \right\}$$

$$f_2'(m_p^2) + f_m'(m_p^2) = \left(\frac{\alpha_{em}}{4\pi} \right) \left(\frac{+2}{m_p^2} \right) \int_0^1 dx \frac{(2-x) x (1-x)}{(1-x)^2 + \frac{\lambda^2}{m_p^2} x}$$

FOR $\lambda^2 = 0$

THIS INTEGRAND $\sim \frac{1}{1-x}$

IR DIVERGENT FOR $x \rightarrow 1$

AROUND $x=1$, λ^2 REGULARIZES INTEGRAL

$$\int_0^1 dx \frac{\dots}{(1-x)^2 + \frac{\lambda^2}{m_p^2} x} \approx \int_0^{1-\lambda/m_p} dx \frac{\dots}{(1-x)^2} \\ + \text{TERMS } O\left(\frac{\lambda^2}{m_p^2}\right)$$

$$f_2'(m_p^2) + f_m'(m_p^2) = \left(\frac{\alpha_{em}}{4\pi} \right) \frac{2}{m_p^2} \underbrace{\int_0^{1-\lambda/m_p} dx \frac{x(2-x)}{(1-x)}}_{-\frac{1}{2} \left(1 + \ln \frac{\lambda^2}{m_p^2} + O\left(\frac{\lambda^2}{m_p^2}\right) \right)}$$

$$\bullet \quad \boxed{\delta_2^{OS} = \left(-\frac{\alpha_{em}}{4\pi} \right) \left\{ \frac{1}{\epsilon} - \gamma_E + \ln \left(\frac{4\pi \mu^2}{m_p^2} \right) + 4 + 2 \ln \frac{\lambda^2}{m_p^2} \right\}}$$

NOTE: IN ON-SHELL
SUBTRACTION SCHEME:

W.F. RENORM.
CONSTANT
CONTAINS IR DIV

$$\hookrightarrow [S^R(p)]_{OS}^{-1}$$

$$[S^R(p)]_{OS}^{-1} = \not{p} - m_p$$

$$+ \not{p} \left(\frac{\alpha_{em}}{4\pi} \right) \left\{ -1 - 2 \int_0^1 dx \, x \ln \left[\left(1 - \frac{p^2}{m_p^2} x \right) (1-x) \right] \right. \\ \left. - 4 - 2 \ln \frac{\lambda^2}{m_p^2} \right\}$$

$$- m_p \left(\frac{\alpha_{em}}{4\pi} \right) \left\{ -2 - 4 \int_0^1 dx \, \ln \left[\left(1 - \frac{p^2}{m_p^2} x \right) (1-x) \right] \right. \\ \left. - 4 - 2 \ln \frac{\lambda^2}{m_p^2} - 4 \right\}$$

$$[S^R(p)]_{OS}^{-1} = (\not{p} - m_p) \left[1 - \left(\frac{\alpha_{em}}{4\pi} \right) 2 \ln \frac{\lambda^2}{m_p^2} \right]$$

$$+ \left(\frac{\alpha_{em}}{4\pi} \right) \left\{ \not{p} \left[-3 + \frac{m_p^2}{p^2} + \left(\frac{m_p^2}{p^2} + 1 \right) \left(\frac{m_p^2}{p^2} - 1 \right) \ln \left(1 - \frac{p^2}{m_p^2} \right) \right] \right. \\ \left. - m_p \left[-2 + 4 \left(\frac{m_p^2}{p^2} - 1 \right) \ln \left(1 - \frac{p^2}{m_p^2} \right) \right] \right\}$$

NOTE : • IR DIV $\sim \ln \frac{\lambda^2}{m_p^2}$ ENTERS DUE TO OUR

CHOICE OF ON-SHELL SUBTRACTION SCHEME

• IR DIV WILL CANCEL OUT OF PHYSICAL OBSERVABLES

5) RENORMALIZED PERTURBATION THEORY ^{RENORM 73}

⇒ RENORMALIZATION CONSTANTS IN QED

↳ QED LAGRANGIAN : WITH PARAMETERS (UNPHYSICAL) DEFINED IN ABSENCE OF INTERACTIONS

$$\mathcal{L}_{\text{QED}} = \bar{\psi}_B (i \gamma^\mu \partial_\mu - m_B) \psi_B + e_B \bar{\psi}_B \gamma^\mu \psi_B A_{B\mu} - \frac{1}{4} F_{B\mu\nu} F^{B\mu\nu} + \text{GAUGE - FIXING}$$

MULTIPLICATIVE RENORMALIZABILITY :

ALL INFINITIES IN QED CAN BE ABSORBED

BY IN RENORMALIZATION CONSTANTS WHICH

REDEFINE BARE QUANTITIES INTO PHYSICAL ONES

$\psi_B = \sqrt{Z_2} \psi$	ELECTRON FIELD RENORM.
$A_B^\mu = \sqrt{Z_3} A^\mu$	PHOTON FIELD RENORM.
$e_B = Z_e e_R$	CHARGE RENORM.
$m_B = Z_m m_R$	MASS RENORM.

ψ, A^μ, e_R, m_R ARE PHYSICAL QUANTITIES

$\Rightarrow$ COUNTERTERM FEYNMAN RULES

$$\begin{aligned}
 \mathcal{L}_{\text{QED}} = & Z_2 \bar{\Psi} (i \gamma^\mu \partial_\mu - Z_m m_R) \Psi \\
 & + Z_e Z_2 \sqrt{Z_3} e_R \bar{\Psi} \gamma^\mu \Psi A_\mu \\
 & - \frac{1}{4} Z_3 F_{\mu\nu} F^{\mu\nu} + \text{GAUGE-FIXING}
 \end{aligned}$$

$\mathcal{L}_{\text{QED}}$ COUNTERTERMS δ (START AT ORDER e_R^2)

$$Z_2 = 1 + \delta_2$$

$$Z_m = 1 + \delta_m$$

$$Z_3 = 1 + \delta_3$$

$$Z_1 = 1 + \delta_1$$

$$\underline{\underline{Z_1 \equiv Z_e Z_2 \sqrt{Z_3}}} \\
 \text{VERTEX RENORM.}$$

$$\begin{aligned}
 \mathcal{L}_{\text{QED}} = & \bar{\Psi} (i \gamma^\mu \partial_\mu - m_R) \Psi + e_R \bar{\Psi} \gamma^\mu \Psi A_\mu \\
 & - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \text{GAUGE FIXING} \\
 & + \delta_2 \bar{\Psi} i \gamma^\mu \partial_\mu \Psi - (\delta_2 + \delta_m) \bar{\Psi} m_R \Psi \\
 & + \delta_1 e_R \bar{\Psi} \gamma^\mu \Psi A_\mu - \delta_3 \frac{1}{4} F_{\mu\nu} F^{\mu\nu}
 \end{aligned}$$

$\mathcal{L}_{\text{QED}}$ CONSISTS OF

- 1) TERMS OF SAME STRUCTURE AS BEFORE BUT NOW WITH PHYSICAL (RENORMALIZED) PARAMETERS
- 2) COUNTERTERMS WHICH APPEAR AS INTERACTIONS IN $\mathcal{L}$

↳ ADDITIONAL FEYNMAN RULES

- $\delta_2 \bar{\Psi} i \gamma^\mu \partial_\mu \Psi - (\delta_2 + \delta_m) m_R \bar{\Psi} \Psi$



$$i \left(\delta_2 \not{P} - (\delta_2 + \delta_m) m_R \right)$$

- $-\delta_3 \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

- $\gamma \xrightarrow{k} \text{circle with cross} \xrightarrow{\mu}$ $i \delta_3 (-g^{\mu\nu} k^2 + k^\mu k^\nu)$

GAUGE FIXING

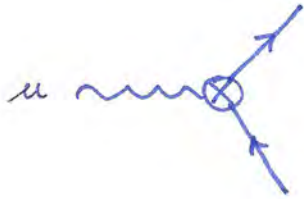
$$+ \frac{1}{2\xi} i \delta_3 (-k^\mu k^\nu)$$

- FEYNMAN GAUGE ($\xi=1$)



$$i \delta_3 (-g^{\mu\nu}) k^2$$

$$\delta_1 e_R \bar{\Psi} \gamma^\mu \Psi A_\mu$$



$$i \delta_1 e_R \gamma^\mu$$

SIGN FOR e^-
($e_R > 0$)

$\Rightarrow$ 2-POINT FUNCTION: ELECTRON

- TREE LEVEL



$$\frac{i}{p - m_R}$$

- 1-LOOP CORR.



$$\frac{i}{p - m_R} (-i \Sigma(p)) \frac{i}{p - m_R}$$



$$+ \frac{i}{p - m_R} i (\delta_2 p - (\delta_2 + \delta_m) m_R) \frac{i}{p - m_R}$$

- 1-PARTICLE IRREDUCIBLE GRAPHS + COUNTERTERM YIELD

$$\Sigma(p) = \delta_2 p + (\delta_2 + \delta_m) m_R$$

$$\equiv \Sigma_R(p)$$

DEFINES FINITE (RENORMALIZED)
SELF ENERGY (SEE P. 68)

- BY SUMMING ALL 1-PARTICLE REDUCIBLE GRAPHS
YIELDS RENORMALIZED (FULL) PROPAGATOR

$$\overline{\overline{p}} = \frac{i}{p - m_R - \Sigma_R(p)} \equiv i S_R(p)$$

δ_2 & δ_m ABSORB DIVERGENCIES IN $\Sigma(p)$

DEFINED IN A PARTICULAR SUBTRACTION SCHEME

WE HAVE DISCUSSED

1) $\overline{MS}$ SCHEME

2) ON-SHELL SCHEME

$$\sum_R^{OS}(p=m_R)=0 \quad \& \quad \frac{d}{dp} \sum_R^{OS}(p=m_R)=0$$

$\Rightarrow$ 2-POINT FUNCTION: PHOTON

- TREE-LEVEL (FEYNMAN GAUGE)

$$\nu \text{ --- } k \text{ --- } \mu \quad - \frac{ig^{\mu\nu}}{k^2}$$

- 1-LOOP CORR.

$$\text{---} \bigcirc \text{---} \quad - \frac{ig^{\mu\alpha}}{k^2} \left(i \Pi_{\alpha\beta}(k) \right) - \frac{ig^{\beta\nu}}{k^2}$$

$$+ \text{---} \otimes \text{---} \quad - \frac{ig^{\mu\alpha}}{k^2} \left(-i\delta_3 k^2 g_{\alpha\beta} \right) - \frac{ig^{\beta\nu}}{k^2}$$

- 1-PARTICLE IRREDUCIBLE GRAPHS + COUNTERTERM

$$\Pi_{\alpha\beta}(k) = \left(k^2 g_{\alpha\beta} - k_\alpha k_\beta \right) \Pi(k^2)$$

$$\Pi(k^2) - \delta_3 \equiv \Pi_R(k^2) \quad \text{DEFINES FINITE (RENORM) PHOTON SELF-ENERGY}$$

δ_3 ABSORBS DIV.

- BY SUMMING ALL 1-PARTICLE REDUCIBLE GRAPHS
+ COUNTERTERMS
YIELDS RENORMALIZED (FULL) PHOTON PROPAGATOR

$$\text{wavy line with } k \text{ above it} = \frac{i(-g^{\mu\nu})}{k^2 [1 - \Pi_R(k^2)]} + k^\mu k^\nu \text{ TERMS}$$

PROPAGATOR HAS POLE AT $k^2=0$ (γ IS MASSLESS)
WITH RESIDUE = 1
ENSURED BY
GAUGE INV.

$$\Downarrow$$

$$\boxed{\Pi_R(0) = 0}$$

$$\Updownarrow$$

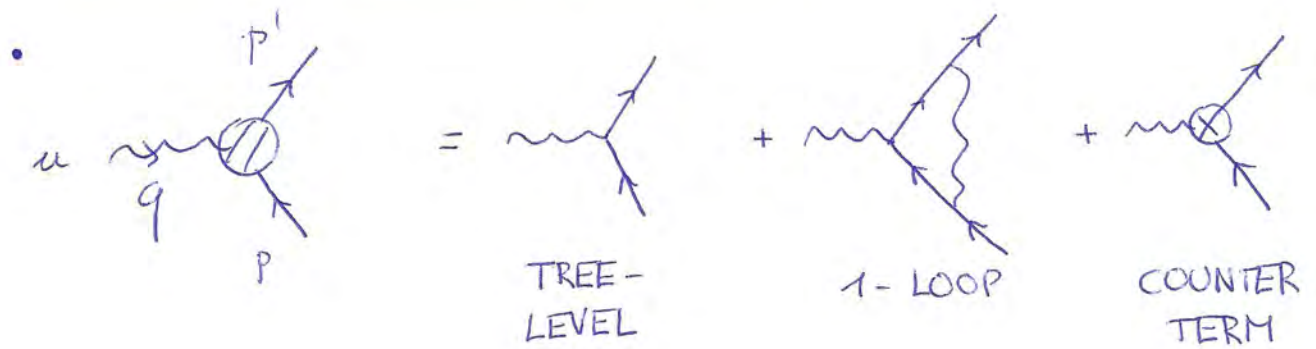
$$\boxed{\delta_3 = \Pi(0)}$$

$\rightarrow$ SEE P. 53

$$\left\| \begin{aligned} \delta_3 &= -\frac{e_R^2}{12\pi^2} \left\{ \frac{1}{\epsilon} - \gamma_E - \frac{1}{2} f'(4) + \ln\left(\frac{4\pi\mu^2}{m_R^2}\right) \right\} \\ &+ O(e_R^4) \end{aligned} \right.$$

$$\left\| \begin{aligned} \Pi_R(k^2) &= -\frac{e_R^2}{2\pi^2} \int_0^1 dx \, x(1-x) \ln\left(\frac{m_R^2}{m_R^2 - k^2 x(1-x)}\right) \\ &+ O(e_R^4) \end{aligned} \right.$$

⇒ 3-POINT FUNCTION : VERTEX



$$ie_R \Gamma^\mu = ie_R \gamma^\mu + ie_R \left[F_1^{1\text{LOOP}}(q^2) \gamma^\mu + F_2^{1\text{LOOP}}(q^2) i \frac{\sigma^{\mu\nu} q_\nu}{2m} \right] + ie_R \delta_1 \gamma^\mu$$

- $F_2^{1\text{LOOP}}(q^2)$ IS FINITE

WE FOUND $F_2^{1\text{LOOP}}(0) = \frac{\alpha_{em}}{2\pi}$

- $\Gamma^\mu = \gamma^\mu F_1(q^2) + i \frac{\sigma^{\mu\nu} q_\nu}{2m} F_2(q^2)$

⇓

$$F_1(q^2) = 1 + F_1^{1\text{LOOP}}(q^2) + \delta_1$$

BECAUSE

$$F_1(q^2=0) = 1$$

CHARGE !

⇕

$$\delta_1 = - F_1^{1\text{LOOP}}(0)$$

$\Rightarrow \underline{Z_1 = Z_2}$ PROOF (1 LOOP)

$\hookrightarrow F_1(q^2)$ TO 1 LOOP



UV DIVERGENT CONTRIBUTION
CAN ONLY COME FROM TERM

$$\mathcal{M}_{(1)}^{\mu, UV}(p', p) = ie \bar{U}(p', s')$$

$$\cdot (-ie^c) 2 \int_0^1 dx \int_0^x dy \mu^{2\epsilon} \int \frac{d^D k}{(2\pi)^D} \frac{\gamma^\alpha \not{k} \gamma^\mu \not{k} \gamma_\alpha}{(k^2 - \Delta + i\epsilon)^3}$$

WITH $\underline{\Delta \equiv m^2 x^2 - q^2 y(x-y)}$

$$\downarrow \cdot \gamma^\alpha \not{k} \gamma^\mu \not{k} \gamma_\alpha = -2(1-\epsilon) \not{k} \gamma^\mu \not{k}$$

$$= -2(1-\epsilon) \gamma^\alpha \gamma^\mu \gamma^\beta k_\alpha k_\beta$$

$$\cdot \int \frac{d^D k}{(2\pi)^D} \frac{k_\alpha k_\beta}{(k^2 - \Delta + i\epsilon)^3} = g_{\alpha\beta} \frac{i (4\pi)^\epsilon}{(4\pi)^2} \frac{\Gamma(\epsilon)}{2 \Gamma(3)}$$

$$\cdot \frac{1}{\Delta^\epsilon}$$

$$\cdot -2(1-\epsilon) \gamma^\alpha \gamma^\mu \gamma_\alpha = 4(1-2\epsilon + O(\epsilon^2)) \gamma^\mu$$

$$F_1^{UV}(0) = \frac{e^2}{(4\pi)^2} (4\pi\mu^2)^\varepsilon \frac{\Gamma(\varepsilon)}{4} 4(1-2\varepsilon)$$

$$\cdot 2 \int_0^1 dx \int_0^x dy \{ 1 - \varepsilon \ln m^2 x^2 \}$$

$$= \frac{\alpha_{em}}{4\pi} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \left(\frac{4\pi\mu^2}{m^2} \right) - 2 \right.$$

$$\left. - 4 \underbrace{\int_0^1 dx \, x \ln x}_{-\frac{1}{4}} \right\}$$

$$= \frac{\alpha_{em}}{4\pi} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \left(\frac{4\pi\mu^2}{m^2} \right) - 1 \right\}$$

$\Downarrow$

$$\hookrightarrow \delta_1^{\overline{MS}} = - F_1^{UV}(0)$$

$$\delta_1^{\overline{MS}} = - \frac{\alpha_{em}}{4\pi} \left[\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi \right] + O(\alpha_{em}^2)$$

$\Downarrow$

$$\boxed{Z_1^{\overline{MS}} = 1 - \frac{\alpha_{em}}{4\pi} \left[\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi \right] + O(\alpha_{em}^2)}$$

$$\hookrightarrow Z_1^{\overline{MS}} = Z_2^{\overline{MS}}$$

- VALID TO ALL ORDERS IN PERTURBATION THEORY!
(IN QED)

$$\boxed{Z_1 = Z_2} \quad (\text{QED})$$

- IS A CONSEQUENCE OF E.M. GAUGE INVARIANCE

E.M. CURRENT DOES NOT GET RENORMALIZED

$$J^\mu = \bar{\psi} \gamma^\mu \psi$$

(WARD - TAKAHASHI ID)

- AS A CONSEQUENCE

$$Z_1 = Z_e Z_2 Z_3^{1/2}$$

$\Downarrow$

$$\boxed{Z_e = Z_3^{-1/2}}$$

CHARGE RENORMALIZATION

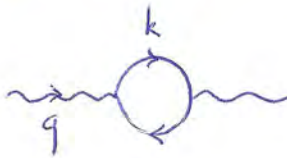
CAN BE OBTAINED FROM CALCULATION
OF VACUUM POL.



6) RENORMALIZABILITY: GENERAL

⇒ SUPERFICIAL DEGREE OF DIVERGENCE (DD)

↳ QED: 1 LOOP

•  $\xrightarrow{k \gg q} \int d^4k \frac{1}{k \not{k}} : DD = 2$

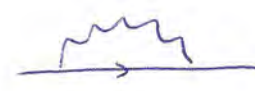
$$DD = \begin{array}{l} \# \text{ MOMENTA IN NUMERATOR} \\ - \# \text{ " " " DENOMINATOR} \end{array}$$

SUPERFICIAL DD GIVES MAX. POSSIBLE UV DIV.

FOR VACUUM POL: GAUGE INV.

$$\Pi^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu) \Pi(q^2)$$

$\Pi(q^2)$ HAS 'ONLY' LOGARITHMIC DIV.
↳ RENORM. CONSTANT Z_3

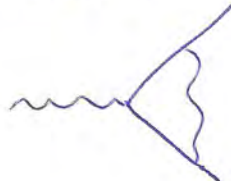
•  $\xrightarrow{k \gg q} \int d^4k \frac{1}{k^2 \not{k}} : DD = 1$

LINEAR DIV.

↳ 2 RENORM. CONSTANTS

Z_2, Z_m

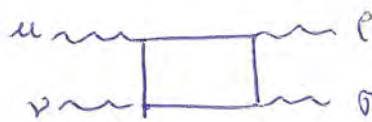
• 3-POINT FUNCTION $\langle \bar{\psi} \psi A \rangle$

•  $\xrightarrow{k \gg q} \int d^4k \frac{1}{\not{k} \not{k} k^2} : DD = 0$

LOGARITHMIC DIV.

↳ RENORM. CONSTANT Z_1

• 4-POINT FUNCTION $\langle A A A A \rangle$



$$\rightarrow \int d^4 k \frac{1}{k^2 k^2 k^2 k^2} \quad DD = 0.$$

$$M^{\mu\nu\rho\sigma} = a \overset{\text{DIV.}}{\left(g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho} \right)} + \text{FINITE TERMS} \sim q_i^\mu$$

BUT GAUGE INV. REQUIRES

$$q_{1\mu} M^{\mu\nu\rho\sigma} = 0, \quad q_{2\nu} M^{\mu\nu\rho\sigma} = 0$$

$$q_{3\rho} M^{\mu\nu\rho\sigma} = 0, \quad q_{4\sigma} M^{\mu\nu\rho\sigma} = 0.$$

ONLY POSSIBLE IF $a = 0$

∴ DUE TO GAUGE INV. $\langle A A A A \rangle$ IS FINITE
DESPITE $DD = 0$

• 4-POINT FUNCTION $\langle \bar{\psi} \psi A A \rangle$



$$\rightarrow \int d^4 k \frac{1}{k^2 k^2 k^2 k^2} \quad DD = -1$$

↓
FINITE

• 4-POINT FUNCTION $\langle \bar{\psi} \psi \bar{\psi} \psi \rangle$



$$\rightarrow \int d^4 k \frac{1}{k^2 k^2 k^2 k^2} \quad DD = -2$$

↓
FINITE

- 5- POINT FUNCTION $\langle \bar{\Psi} \Psi A A A \rangle$

$$\text{Diagram: A box with two vertical wavy lines on the left and three horizontal wavy lines on the right.} \rightarrow \int d^4 k \frac{1}{k^2 k k k k}$$

$$DD = -2$$

$$\downarrow$$

$$\text{FINITE}$$

- ALL m -POINT FUNCTIONS WITH $m \geq 5$ HAVE $DD < 0 \Rightarrow \text{FINITE}$

- AT 1-LOOP : ONLY $\langle A A \rangle$

$$\langle \bar{\Psi} \Psi \rangle$$

$$\langle \bar{\Psi} \Psi A \rangle$$

ARE DIVERGENT $\Rightarrow$ RENORMALIZATION
CONSTANTS Z_3, Z_2, Z_m, Z_1

$\hookrightarrow$ QED : 2 LOOP

- $\langle A A \rangle$

$$\text{Diagram: A circle with a vertical wavy line on the left and a horizontal wavy line on the right.} \rightarrow \int d^8 k \frac{1}{k k k^2 k k}$$

$$DD = 2$$

- $\langle \bar{\Psi} \Psi \rangle$

$$\text{Diagram: A horizontal wavy line with a vertical wavy line attached to its center.} \rightarrow \int d^8 k \frac{1}{k k k k^2 k^2}$$

$$DD = 1$$

- $\langle \bar{\Psi} \Psi A \rangle$

$$\text{Diagram: A triangle with a horizontal wavy line on the left and a vertical wavy line on the right.} \rightarrow \int d^8 k \frac{1}{k k k k k^2 k^2}$$

$$DD = 0$$

FOR A THEORY LIKE QED

DD IS SAME FOR 1-LOOP, 2-LOOP, ..., n-LOOP GRAPHS

↪ DIVERGENCES CAN BE ABSORBED IN SAME 4
 RENORMALIZATION CONSTANTS Z_3, Z_2, Z_m, Z_1
 $\updownarrow$
 Z_e

e.g.



$\Rightarrow O(\alpha_{em})$ TERM IN Z_3



$\Rightarrow O(\alpha_{em}^2)$ TERM IN Z_3

...

THEORIES FOR WHICH DIV. CAN BE ABSORBED
 BY A FINITE # OF COUNTERTERMS (4 FOR QED)
 ARE CALLED RENORMALIZABLE $\Rightarrow$ INFINITE # OF PREDICTIONS !

FOR QED ($Z_1 = Z_2$)

WE ONLY NEED TO MEASURE 2 QUANTITIES ! (2 FUNDAMENTAL PARAMETERS)

$\Rightarrow e_R$: RENORM. CHARGE $\rightarrow$ DETERMINED FROM COULOMB LAW AT LARGE DISTANCE

$\downarrow$
 FIXES $Z_e = Z_3^{-1/2}$

$\alpha_{em} \equiv \frac{e_R^2}{4\pi} = \frac{1}{137}$

$\Rightarrow m_p$: RENORM. MASS (IN ON-SHELL SCHEME)

$\parallel$
 POLE MASS $m_p^{e^-} = 0.511 \text{ MeV}$

PROPAGATOR HAS POLE AT $p = m_p \rightarrow$ FIXES Z_m
 WITH RESIDUE 1 $\rightarrow$ FIXES Z_2



GENERAL THEORY

HOW TO CALCULATE DD FOR AN ARBITRARY DIAGRAM ?

EX.



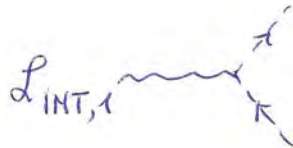
- AT EACH VERTEX OF TYPE i
WE MAY HAVE DERIVATIVE INTERACTIONS
VERTEX TYPE $i \Rightarrow \underline{d_i = \# \text{ DERIVATIVES IN } \mathcal{L}_{\text{INT}}}$

→ SPINOR QED $\mathcal{L}_{\text{INT}} = e \bar{\psi} \gamma^\mu \psi A_\mu \quad (q = -e)$

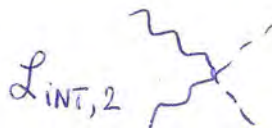


$$d = 0$$

→ SCALAR QED
2 TYPES



$$-iq(P+P')^\mu \Rightarrow d_1 = 1.$$



$$2iq^2 g^{\mu\nu} \Rightarrow d_2 = 0.$$

→ QCD : HAS 3 TYPES OF INTERACTIONS



$$\mathcal{L}_{\text{INT},1}$$

$$d_1 = 0$$



$$\mathcal{L}_{\text{INT},2}$$

$$d_2 = 1$$



$$\mathcal{L}_{\text{INT},3}$$

$$d_3 = 0$$

- EVERY INTERNAL FERMION LINE

$$\int d^4 p_i \quad \longrightarrow \quad \sim \int d^4 p_i \frac{1}{p_i} \quad DD = 3$$

CONNECTS 2 VERTICES $\Rightarrow$ ATTRIBUTE $DD = \frac{3}{2}$ TO EACH OF THE 2 VERTICES

$\therefore$ $f_i = \#$ FERMION LINES ENTERING VERTEX OF TYPE i

CONTRIBUTE $DD = \frac{3}{2} \cdot f_i$ AT EACH VERTEX

- EVERY INTERNAL BOSON LINE

$$\int d^4 p_i \quad \rightsquigarrow \quad \sim \int d^4 p_i \frac{1}{p_i^2} \quad DD = 2$$

CONNECTS 2 VERTICES $\Rightarrow$ ATTRIBUTE $DD = 1$ TO EACH OF THE 2 VERTICES

$\therefore$ $b_i = \#$ BOSON LINES ENTERING VERTEX OF TYPE i

CONTRIBUTE $DD = b_i$ AT EACH VERTEX

- ENERGY - MOMENTUM CONSERVATION

AT EACH VERTEX : THERE IS 1 ENERGY - MOMENTUM CONSERVATION CONDITION



$$\delta^4 \left(\sum_i p_i \right)$$

REDUCES DD AT EACH VERTEX BY 4

$\therefore$ CONTRIBUTES $DD = -4$ AT EACH VERTEX

- AT EACH VERTEX OF TYPE i

δ_i = INDEX OF DIVERGENCE OF $\mathcal{L}_{\text{INT},i}$

$$\delta_i = d_i + \frac{3}{2} f_i + b_i - 4 \rightarrow \text{GIVES CONTRIBUTION TO DD AT VERTEX OF TYPE } i$$

→ SPINOR QED



$$\begin{aligned} b &= 0 \\ f &= 2 \\ d &= 0 \end{aligned}$$

$$\Rightarrow \delta = 0$$

→ SCALAR QED



$$\mathcal{L}_{\text{INT},1}: \begin{aligned} b_1 &= 3 \\ f_1 &= 0 \\ d_1 &= 1 \end{aligned}$$

$$\Rightarrow \delta_1 = 0$$



$$\mathcal{L}_{\text{INT},2}: \begin{aligned} b_2 &= 4 \\ f_2 &= 0 \\ d_2 &= 0 \end{aligned}$$

$$\Rightarrow \delta_2 = 0$$

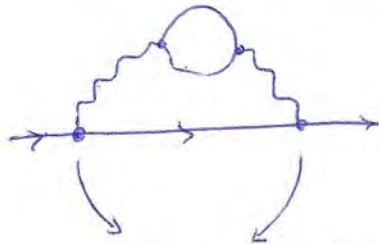
• CONTRIBUTION TO DD WHEN SUMMING OVER ALL VERTICES IN A DIAGRAM

$$DD = \sum_i n_i \delta_i$$

$\sum_i$: SUMS OVER ALL TYPES OF INTERACTIONS
 SPINOR QED ($i=1$)
 SCALAR QED ($i=1,2$)
 QCD ($i=1,2,3$)

n_i : # VERTICES OF TYPE i

- CORRECTION FOR EXTERNAL LINES



AT THESE VERTICES WE HAVE EXTERNAL LINES

→ FOR EXTERNAL LINES : NO PROPAGATORS
WE SHOULD CORRECT TERMS $\frac{3}{2}f_i$ & b_i
FOR EXTERNAL LINES

→ WE SUBTRACTED ONE ENERGY-MOMENTUM
TOO MUCH, BECAUSE ONE JUST CORRESPONDS
TO OVERALL ENERGY-MOMENTUM CONSERVATION
AND DOES NOT LEAD TO A REDUCTION OF DD

- FINAL FORMULA : GENERAL DIAGRAM / GENERAL THEORY

$$DD = \sum_i n_i \delta_i + \left(4 - \frac{3}{2}F - B \right)$$

$$\delta_i = d_i + \frac{3}{2}f_i + b_i - 4$$

F = # EXTERNAL FERMION LINES IN DIAGRAM

B = # EXTERNAL BOSON LINES IN DIAGRAM

⇒ RENORMALIZABLE, NON-RENORMALIZABLE,
SUPER-RENORMALIZABLE THEORIES

↳ $\delta_i = 0$: RENORMALIZABLE THEORIES

• IF $\delta_i = d_i + \frac{3}{2} f_i + b_i - 4 = 0$

⇓

$$DD = 4 - \frac{3}{2} F - B$$

IN RENORMALIZABLE THEORY :

SUPERFICIAL DD ONLY DEPENDS ON EXTERNAL LINES
NOT ON INTERNAL LINES

∴ NO NEW DIVERGENCES APPEAR WHEN
 CALCULATING HIGHER LOOP DIAGRAMS
 COMPARED TO 1-LOOP DIAGRAMS

⇓

WE ONLY NEED TO RENORMALIZE FINITE # PARAMETERS
 & ALL OTHER QUANTITIES CAN (IN PRINCIPLE) BE PREDICTED

• EXAMPLES ($\delta = 0$) : QED, QCD, STANDARD ELECTROWEAK THEORY

≡ STANDARD MODEL OF
 PARTICLE PHYSICS

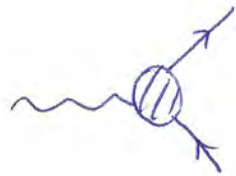
• QED : ONLY DIVERGENCES



$F = 0, B = 2 \Rightarrow DD = 2$



$$F=2, B=0 \Rightarrow DD=1$$



$$F=2, B=1 \Rightarrow DD=0$$

- IN RENORMALIZABLE THEORY: COUPLINGS ARE DIMENSIONLESS ∇_0

$$\mathcal{L}_{\text{INT}} = -g \bar{\psi} \gamma^\mu \psi A_\mu$$

$$[\mathcal{L}_{\text{INT}}] = 4$$

$$[\psi] = \frac{3}{2}$$

$$[A^\mu] = 1$$

$$[g] = 4 - \frac{3}{2} f_i - b_i - d_i$$

$$\downarrow$$

$$\downarrow$$

$$\downarrow$$

$$2$$

$$1$$

$$0$$

SPINOR QED

$$[g] = -\delta_i = 0$$

$$\uparrow$$

IN RENORMALIZABLE THEORY

NOTE: INTERACTIONS WITH $\delta = 0$ ARE ALSO CALLED 'MARGINAL' (DIMENSION 0)

↳ $\delta_i > 0$: NON-RENORMALIZABLE THEORIES

$$\bullet \quad DD = \underbrace{\sum_i m_i \delta_i}_{> 0} + \left(4 - \frac{3}{2} F - B\right)$$

MORE & MORE DIVERGENCES APPEAR
WHEN CALCULATING HIGHER LOOPS

⇓
ONE NEEDS AN INFINITE # COUNTERTERMS
TO ABSORB ALL DIVERGENCES

• EXAMPLE ($\delta_i > 0$)

$$\mathcal{L}_{\text{INT}} = g \phi^2 \square \phi^3 \quad (\phi: \text{SCALAR FIELD})$$

$$\delta_i = \underbrace{d_i}_2 + \frac{3}{2} \underbrace{f_i}_0 + \underbrace{b_i}_5 - 4$$

$$\underline{\underline{\delta_i = 3}}$$

$$[\mathcal{L}] = 4 \quad (\text{ALWAYS})$$

$$[g] = -\delta_i = -3$$

• IN NON-RENORMALIZABLE THEORIES
COUPLING'S HAVE NEGATIVE MASS DIMENSIONS

NOTE: COUPLINGS WITH NEGATIVE MASS DIMENSIONS
ARE CALLED 'IRRELEVANT'

↳ $\delta_i < 0$: SUPER-RENORMALIZABLE THEORY

$$\bullet \quad DD = \underbrace{\sum_i m_i \delta_i}_{< 0} + (4 - \frac{3}{2}F - B)$$

HIGHER LOOPS BECOME
MORE CONVERGENT

• EXAMPLE ($\delta_i < 0$)

$$\mathcal{L}_{\text{INT}} = g \phi^3 \quad (\phi : \text{SCALAR FIELD})$$

$$\delta_i = \underbrace{d_i}_0 + \frac{3}{2} \underbrace{f_i}_0 + \underbrace{b_i}_3 - 4$$

$$\underline{\underline{\delta_i = -1}}$$

$$[g] = -\delta_i = +1$$

• IN SUPER-RENORMALIZABLE THEORIES
COUPLINGS HAVE POSITIVE MASS DIMENSIONS

NOTE : COUPLINGS WITH POSITIVE MASS DIMENSIONS
ARE CALLED 'RELEVANT'

$\Rightarrow$ NON-RENORMALIZABLE THEORIES AS EFFECTIVE FIELD THEORIES

$\hookrightarrow$ EXAMPLE : SCALAR FIELD THEORY WITH DERIVATIVE INTERACTIONS

$$\bullet \quad \mathcal{L} = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 + \underbrace{\frac{g}{4!} \phi^2 \Box \phi^2}_{\mathcal{L}_{\text{INT}}}$$

$$\mathcal{L}_{\text{INT}} \Rightarrow \delta = d + \frac{3}{2} f + b - 4$$

$$= 2 + 0 + 4 - 4 = 2 \Rightarrow \text{NON-RENORMALIZABLE THEORY}$$

$$[g] = -\delta = -2.$$

4-POINT FUNCTION

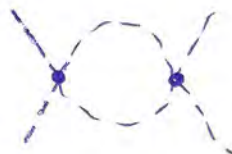
$\rightarrow$ TREE-LEVEL



$$\mathcal{M}^{(0)} \sim g p^2$$

WHERE p STANDS FOR A GENERIC EXTERNAL MOMENTUM

$\rightarrow$ 1-LOOP



$$DD = 2 \cdot \underset{\substack{\parallel \\ 2}}{\delta} + 4 - \underset{\substack{\parallel \\ 4}}{B} = 4$$

USE UV REGULATOR Λ (e.g. PAULI-VILLARS MASS SCALE)

$$\mathcal{M}^{(1)} = g^2 (c_1 \Lambda^4 + c_2 \Lambda^2 p^2 + c_3 p^4 \ln \Lambda + \dots)$$

$C_2 \Lambda^2 p^2$ DIVERGENCE CAN BE REMOVED

BY RENORMALIZING g AS ITS TREE-LEVEL CONTRIBUTION
IS $\sim p^2$

(BUT) $C_1 \Lambda^4$ & $C_3 p^4 \ln \Lambda$ TERMS AT 1-LOOP
REQUIRE NEW TREE LEVEL TERMS IN ORDER
TO ABSORB THESE DIVERGENCES

• CONSIDER THE EXTENDED $\mathcal{L}$

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 + \lambda_R Z_\lambda \phi^4 + g_R Z_g \phi^2 \square \phi^2 + k_R Z_k \phi^2 \square^2 \phi^2 + \dots$$

$$\downarrow \quad Z_\lambda = 1 + \delta_\lambda, \quad Z_g = 1 + \delta_g, \quad Z_k = 1 + \delta_k$$

$$\begin{aligned} &= \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 \\ &+ \lambda_R \phi^4 + g_R \phi^2 \square \phi^2 + k_R \phi^2 \square^2 \phi^2 + \dots \\ &+ \lambda_R \delta_\lambda \phi^4 + g_R \delta_g \phi^2 \square \phi^2 + k_R \delta_k \phi^2 \square^2 \phi^2 + \dots \end{aligned}$$

$\downarrow$ LAST LINE CORRESPONDS WITH 1-LOOP TERMS



$$\sim \lambda_R \delta_\lambda + g_R \delta_g p^2 + k_R \delta_k p^4$$

(POLYNOMIAL IN EXTERNAL MOMENTA)

$\lambda_R \delta_\lambda$ WILL ABSORB $C_1 \Lambda^4$ DIV.

$g_R \delta_g p^2$ WILL ABSORB $C_2 \Lambda^2 p^2$ DIV.

$K_R \delta_K p^4$ WILL ABSORB $C_3 p^4 \ln \Lambda$ DIV.

NOTE: ALSO K_R TERM WILL GENERATE NEW INFINITIES WHEN USED IN LOOP



CAN ALSO BE REMOVED WHEN ADDING ALL POSSIBLE TERMS TO $\mathcal{L}$, CONSISTENT WITH SYMMETRIES

THIS WILL BE POSSIBLE PROVIDED THAT DIVERGENCES MULTIPLY FUNCTIONS THAT ARE POLYNOMIALS IN EXTERNAL MOMENTA

• THEOREM (WEINBERG, 1995)

DIVERGENCES FROM LOOP INTEGRALS ARE POLYNOMIAL TERMS IN EXTERNAL MOMENTA

↳ CAN BE ABSORBED BY TREE-LEVEL TERMS

PROOF:

GENERAL DIVERGENT INTEGRAL IS OF FORM

$$I(p) = \int_0^\infty dk \frac{k}{(k+p)}$$

MUST HAVE AT LEAST 1 DENOMINATOR WITH p IN IT
(OTHERWISE IT WILL YIELD A RESULT WHICH MULTIPLIES A POLYNOMIAL IN p)

IF ONE DIFFERENTIATES ENOUGH w.r.t p
 THEN INTEGRAL BECOMES CONVERGENT
 IN THIS CASE : ONE NEED TO DIFF. TWICE :

$$I''(p) = \int_0^\infty dk \frac{2k}{(k+p)^3} = \frac{1}{p} \underbrace{\int_0^\infty d\tilde{k} \frac{2\tilde{k}}{(1+\tilde{k})^3}}_1$$

$$I''(p) = \frac{1}{p}$$

↓ INTEGRATE

$$I'(p) = \ln p - \underbrace{\ln \Lambda}_{\text{INTEGRATION CONSTANT}}$$

↓ INTEGRATE

$$I(p) = p \ln p - p - p \ln \Lambda + \underbrace{C_1 \Lambda}_{\text{INTEGRATION CONSTANT}}$$

$$I(p) = p \ln p - p(1 + \ln \Lambda) + C_1 \Lambda$$

↑
FINITE TERM
IS

NON-ANALYTIC IN p :

PREDICTION OF
NON-RENORM. THEORY!

↓
DIVERGENT TERMS
ARE POLYNOMIALS IN p :
CAN BE ABSORBED
BY TREE-LEVEL TERMS IN $\mathcal{L}$

◦◦ THE NON-ANALYTIC TERMS IN EXTERNAL MOMENTA
 (e.g. DUE TO BRANCH CUTS, POLES)
 ARE FULLY PREDICTED BY NON-RENORMALIZABLE
 THEORIES! $\Rightarrow$ QUANTUM PREDICTIONS

• PREDICTIVE POWER

MAKE COUPLINGS OF NON-RENORM. TERMS IN $\mathcal{L}$ DIMENSIONLESS BY INTRODUCING A MASS SCALE M

$$\mathcal{L} = + \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) + \lambda \phi^4 + \frac{g_1}{M^2} \phi^2 \square \phi^2 + \frac{g_2}{M^4} \phi^2 \square^2 \phi^2 + \dots$$

WHEN CALCULATING 4-POINT FUNCTION AS FUNCTION OF ENERGY E (e.g. C.M. ENERGY) WE OBTAIN



$$\langle \phi^4 \rangle = \lambda_R + g_{1R} \frac{E^2}{M^2} + g_{2R} \frac{E^4}{M^4} + \dots$$

+ NON-ANALYTIC TERMS
IN TERMS OF RENORM. COUPLINGS

FOR $E \ll M$

ONLY A FINITE # TERMS ARE IMPORTANT IN EXPANSION



IF ONE CAN FIX THEM BY A FINITE # MEASUREMENTS



▽
0 THEORY IS FULLY PREDICTIVE
FOR $E \ll M$
(EFFECTIVE FIELD THEORY)