III. ABELIAN GAUGE THEORY AND ITS QUANTIZATION

1) ABELIAN GAUGE THEORY, QED

2) QUANTIZATION OF PHOTON FIELD

3) FEYNMAN PROPAGATOR FOR PHOTONS

1) ABELIAN GAUGE THEORY (QED)

⇒ LOCAL PHASE TRANSFORMATION

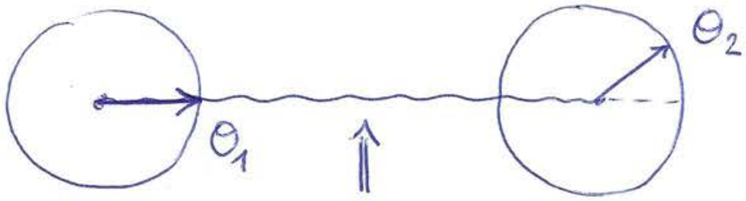
↳ FERMION (SPIN 1/2)

$$\mathcal{L}_{DIRAC} = \bar{\psi}(x)\left(i\gamma^\mu\partial_\mu - m\right)\psi(x)$$

↳ LOCAL PHASE TF. : $U(1)$ GROUP

$$\psi(x) \xrightarrow{U(1)} e^{i\theta(x)}\,\psi(x)$$

$\theta(x)$ → DEPENDS ON SPACE-TIME POINT



$$\Delta\theta = \theta_2 - \theta_1$$

SIGNAL WHICH COMMUNICATES PHASE DIFFERENCE ("PHOTON")

↳ TRANSFORMATION OF $\mathcal{L}_{DIRAC}$ UNDER LOCAL PHASE TF.

$$\partial_\mu\psi(x) \longrightarrow e^{i\theta(x)}\left[\partial_\mu\psi(x) + i(\partial_\mu\theta)\psi\right]$$

$$\bar{\psi}\gamma^\mu\partial_\mu\psi \longrightarrow \bar{\psi}\gamma^\mu\partial_\mu\psi + i(\partial_\mu\theta)\bar{\psi}\gamma^\mu\psi$$

$$\bar{\psi}\psi \longrightarrow \bar{\psi}\psi$$

$$\mathcal{L}_{DIRAC} \longrightarrow \mathcal{L}_{DIRAC} - (\partial_\mu\theta)\overbrace{\bar{\psi}\gamma^\mu\psi}^{\text{VECTOR}}$$

↳ INVARIANCE UNDER LOCAL PHASE TF.

- $$\mathcal{L}_{DIRAC} \xrightarrow{U(1)} \mathcal{L}_{DIRAC} \underbrace{- (\partial_\mu \theta) \bar{\psi} \gamma^\mu \psi}_{\text{EXTRA TERM}}$$

EXTRA TERM $\sim (\partial_\mu \theta)$

DIFFERENCE IN PHASE BETWEEN DIFFERENT SPACE-TIME POINTS

- TO MAKE $\mathcal{L}$ INVARIANT UNDER LOCAL PHASE TF $\Rightarrow$ INTRODUCE VECTOR FIELD (GAUGE FIELD) WHICH COMPENSATES FOR DIFFERENT CHOICES OF PHASE BETWEEN DIFFERENT SPACE-TIME POINTS

- INTRODUCE COVARIANT DERIVATIVE

$$\partial_\mu \underset{\text{REPLACE}}{\Longrightarrow} D_\mu = \partial_\mu + i e A_\mu$$

VECTOR FIELD

ELECTRIC CHARGE
COUPLING OF VECTOR FIELD TO DIRAC FIELD

$$A^\mu \xrightarrow{U(1)} A'^\mu$$

CHOOSE A'^μ SUCH THAT TOTAL $\mathcal{L}$ IS INVARIANT UNDER LOCAL PHASE TF.

- $$D_\mu \psi = \partial_\mu \psi + ie A_\mu \psi$$

$$\xrightarrow{U(1)} e^{i\Theta(x)} \left[\partial_\mu \psi + i(\partial_\mu \Theta)\psi + ie A'_\mu \psi \right]$$

$$= e^{i\Theta(x)} \left[\partial_\mu \psi + ie\left(A'_\mu + \frac{1}{e}\partial_\mu \Theta\right)\psi \right]$$

CHOOSE $A'_\mu = A_\mu - \frac{1}{e}\partial_\mu \Theta$

$$\Downarrow$$

$$D_\mu \psi \xrightarrow{U(1)} e^{i\Theta(x)} \left[\partial_\mu \psi + ie A_\mu \psi \right]$$

$$\underline{\underline{D_\mu \psi \xrightarrow{U(1)} e^{i\Theta(x)} D_\mu \psi}}$$

$$\bar{\psi} i \gamma^\mu D_\mu \psi \xrightarrow{U(1)} \bar{\psi} i \gamma^\mu D_\mu \psi$$

INVARIANT UNDER LOCAL U(1) TF !

$\therefore$

LAGRANGIAN $\mathcal{L} = \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi$ IS INVARIANT UNDER LOCAL PHASE TF : $\psi(x) \xrightarrow{U(1)} e^{i\Theta(x)} \psi(x)$ $A^\mu \xrightarrow{U(1)} A^\mu - \frac{1}{e}\partial^\mu \Theta$

$\Rightarrow$ GEOMETRIC INTERPRETATION OF COVARIANT DERIVATIVE

$\hookrightarrow$ NORMAL DERIVATIVE

dx^μ : INFINITESIMAL DISTANCE BETWEEN 2 SPACE-TIME POINTS

$x^\mu \qquad x^\mu + dx^\mu$

$$\psi(x+dx) - \psi(x) = dx^\mu \, \partial_\mu \psi$$

$\hookrightarrow$ COVARIANT DERIVATIVE

$$dx^\mu D_\mu \psi = dx^\mu \left(\partial_\mu \psi + ie A_\mu \psi \right)$$

$\downarrow$ CHOOSE $A_\mu = -\frac{1}{e} \partial_\mu \theta$ (PURE GAUGE FIELD)

$$= dx^\mu \left(\partial_\mu \psi - i (\partial_\mu \theta) \psi \right)$$

$$= \psi(x+dx) - \psi(x) - i \left[\theta(x+dx) - \theta(x) \right] \psi(x)$$

$$= \psi(x+dx) - \left[1 + i \left(\theta(x+dx) - \theta(x) \right) \right] \psi(x)$$

$$\simeq \psi(x+dx) - \exp\left\{ i \left[\theta(x+dx) - \theta(x) \right] \right\} \psi(x)$$

$\Uparrow$

INFINITESIMAL

↳ PARALLEL TRANSPORT

- $\underline{\Phi}(x+dx, x) = \exp\{i[\theta(x+dx) - \theta(x)]\}$

↳ FACTOR WHICH COMPENSATES FOR PHASE DIFFERENCE BETWEEN 2 NEIGHBOURING POINTS.

$$\underline{\Phi}(x+dx, x) = \exp\{i\, dx_\mu\, \partial^\mu \theta\}$$

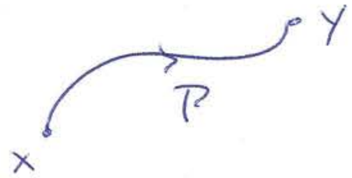
↳ PARALLEL TRANSPORT

$$\underline{\Phi}(x+dx, x) = \exp\{-ie\, dx_\mu\, A^\mu\}$$

$$dx_\mu D^\mu \psi(x) = \psi(x+dx) - \underline{\Phi}(x+dx, x)\,\psi(x).$$

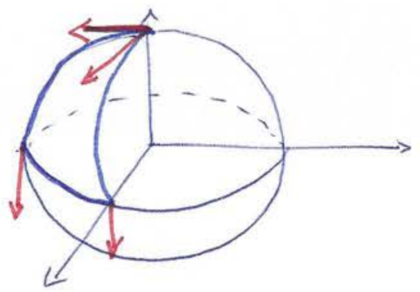
∴ GAUGE FIELD IS EQUIVALENT TO A PHASE DIFFERENCE FOR ψ

- FOR FINITE PATH



$$\underset{P}{\underline{\Phi}}(y, x) = \exp\{-ie \int_P dx_\mu\, A^\mu\}.$$

↳ GEOMETRIC ANALOGY (PERSON LIVING ON SURFACE OF SPHERE GOING AROUND CLOSED CURVE WITH ARROW)



$\Delta\theta = 90°$

90° PHASE DIFFERENCE !

$\Rightarrow$ INTERACTION BETWEEN MATTER & GAUGE FIELD

↳ SYMMETRY OF $\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$ UNDER LOCAL $U(1)$ IS CALLED ABELIAN GAUGE SYMMETRY

$\downarrow$

REFERS TO GAUGE GROUP $U(1)$

↳ COUPLING TO GAUGE (PHOTON) FIELD

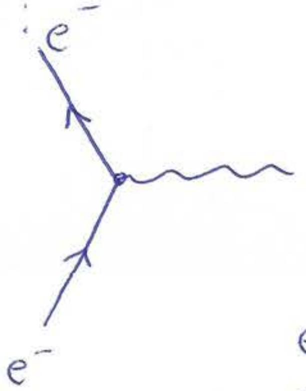
$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$$

$$= \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - e(\bar{\psi}\gamma^\mu\psi)A_\mu$$

$$= \mathcal{L}_{DIRAC} + \underbrace{\mathcal{L}_{INT}}_{\text{INTERACTION LAGRANGIAN}}$$

$$\boxed{\mathcal{L}_{INT} = -e(\bar{\psi}\gamma^\mu\psi)A_\mu}$$

GRAPHICALLY:



γ (DESCRIBED BY A_μ FIELD).

e IS STRENGTH OF COUPLING BETWEEN FIELDS

⇒ QUANTUM ELECTRODYNAMICS (QED)

↳ INTERACTION BETWEEN MATTER (SPIN 1/2) AND GAUGE FIELDS (PHOTONS) IS DESCRIBED BY

$$\boxed{\mathcal{L}_{QED} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - e\bar{\psi}\gamma^\mu\psi A_\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}}$$

- FREE MATTER FIELD ψ

$$\mathcal{L}_{DIRAC} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

- INTERACTION BETWEEN MATTER & GAUGE FIELDS

$$\mathcal{L}_{INT} = -e\bar{\psi}\gamma^\mu\psi A_\mu$$

$$\equiv -\underbrace{J^\mu_{em}(x)}_{\text{ELECTROMAGNETIC CURRENT}} A_\mu$$

$$J^\mu_{em} = e\bar{\psi}\gamma^\mu\psi$$

- FREE PHOTON FIELD

FIELD TENSOR $\underline{\underline{F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu}}$.

UNDER $U(1)$ $\quad A^\mu \xrightarrow{U(1)} A^\mu - \frac{1}{e}\partial^\mu\theta$

$$F^{\mu\nu} \xrightarrow{U(1)} F^{\mu\nu}$$

$$\mathcal{L}_{em} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

↳ $\mathcal{L}_{QED}$ IS INVARIANT UNDER LOCAL $U(1)$!

$$\psi(x) \xrightarrow{U(1)} e^{i\Theta(x)} \psi(x)$$

$$A^\mu(x) \xrightarrow{U(1)} A^\mu(x) - \frac{1}{e} \partial^\mu \Theta$$

$U(1)$ GAUGE SYMMETRY

↳ FIELD EQUATIONS FOR A^μ

$$\frac{\partial \mathcal{L}}{\partial \phi_r} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} = 0$$

$$\downarrow \quad \text{FOR } \phi_r = A_\nu$$

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = - J^\nu_{em}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = - \frac{1}{2} \cdot 2 \cdot F^{\mu\nu}$$

$$\boxed{\partial_\mu F^{\mu\nu} = J^\nu_{em}}$$

INHOMOGENEOUS MAXWELL EQ !

↳ EM CURRENT : SOURCE TERM IN MAXWELL EQ.

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu (\partial_\mu A^\mu) = J^\nu_{em}$$

↳ GAUGE INVARIANCE

$$A^\mu \xrightarrow{U(1)} A^\mu - \frac{1}{e}\partial^\mu \theta$$

FREEDOM TO CONSTRAIN A^μ

POSSIBLE CHOICES OF GAUGE ARE

- $\partial_\mu A^\mu = 0$: LORENZ GAUGE (COVARIANT)
- $\bar{\nabla}\cdot\bar{A} = 0$ $(A^0 = 0)$: COULOMB GAUGE (NON-COVARIANT)
  ↳ USED FOR FREE FIELDS ($J^\nu_{em} = 0$, ABSENCE OF SOURCES)
- $n_\mu A^\mu = 0$: AXIAL GAUGE

  WITH $n_\mu n^\mu = -1$

  e.g. $n^\mu (0,0,0,1)$

IN FOLLOWING WE WILL OFTEN USE LORENZ GAUGE $\partial_\mu A^\mu = 0$

↓

$$\Box A^\nu = J^\nu_{em}$$

$$\partial_\mu A^\mu = 0 \qquad \text{CONSTRAINT}$$

GAUGE INVARIANCE ENSURES THAT RESULTS FOR PHYSICAL OBSERVABLES ARE INVARIANT UNDER CHOICE OF GAUGE

2) QUANTIZATION OF PHOTON FIELD

⟹ PRESENCE OF CONSTRAINTS : ISSUES

↳ $$\Box A^\nu = J^\nu_{em}$$

$$\partial_\nu A^\nu = 0 \qquad \text{(LORENZ GAUGE)}$$

HOW TO QUANTISE A^ν IN PRESENCE OF CONSTRAINT $\partial_\nu A^\nu = 0$.

↳ ISSUE 1 : CANONICAL MOMENTA

$$A^\nu \longrightarrow \Pi^\nu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\nu} = -F^{0\nu}$$

$$\Pi^0 = 0$$

$$\Pi^i = -F^{0i} = -\partial^0 A^i + \partial^i A^0$$

$$= \left(-\bar{\nabla} A^0 - \frac{\partial \bar{A}}{\partial t}\right)^i$$

$$= E^i \quad \text{(ELECTRIC FIELD)}$$

FOR A^1, A^2, A^3 OK

FOR A^0 ⟹ WE CANNOT IMPOSE COMMUTATION RELATIONS BECAUSE $\Pi^0 = 0$

↳ ISSUE 2 : PHOTON PROPAGATOR

- KLEIN-GORDON (SPIN-0)

$$(\Box + m^2)\, \phi(x) = 0$$

$\downarrow$ FOR SOLUTION $\phi(x) \sim e^{-ik.x}$

$$(-k^2 + m^2)\, \phi = 0$$

SCALAR PROPAGATOR $\dfrac{i}{k^2 - m^2 + i\varepsilon}$

IS INVERSE OF OPERATOR IN FREE FIELD EQUATION (GREEN'S FUNCTION)

$$(\Box_x + m^2)\, \Delta_F(x) = -\delta^4(x)$$

- FREE FIELD EQUATION FOR A^ν

$$\Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = 0$$

$\downarrow$ FOR SOLUTION $\sim e^{-ik.x}$

$$\underbrace{(-k^2 g^{\mu\nu} + k^\mu k^\nu)}_{}\, A_\nu = 0$$

WE LOOK FOR INVERSE OF THIS OPERATOR

$$\left(-k^2 g^{\mu\nu} + k^\mu k^\nu\right)\left(A g_{\nu\lambda} + B k_\nu k_\lambda\right) = g^\mu_\lambda$$

$$-Ak^2 g^\mu_\lambda + A k^\mu k_\lambda - \cancel{k^2 B k^\mu k_\lambda} + \cancel{k^2 B k^\mu k_\lambda} = g^\mu_\lambda$$

$$\begin{cases} -Ak^2 = 1 \\ A = 0 \end{cases}$$

$\hookrightarrow$ IMPOSSIBLE FOR ANY CHOICE OF A

$\therefore$ OPERATOR IN FIELD EQUATION FOR A^ν HAS NO INVERSE

⇒ GAUGE FIXING

↳ TO SOLVE ABOVE PROBLEMS WHEN QUANTIZING THE ABELIAN GAUGE THEORY CONSIDER ALTERNATIVE $\mathcal{L}$

$$\mathcal{L}_{em} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$ DESCRIBES FREE PHOTON FIELD

$$\mathcal{L}_{em} \longrightarrow \boxed{\mathcal{L}'_{em} = \underbrace{-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}}_{\mathcal{L}_{em}} + \underbrace{- \frac{1}{2} (\partial_\mu A^\mu)^2}_{\mathcal{L}_{GF}}}$$

(GAUGE FIXING)

↳ EQUATIONS OF MOTION

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = - F^{\mu\nu} - (\partial_\alpha A^\alpha)\, g^{\mu\nu}$$

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = 0$$

$$\partial_\mu F^{\mu\nu} + \partial^\nu (\partial_\alpha A^\alpha) = 0$$

$$\Box A^\nu - \partial^\nu (\cancel{\partial_\mu A^\mu}) + \partial^\nu (\cancel{\partial_\alpha A^\alpha}) = 0$$

$$\Box A^\nu = 0$$ ⤻ FIELD EQ. CORRESPONDING WITH $\mathcal{L}'_{em}$ CORRESPOND WITH CHOICE OF LORENZ GAUGE

↳ CANONICAL MOMENTA

$$A^\nu \rightarrow \Pi^\nu = -F^{0\nu} - (\partial_\alpha A^\alpha)\, g^{0\nu}$$

$$\begin{cases} \Pi^0 = -\partial_\alpha A^\alpha \\ \Pi^i = -F^{0i} \end{cases}$$

, OK ISSUE 1 IF $\partial_\alpha A^\alpha$ CONSIDERED AS OPERATOR

↳ EQUAL TIME COMMUTATION RELATIONS (ETCR)

TO QUANTIZE THE THEORY, WE IMPOSE

$$\begin{aligned} &[A^\mu(\bar{x},t), A^\nu(\bar{x}',t)]_- = 0 \\ &[\Pi^\mu(\bar{x},t), \Pi^\nu(\bar{x}',t)]_- = 0 \\ &[A^\mu(\bar{x},t), \Pi^\nu(\bar{x}',t)]_- = i\, g^{\mu\nu}\, \delta^3(\bar{x}-\bar{x}') \end{aligned}$$

WITH $[A,B]_- = AB - BA$ COMMUTATOR

SPIN 1 FIELD DESCRIBES BOSONS (PHOTONS)

A^μ, Π^μ CONSIDERED AS OPERATORS

$$\rightsquigarrow [A^\mu(\bar{x},t), \Pi^0(\bar{x}',t)]_- = i\, g^{\mu 0}\, \delta^3(\bar{x}-\bar{x}')$$

$$= -[A^\mu(\bar{x},t), \dot{A}^0(\bar{x}',t)]_- - \bar{\nabla}_{x'} \cdot \underbrace{\cancel{[A^\mu(\bar{x},t), \bar{A}(\bar{x}',t)]_-}}_{0}$$

$$\rightsquigarrow \quad \left[A^\mu(\bar{x},t) \, , \, \underbrace{\Pi^i(\bar{x}',t)}_{-\dot{A}^i + \partial^i A^0} \right]_- = i g^{\mu i} \, \delta^3(\bar{x} - \bar{x}')$$

$$= - \left[A^\mu(\bar{x},t), \, \dot{A}^i(\bar{x}',t) \right]_- - \bar{\nabla}_{x'}^i \cdot \underbrace{\cancel{\left[A^\mu(\bar{x},t), \, A^0(\bar{x}',t) \right]_-}}_{0}$$

$$\therefore \quad \left\| \; \left[A^\mu(\bar{x},t), \, \dot{A}^\nu(\bar{x}',t) \right]_- = -i g^{\mu\nu} \delta^3(\bar{x} - \bar{x}') \right.$$

$$\hookrightarrow \mu = 0 : \quad \boxed{\left[A^\mu(\bar{x},t), \, \dot{A}^0(\bar{x}',t) \right]_- = -i \, \delta^3(\bar{x} - \bar{x}')}$$

$$\hookrightarrow \mu = i : \quad \boxed{\left[A^\mu(\bar{x},t), \, \dot{A}^i(\bar{x}',t) \right]_- = +i \, \delta^3(\bar{x} - \bar{x}')}$$

↳ PHOTON PROPAGATOR (IN LORENZ GAUGE)

$$\Box A^\mu = 0$$

$$\underbrace{\left(-k^2 g^{\mu\nu} \right)}_{\text{INVERSE EXISTS}} A_\nu = 0$$

$$\left(-k^2 g^{\mu\nu} \right) \cdot \left(A \, g_{\nu\lambda} + B \, k_\nu k_\lambda \right) = g^\mu{}_\lambda$$

$$A = -\frac{1}{k^2}$$

$$B = 0$$

$$\Longrightarrow \quad \text{PHOTON PROPAGATOR} \sim \frac{-g^{\mu\nu}}{k^2 + i\varepsilon}$$

$\Rightarrow$ NORMAL MODE EXPANSION OF $A^\mu(\bar{x},t)$

↳ UPON QUANTIZATION

CLASSICAL FIELDS A^μ

ARE RE-INTERPRETED AS FIELD OPERATORS $\hat{A}^\mu$

THAT SATISFY ETCR

(NOTE: FOR SIMPLICITY WE WILL DROP ^ NOTATION

IN QFT THE FIELDS ARE UNDERSTOOD

AS OPERATORS)

↳ POLARIZATION VECTORS:

A NORMAL MODE SOLUTION IS CHARACTERIZED

BY $\sim e^{ik.x}\, \varepsilon^\mu(\bar{k},\lambda) \qquad \lambda = 0,1,2,3$

WITH $k(\omega_{\bar{k}}, \underbrace{0,0,|\bar{k}|}_{\bar{k}})$ CONVENIENT CHOICE $\bar{k} = |\bar{k}|\,\bar{e}_z$

$\underline{\underline{\omega_{\bar{k}} = |\bar{k}|}}$

$\varepsilon^\mu(\bar{k},\lambda=0) = (1,0,0,0)$ SCALAR POL.

$\left.\begin{aligned} \varepsilon^\mu(\bar{k},\lambda=1) &= (0,1,0,0) \\ \varepsilon^\mu(\bar{k},\lambda=2) &= (0,0,1,0) \end{aligned}\right\}$ TRANSVERSE POL. $0 = \bar{k}\cdot\bar{\varepsilon}(\bar{k},\lambda=\tfrac{1}{2})$

$\varepsilon^\mu(\bar{k},\lambda=3) = (0,0,0,1)$ LONGITUDINAL POL.

NORMALIZATION $\quad \varepsilon^\mu(\bar{k},\lambda)\,\varepsilon^*_\mu(\bar{k},\lambda') = -\zeta_\lambda\,\delta_{\lambda\lambda'}$

$(\zeta_0 \equiv -1,\ \zeta_i \equiv +1)$

↳ NORMAL MODE EXPANSION

$$\boxed{A^\mu(\bar{x},t) = \int \frac{d^3\bar{k}}{(2\pi)^3\sqrt{2\omega_{\bar{k}}}} \sum_{\lambda=0}^{3} \left\{ a(\bar{k},\lambda)\,\varepsilon^\mu(\bar{k},\lambda)\, e^{-ik.x} + a^+(\bar{k},\lambda)\,\varepsilon^{\mu *}(\bar{k},\lambda)\, e^{+ik.x} \right\}}$$

$\omega_{\bar{k}} = |\bar{k}|$ PARTICLE WITH MASS 0 (PHOTON)
(→ MOVES WITH SPEED OF LIGHT)

$a(\bar{k},\lambda)$ ANNIHILATES PHOTON WITH MOMENTUM $\bar{k}$ & POLARIZATION λ

$a^+(\bar{k},\lambda)$ CREATES PHOTON " "

⟹ COMMUTATION RELATIONS FOR a, a^+

↳ $\int d^3\bar{x}\; e^{-i\bar{k}.\bar{x}} \left\{ A^\mu(\bar{x},t) + \frac{i}{\omega_{\bar{k}}} \dot{A}^\mu(\bar{x},t) \right\}$

$$= 2\,\frac{1}{\sqrt{2\omega_{\bar{k}}}} \sum_{\lambda=0}^{3} a(\bar{k},\lambda)\,\varepsilon^\mu(\bar{k},\lambda)\; e^{-i\omega_{\bar{k}}t}$$

↳ $\int d^3\bar{x}\; e^{+i\bar{k}.\bar{x}} \left\{ A^\mu(\bar{x},t) - \frac{i}{\omega_{\bar{k}}} \dot{A}^\mu(\bar{x},t) \right\}$

$$= 2\,\frac{1}{\sqrt{2\omega_{\bar{k}}}} \sum_{\lambda=0}^{3} a^+(\bar{k},\lambda)\,\varepsilon^{\mu *}(\bar{k},\lambda)\; e^{i\omega_{\bar{k}}t}$$

$\downarrow$ USING $\varepsilon^{\mu}(\bar{k},\lambda)\,\varepsilon^{*}_{\mu}(\bar{k},\lambda') = -\zeta_{\lambda}\,\delta_{\lambda\lambda'}$

$$a(\bar{k},\lambda) = -\zeta_{\lambda}\frac{1}{2}\sqrt{2\omega_{\bar{k}}}\;e^{+i\omega_{\bar{k}}t}\int d^3\bar{x}\;e^{-i\bar{k}.\bar{x}}\;\varepsilon^{*}_{\mu}(\bar{k},\lambda)\left\{A^{\mu}+\frac{i}{\omega_{\bar{k}}}\dot{A}^{\mu}\right\}$$

$$a^{+}(\bar{k},\lambda) = -\zeta_{\lambda}\frac{1}{2}\sqrt{2\omega_{\bar{k}}}\;e^{-i\omega_{\bar{k}}t}\int d^3\bar{x}\;e^{+i\bar{k}.\bar{x}}\;\varepsilon_{\mu}(\bar{k},\lambda)\left\{A^{\mu}-\frac{i}{\omega_{\bar{k}}}\dot{A}^{\mu}\right\}$$

$\hookrightarrow\ \left[a(\bar{k},\lambda),\,a^{+}(\bar{k}',\lambda')\right]_{-}$

$$= \zeta_{\lambda}\zeta_{\lambda'}\,\frac{1}{4}(2\omega_{\bar{k}})^{1/2}(2\omega_{\bar{k}'})^{1/2}e^{i(\omega_{\bar{k}}-\omega_{\bar{k}'})t}\;\varepsilon^{*}_{\mu}(\bar{k},\lambda)\,\varepsilon_{\nu}(\bar{k}',\lambda')$$

$$\cdot\int d^3\bar{x}\int d^3\bar{x}'\;e^{-i\bar{k}.\bar{x}}\;e^{+i\bar{k}'.\bar{x}'}$$

$$\cdot\underbrace{\left[A^{\mu}(\bar{x},t)+\frac{i}{\omega_{\bar{k}}}\dot{A}^{\mu}(\bar{x},t),\;A^{\nu}(\bar{x}',t)-\frac{i}{\omega_{\bar{k}'}}\dot{A}^{\nu}(\bar{x}',t)\right]_{-}}_{-\frac{1}{\omega_{\bar{k}'}}g^{\mu\nu}\delta^3(\bar{x}-\bar{x}')\;-\;\frac{1}{\omega_{\bar{k}}}g^{\mu\nu}\delta^3(\bar{x}-\bar{x}')}$$

$$= \zeta_{\lambda}\zeta_{\lambda'}\,\frac{1}{4}(2\omega_{\bar{k}})^{1/2}(2\omega_{\bar{k}'})^{1/2}(-1)\left(\frac{1}{\omega_{\bar{k}}}+\frac{1}{\omega_{\bar{k}'}}\right)e^{i(\omega_{\bar{k}}-\omega_{\bar{k}'})t}$$

$$\cdot\,\varepsilon^{*}_{\mu}(\bar{k},\lambda)\,\varepsilon^{\mu}(\bar{k}',\lambda')\underbrace{\int d^3\bar{x}\;e^{-i(\bar{k}-\bar{k}').\bar{x}}}_{(2\pi)^3\delta^3(\bar{k}-\bar{k}')}$$

$$[a(\bar{k},\lambda),\ a^{+}(\bar{k}',\lambda')]_{-}$$

$$= -\zeta_\lambda \zeta_{\lambda'}\ \frac{1}{4}\,(2\omega_{\bar{k}})\left(\frac{2}{\omega_{\bar{k}}}\right)\cdot(2\pi)^3\ \underbrace{\varepsilon^{*}_{\mu}(\bar{k},\lambda)\,\varepsilon^{\mu}(\bar{k},\lambda')}_{-\zeta_\lambda\,\delta_{\lambda\lambda'}}$$

$$\downarrow\ \ \zeta_\lambda^2 = 1$$

$$= \zeta_\lambda\ \delta_{\lambda\lambda'}\ (2\pi)^3\ \delta^3(\bar{k}-\bar{k}')$$

$\therefore$

$$\boxed{\begin{gathered}[a(\bar{k},\lambda),\ a^{+}(\bar{k}',\lambda')]_{-} = \zeta_\lambda\,\delta_{\lambda\lambda'}(2\pi)^3\,\delta^3(\bar{k}-\bar{k}')\\ \text{ANALOGOUSLY}\\ [a(\bar{k},\lambda),\ a(\bar{k}',\lambda')]_{-} = 0\\ [a^{+}(\bar{k},\lambda),\ a^{+}(\bar{k}',\lambda')]_{-} = 0\end{gathered}}$$

$\zeta_0 = -1,\ \zeta_i = +1$
$(i = 1,2,3)$

FOR $\lambda = 1,2,3$

$$[a(\bar{k},\lambda),\ a^{+}(\bar{k}',\lambda)]_{-} = \delta_{\lambda\lambda'}\ (2\pi)^3\,\delta^3(\bar{k}-\bar{k}')$$

FOR $\lambda = 0$

$$[a(\bar{k},0),\ a^{+}(\bar{k}',0)]_{-} = -\ \ (2\pi)^3\,\delta^3(\bar{k}-\bar{k}')$$

FOR $\lambda = 1,2,3$ $\longrightarrow$ STANDARD BOSON COMMUTATION RELATIONS

$\lambda = 0$ $\longrightarrow$ SIGN CHANGE

↳ IF WE CONSIDER VACUUM STATE AS STATE WITHOUT PHOTONS

i.e. $a(\bar{k},\lambda)\,|0\rangle = 0$ $\qquad \lambda = 0, 1, 2, 3$

$\Downarrow$

1 - PHOTON STATE WITH POLARIZATION λ

$$|1\gamma,\lambda\rangle = \int d^3\bar{k}\; f(\bar{k})\; a^{+}(\bar{k},\lambda)\,|0\rangle$$

↳ WAVE PACKET

$$\int d^3\bar{k}\; |f(\bar{k})|^2 < \infty$$

NORMALIZATION OF 1γ STATE

$$\langle 1\gamma,\lambda\,|\,1\gamma,\lambda\rangle$$

$$= \int d^3\bar{k} \int d^3\bar{k}'\; f^{*}(\bar{k})\, f(\bar{k}')\; \underbrace{\langle 0|\, a(\bar{k},\lambda)\, a^{+}(\bar{k}',\lambda)\,|0\rangle}_{\zeta_\lambda\,(2\pi)^3\,\delta^3(\bar{k}-\bar{k}')}$$

$$= \zeta_\lambda\,(2\pi)^3 \int d^3\bar{k}\; |f(\bar{k})|^2$$

$$= \begin{cases} > 0 & \text{FOR } \lambda = 1,2,3 \\ < 0 & \text{FOR } \lambda = 0 \ ! \ \text{NEGATIVE NORM STATE} \end{cases}$$

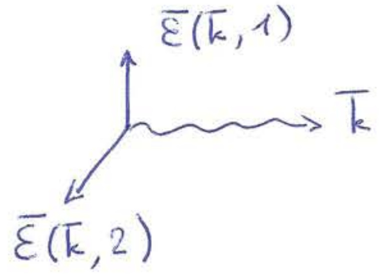
↳ HAMILTONIAN :

$$H = \int \frac{d^3\bar{k}}{(2\pi)^3}\, \omega_{\bar{k}} \left\{ - a^{+}(\bar{k},0)\, a(\bar{k},0) + \sum_{i=1}^{3} a^{+}(\bar{k},i)\, a(\bar{k},i) \right\}$$

STATES WITH $\lambda = 0$ LEAD TO NEGATIVE ENERGY !

↳ PHYSICAL STATES :

FREE MAXWELL FIELD HAS ONLY 2 TRANSVERSE COMPONENTS



↳ $\lambda = 0$, $\lambda = 3$ STATES SHOULD NOT APPEAR IN PHYSICAL STATES UPON QUANTIZATION

$|\Psi\rangle$ PHYSICAL VACUUM (HAS NO TRANSVERSE PHOTONS)

WE REQUIRE

$$\underline{\underline{\partial_\mu A^{\mu(+)} |\Psi\rangle = 0}} \qquad \text{(GUPTA - BLEULER)}$$

(+) STANDS FOR POS. FREQUENCY (ANNIHILATING) PART IN A^μ

NOTE : THIS IS A <u>WEAKER</u> CONDITION THAN CONSIDERING $\partial_\mu \hat{A}^\mu = 0$ AS OPERATOR CONDITION

$$\partial_\mu A^{\mu(+)} | \Psi \rangle = 0$$

$$\downarrow$$

$$\partial_\mu \int \frac{d^3\bar{k}}{(2\pi)^3 \sqrt{2\omega_{\bar{k}}}} \sum_\lambda a(\bar{k},\lambda)\, \varepsilon^\mu(\bar{k},\lambda)\, e^{-ik.x} \, | \Psi \rangle = 0$$

$$\int \frac{d^3\bar{k}}{(2\pi)^3 \sqrt{2\omega_{\bar{k}}}} \sum_\lambda (-i)\, a(\bar{k},\lambda)\, e^{-ik.x} \, k_\mu \varepsilon^\mu(\bar{k},\lambda) | \Psi \rangle = 0$$

$$\downarrow \quad \begin{aligned} & k_\mu \varepsilon^\mu(\bar{k},\lambda) \\ &= \omega_{\bar{k}}\, \varepsilon^0(\bar{k},\lambda) - |\bar{k}|\, \varepsilon^3(\bar{k},\lambda) \\ &= \omega_{\bar{k}} (\delta_{\lambda 0} - \delta_{\lambda 3}) \end{aligned}$$

$$\forall \bar{k} : \quad \underline{\underline{\big(a(\bar{k},0) - a(\bar{k},3) \big) | \Psi \rangle = 0.}}$$

$$\Downarrow$$

$$\langle \Psi | a^\dagger(\bar{k},3)\, a(\bar{k},3) | \Psi \rangle = \langle \Psi | a^\dagger(\bar{k},0)\, a(\bar{k},0) | \Psi \rangle$$

- PHYSICAL VACUUM STATE HAS EQUAL NUMBER OF SCALAR ($\lambda = 0$) AND LONGITUDINAL ($\lambda = 3$) PHOTONS

- THE COMBINED ENERGY OF SCALAR & LONGITUDINAL PHOTONS IS ZERO

$$\langle \Psi | H | \Psi \rangle = \int \frac{d^3 \bar{k}}{(2\pi)^3} \, \omega_{\bar{k}} \sum_{\lambda=1,2} \langle \Psi | a^{+}(\bar{k},\lambda) a(\bar{k},\lambda) | \Psi \rangle$$

- PHYSICAL QUANTITIES ONLY INVOLVE TRANSVERSE PHOTONS ($\lambda = 1, 2$).

- ALTERING THE ALLOWED ADMIXTURES OF SCALAR AND LONGITUDINAL PHOTONS IS EQUIVALENT TO A GAUGE TF. BETWEEN 2 POTENTIALS, BOTH OF WHICH ARE IN LORENZ GAUGE

  ($\rightarrow$ EXERCISE)

3) FEYNMAN PROPAGATOR FOR PHOTONS

⇒ 2 POINT CORRELATION FUNCTIONS.

↳ $\langle 0 | A^\mu(x) \, A^\nu(y) | 0 \rangle = i D^{\mu\nu}(x-y)$

WITH $|0\rangle$ STATE WITHOUT ANY PHOTONS.

$$i D^{\mu\nu}(x-y)$$

$$= \int \frac{d^3\bar{k}}{(2\pi)^3\sqrt{2\omega_{\bar{k}}}} \int \frac{d^3\bar{k}'}{(2\pi)^3\sqrt{2\omega_{\bar{k}'}}} \sum_{\lambda=0}^{3} \sum_{\lambda'=0}^{3}$$

$$\langle 0 | \left(a(\bar{k},\lambda)\, \varepsilon^\mu(\bar{k},\lambda)\, e^{-ik.x} + \cancel{a^+(\bar{k},\lambda)}\, \varepsilon^{*\mu}(\bar{k},\lambda)\, e^{+ik.x} \right)$$

$$\cdot \left(\cancel{a(\bar{k}',\lambda')}\, \varepsilon^\nu(\bar{k}',\lambda')\, e^{-ik'.y} + a^+(\bar{k}',\lambda')\, \varepsilon^{*\nu}(\bar{k}',\lambda')\, e^{+ik'.y} \right)$$

$$|0\rangle$$

$$= \int \frac{d^3\bar{k}}{(2\pi)^3} \frac{1}{2\omega_{\bar{k}}} \sum_{\lambda=0}^{3} \zeta_\lambda \, \varepsilon^\mu(\bar{k},\lambda)\, \varepsilon^{*\nu}(\bar{k},\lambda)\, e^{-ik.(x-y)} \Big|_{k^0=\omega_{\bar{k}}}$$

↳ POLARIZATION SUM

$$\boxed{\sum_{\lambda=0}^{3} \zeta_\lambda \, \varepsilon^\mu(\bar{k},\lambda)\, \varepsilon^{*\nu}(\bar{k},\lambda) = - g^{\mu\nu}}$$

FOR CHOICE $k^\mu(|\bar{k}|, 0, 0, |\bar{k}|)$

$$\varepsilon^\mu(\bar{k},\lambda) = g^\mu_\lambda$$

$$\sum_{\lambda=0}^{3} \zeta_\lambda \, \varepsilon^\mu(\bar{k},\lambda)\, \varepsilon^{*\nu}(\bar{k},\lambda)$$

$$= - g^\mu_0 g^\nu_0 + g^\mu_1 g^\nu_1 + g^\mu_2 g^\nu_2 + g^\mu_3 g^\nu_3$$

FOR $\mu \neq \nu \to 0$

$\mu = \nu = 0 \to -1 = -g^{00}$

$\mu = \nu = i \to +1 = -g^{ii}$
$(1,2,3)$

↳ $i D^{\mu\nu}(x-y)$

$$= (-g^{\mu\nu}) \int \frac{d^3\bar{k}}{(2\pi)^3 2\omega_{\bar{k}}} e^{-ik\cdot(x-y)} \Big|_{k^0 = \omega_{\bar{k}}}$$

$\omega_{\bar{k}} = |\bar{k}|$

$$\boxed{D^{\mu\nu}(x) = (-g^{\mu\nu})\, \Delta(x)}$$

↳ 2 POINT CORRELATOR FOR KLEIN-GORDON THEORY (WITH $m = 0$)

$\Rightarrow$ **FEYNMAN PROPAGATOR**

$\hookrightarrow$ $$\boxed{\langle 0 | T \, A^\mu(x) \, A^\nu(y) \, | \, 0 \rangle \equiv i \, D_F^{\mu\nu}(x-y)}$$

WITH (BOSON FIELDS)

$$T \; A^\mu(x) \, A^\nu(y) = \theta(x^0 - y^0) \, A^\mu(x) \, A^\nu(y) + \theta(y^0 - x^0) \, A^\nu(y) \, A^\mu(x).$$

$\hookrightarrow$ $$\boxed{D_F^{\mu\nu}(x) = (-g^{\mu\nu}) \, \Delta_F(x)}$$

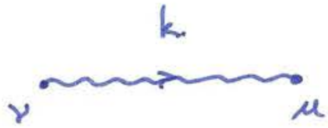
WITH $\Delta_F(x) = \theta(x^0) \, \Delta(x) + \theta(-x^0) \, \Delta(-x)$

$\hookrightarrow$ FEYNMAN PROPAGATOR FOR SPIN-0 PARTICLE WITH $m = 0$

$\hookrightarrow$ IN MOMENTUM SPACE

$$D_F^{\mu\nu}(x) = \int \frac{d^4 k}{(2\pi)^4} \, e^{-ik.x} \, \tilde{D}_F^{\mu\nu}(k)$$

$$\boxed{\tilde{D}_F^{\mu\nu}(k) = \frac{-g^{\mu\nu}}{k^2 + i\varepsilon}}$$



FEYNMAN PROPAGATOR FOR PHOTON IN LORENZ GAUGE (ALSO CALLED FEYNMAN GAUGE)

⇒ MORE GENERAL GAUGES

↳ $\mathcal{L} = \mathcal{L}_{em} + \mathcal{L}_{G.F.}$

$$= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} \left(\partial_\mu A^\mu\right)^2$$

<u>NOTE</u> : CHOICE $\xi = 1$ CORRESPONDS WITH LORENZ (FEYNMAN) GAUGE

↳ FIELD EQUATIONS.

$$\partial_\mu F^{\mu\nu} + \frac{1}{\xi} \partial^\nu \left(\partial_\mu A^\mu\right) = 0$$

$$\Box A^\nu - \left(1 - \frac{1}{\xi}\right) \partial^\nu \left(\partial_\mu A^\mu\right) = 0.$$

↳ PROPAGATOR

$$\underbrace{\left[-k^2 g^{\mu\nu} + \left(1 - \frac{1}{\xi}\right) k^\mu k^\nu \right]} A_\mu = 0$$

PROPAGATOR IS INVERSE OF THIS OPERATOR

$$\left(-k^2 g^{\mu\nu} + \left(1-\frac{1}{\xi}\right) k^\mu k^\nu\right)\left(A\, g_{\nu\lambda} + B\, k_\nu k_\lambda\right) = g^\mu_\lambda$$

$$g^\mu_\lambda = -A k^2 g^\mu_\lambda + \left[\left(A + Bk^2\right)\left(1-\frac{1}{\xi}\right) - Bk^2\right] k^\mu k_\lambda$$

$$\Downarrow$$

$$\begin{cases} -Ak^2 = 1 \\ \left(A + Bk^2\right)\left(1 - \frac{1}{\xi}\right) - Bk^2 = 0 \end{cases}$$

$$A = -1/k^2$$

$$A\left(1-\frac{1}{\xi}\right) = \frac{Bk^2}{\xi} \Rightarrow B = -\frac{(1-\xi)}{k^2} A$$

PHOTON FEYNMAN PROPAGATOR

$$\boxed{\tilde{D}_F^{\mu\nu}(k) = \frac{1}{k^2 + i\varepsilon}\left(-g^{\mu\nu} + (1-\xi)\frac{k^\mu k^\nu}{k^2}\right)}$$