

I.

KLEIN-GORDON FIELD

- 1) INTRODUCTION AND MOTIVATION
- 2) ELEMENTS OF CLASSICAL FIELD THEORY
- 3) QUANTIZATION OF KLEIN-GORDON FIELD
- 4) FEYNMAN PROPAGATOR FOR KLEIN-GORDON FIELD

1) INTRODUCTION AND MOTIVATION

⇒ SPECIAL RELATIVITY

↳ SPACE-TIME POINT DENOTED BY

$$x^\mu(t, \vec{x}) \quad \text{"CONTRAVARIANT"}$$

NOTATION: THROUGHOUT THESE NOTES WE
USE NATURAL UNITS $\hbar = c = 1$

OR $x_\mu(t, -\vec{x}) \quad \text{"COVARIANT"}$

CONNECTED BY

$$x_\mu = g_{\mu\nu} x^\nu \quad \text{OR} \quad x^\mu = g^{\mu\nu} x_\nu$$

FOR MINKOWSKI SPACE-TIME (FLAT SPACE-TIME)
WE USE METRIC

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

↳ RELATIVISTIC ENERGY-MOMENTUM : 4-MOMENTUM

$$P^\mu(E, \vec{p})$$

$$P^2 \equiv P^\mu P_\mu = E^2 - \vec{p}^2 = m^2 \Leftrightarrow E^2 = \vec{p}^2 + m^2$$

P^2 IS LORENTZ INVARIANT

↳ REST MASS OF
PARTICLE

⇒ RELATIVISTIC QUANTUM MECHANICS

↳ NON-RELATIVISTIC : SCHRÖDINGER EQ.

$$H \Psi(\vec{x}, t) = i \frac{\partial \Psi}{\partial t} \quad \text{DESCRIBES QUANTUM MECHANICAL PARTICLE}$$

$$H = T + V = -\frac{1}{2m} \nabla^2 + V$$

↳ RELATIVISTIC EQUATION SPIN 0 : KLEIN-GORDON

$$\left(\partial_\mu \partial^\mu + m^2 \right) \phi(x) = 0$$

WITH x STANDS FOR x^μ

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \vec{\nabla} \right)$$

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\vec{\nabla} \right)$$

$$\partial_\mu \partial^\mu \equiv \square = \frac{\partial^2}{\partial t^2} - \nabla^2 \quad (\text{D'ALAMBERTIAN})$$

FOR FREE PARTICLE : SOLUTIONS $\rightarrow$ PLANE WAVES

$$\phi(x) = C e^{-i p \cdot x} \quad \text{WITH } p \cdot x = p_\mu x^\mu$$

↓

$$p^2 = m^2$$

↳ RELATIVISTIC EQUATION SPIN $1/2$: **DIRAC**

FOR FREE PARTICLE :

$$\underline{(i \gamma^\mu \partial_\mu - m) \psi(x) = 0}$$

WITH γ^μ : DIRAC MATRICES (4×4 MATRICES)

$$\gamma^\mu = (\gamma^0, \vec{\gamma})$$

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu}$$

$$\gamma^0 = \begin{pmatrix} \mathbb{I}_{2 \times 2} & 0 \\ 0 & -\mathbb{I}_{2 \times 2} \end{pmatrix}$$

$$\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

$$\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$$

PAULI MATRICES

AND $\psi(x)$: 4-COMPONENT SPINOR

→ POSITIVE ENERGY SOLUTIONS

$$\psi_+(x) = C e^{-i p \cdot x} U(\vec{p}, s) \quad s = \pm \frac{1}{2}$$

$$\hookrightarrow (\not{p} - m) U(\vec{p}, s) = 0$$

$$\Rightarrow U(\vec{p}, s) = \sqrt{E+m} \begin{pmatrix} \chi_s \\ \frac{\vec{\sigma} \cdot \vec{p}}{E+m} \chi_s \end{pmatrix}$$

FOR SPIN QUANTIZED ALONG Z-AXIS

$$\chi_{+\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\text{NORMALIZATION} \quad \bar{U}(\vec{p}, s) U(\vec{p}, s') = 2m \delta_{ss'}$$

→ NEGATIVE ENERGY SOLUTIONS.

$$\psi_-(x) = C e^{+ip \cdot x} u(\vec{p}, s) \quad \dots \quad s = \pm \frac{1}{2}$$

$$\hookrightarrow (\not{p} + m) u(\vec{p}, s) = 0$$

$$\Rightarrow u(\vec{p}, s) = \sqrt{E+m} \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{p}}{E+m} \chi'_s \\ \chi'_s \end{pmatrix}$$

DIRAC INTERPRETED NEGATIVE - ENERGY SOLUTIONS

AS ANTI - PARTICLES $e^- \leftrightarrow e^+$

WHICH WERE THEREAFTER CONFIRMED EXPERIMENTALLY

IF ENOUGH ENERGY IS AVAILABLE $E > 2m$

$e^- e^+$ PAIR CAN BE CREATED

→ LIMITATION OF EQUATION DESCRIBING
SINGLE PARTICLES

⇒

PARTICLE CREATION / ANNIHILATION

↳ e.g. KLEIN-GORDON EQ.

DESCRIBES FREE PARTICLES OF MOMENTUM $\vec{p}$
ENERGY $E = \sqrt{\vec{p}^2 + m^2}$

$$\phi(x) = \sum_{\vec{p}} \left[a(\vec{p}) e^{-i\vec{p} \cdot x} + a^*(\vec{p}) e^{+i\vec{p} \cdot x} \right]$$

ϕ : REAL , $a(\vec{p}) \in \mathbb{C}$ (COMPLEX)

$a(\vec{p})$ DESCRIBES WEIGHT OF MODE $\vec{p}$
IN SOLUTION $\phi(x)$

↳ SECOND QUANTIZATION

$$\left. \begin{array}{l} a(\vec{p}) \Rightarrow \hat{a}(\vec{p}) \\ a^*(\vec{p}) \Rightarrow \hat{a}^+(\vec{p}) \end{array} \right\} \text{ OPERATORS.}$$

$\hat{a}(\vec{p})$ DESCRIBES ANNIHILATION OF PARTICLE
WITH MOMENTUM $\vec{p}$

$\hat{a}^+(\vec{p})$ DESCRIBES CREATION OF PARTICLE
WITH MOMENTUM $\vec{p}$

$\hat{a}(\vec{p})$ & $\hat{a}^+(\vec{p})$ SATISFY COMMUTATION
RELATIONS
(SEE FURTHER)

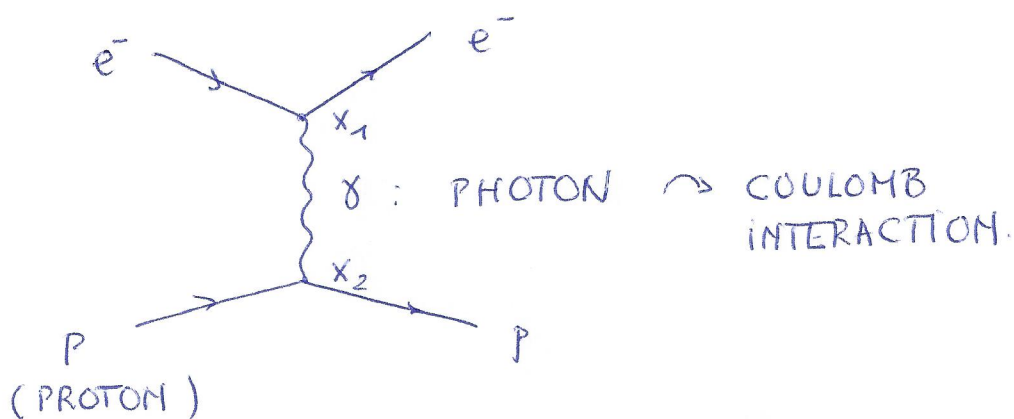
⇒ RELATIVISTIC QUANTUM FIELD THEORY

$\phi(x)$: WILL THEN ALSO BE INTERPRETED AS FIELD OPERATOR WHICH CAN CREATE / ANNIHILATE PARTICLES IN SPACE-TIME POINT x .

IMPORTANCE :

↳ DESCRIPTION OF SCATTERING PROCESSES.

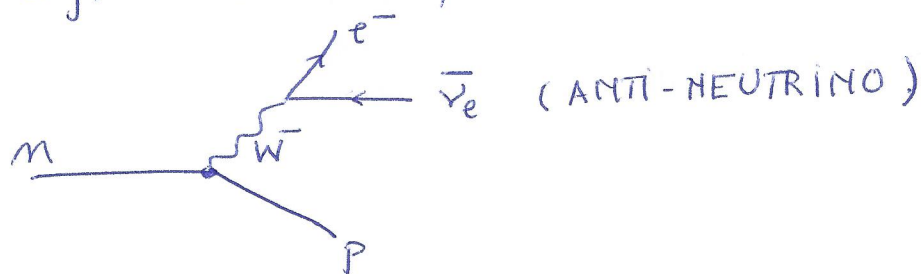
e.g. $e^- p$ SCATTERING



IN MODERN PHYSICS :

INTERACTIONS = EXCHANGE OF "FORCE-CARRYING" PARTICLES

- 1) ELECTROMAGNETIC : PHOTON (MASSLESS)
↳ LONG RANGE
 - 2) WEAK INTERACTIONS : W^+, W^-, Z GAUGE BOSONS (MASSIVE)
↳ SHORT RANGE
- e.g. NEUTRON β -DECAY



3) STRONG INTERACTIONS

MATTER : SPIN $\frac{1}{2}$ QUARKS

UP u $e_u = +\frac{2}{3} e$

DOWN d $e_d = -\frac{1}{3} e$

e : PROTON CHARGE

+ 4 HEAVIER s, c, b, t

- PROTON p CONSISTS OF uud

(BUT ALSO $uud u\bar{u}$, $uud d\bar{d}$,
 $uud u\bar{u} u\bar{u}$, ...).

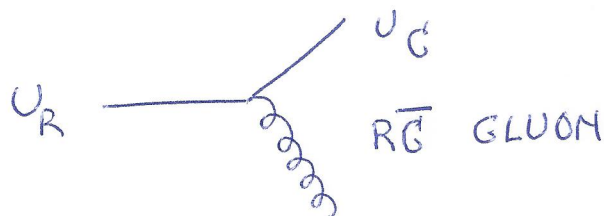
RELATIVISTIC BOUND STATE SYSTEM
STILL NOT SOLVED !

- QUARKS COME IN 3 COLORS (RED, GREEN, BLUE)

u_R, u_G, u_B

FORCE CARRIERS : SPIN 1 MASSLESS GLUONS

COLOR IS CONSERVED



8 GLUONS $R\bar{G}, R\bar{B}, G\bar{R}, G\bar{B}, B\bar{R}, B\bar{G},$
 $R\bar{R} - G\bar{G}, R\bar{R} + G\bar{G} - 2 B\bar{B}$

4) GRAVITATIONAL : GRAVITON (SPIN 2) IS EXCHANGED
BETWEEN 2 MASSES



DESCRIPTION OF QUANTUM CORRECTIONS

e.g. e^- ANOMALOUS MAGNETIC MOMENT

$$\vec{\mu} = \frac{e}{2m} \cdot g \cdot \vec{S} \quad \vec{S} = \frac{\vec{\sigma}}{2} \text{ SPIN VECTOR}$$

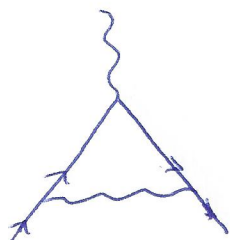
DIRAC THEORY g -FACTOR $g = 2$

QUANTUM FIELD THEORY LEADS TO CORRECTIONS

$$g = 2 \left(1 + \underset{\uparrow}{\alpha} \right)$$

ANOMALOUS MAGN. MOMENT

- 1st ORDER CORRECTION TO e^- α



1 LOOP DIAGRAM

$$\alpha^{(1)} = \frac{\alpha}{2\pi} \quad (\text{SCHWINGER}) \quad 1948$$

$$\approx 0.00116$$

$$\approx 1 \cdot 10^{-3}$$

$$\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}$$

FINE-STRUCTURE
CONSTANT

- α IS CALCULATED NOWADAYS UP TO
5th ORDER α^5 (5 LOOPS !)

THEORY PRECISION ON α : $\sim 5 \cdot 10^{-10}$!

EXP. " " : $\sim 6 \cdot 10^{-10}$

2) ELEMENTS OF CLASSICAL FIELD THEORY ^{KC 9}

$\Rightarrow$ LAGRANGIAN FIELD THEORY

$\hookrightarrow$ SYSTEM SPECIFIED BY FIELDS $\Phi_\kappa(x)$

x DENOTES $x^\mu (t, \vec{x})$

κ DENOTES COMPONENTS OF FIELD

$\hookrightarrow$ SCALAR FIELD ($\kappa = 1$) $\Phi(x)$

e.g. TEMPERATURE AT GIVEN SPACE-TIME POINT

$\hookrightarrow$ VECTOR FIELD ($\kappa = 1, 2, 3, 4$) $A^\mu(x)$

e.g. ELECTROMAGNETIC 4-VECTOR POTENTIAL

$\hookrightarrow$ DYNAMICS DESCRIBED BY LAGRANGIAN DENSITY

$$\mathcal{L}(\Phi_\kappa, \partial_\mu \Phi_\kappa)$$

LAGRANGIAN $L \equiv \int d^3\vec{x} \mathcal{L}$

THROUGH VARIATIONAL PRINCIPLE

ACTION $S = \int_{-\infty}^{\infty} d^4x \mathcal{L}(\Phi_\kappa, \partial_\mu \Phi_\kappa).$

CONSIDER VARIATIONS $\delta\phi_\kappa$ OF ϕ_κ

SUCH THAT $\delta\phi_\kappa = 0$ ON BOUNDARY $\Gamma(\Omega)$

REQUIRE THAT S IS MINIMIZED $\delta S = 0$



FIELD EQUATIONS (EULER-LAGRANGE EQUATIONS)

FOR ϕ_κ

UNDER VARIATIONS $\delta\phi_\kappa$

$$\delta S = \int_{\Omega} d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_\kappa} \delta\phi_\kappa + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_\kappa)} \delta(\partial_\mu \phi_\kappa) \right\}$$

$$= \int_{\Omega} d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_\kappa} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_\kappa)} \right) \right\} (\delta\phi_\kappa)$$

$$+ \underbrace{\int_{\Omega} d^4x \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_\kappa)} \delta\phi_\kappa \right)}_{\text{BOUNDARY TERM}}$$

BOUNDARY TERM

VANISHES BECAUSE $\delta\phi_\kappa = 0$ ON $\Gamma(\Omega)$

∴ BY REQUIRING THAT

$\delta S = 0$ FOR ARBITRARY $\delta\phi_\kappa$



$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi_\kappa} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_\kappa)} \right) = 0}$$

EULER -
LAGRANGE
EQUATIONS



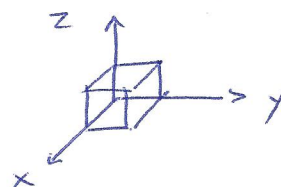
HAMILTONIAN FIELD THEORY

↳ CONJUGATE MOMENTA :

- $\phi_{\kappa}(t, \vec{x})$ FIELD

DIVIDE SPACE IN CELLS (DISCRETIZATION)

POSITION OF CELL : $i = 1, 2, 3, 4, \dots$



GENERALIZED COORDINATES

$$q_{\kappa i}(t) \equiv \phi_{\kappa}(t, i) \quad i = 1, 2, 3, 4, \dots$$

- LAGRANGIAN (DISCRETIZED)

$$L = \int d^3\vec{x} \mathcal{L}$$

$$= \sum_i (\delta\vec{x}_i) \cdot \mathcal{L}(\phi_{\kappa}(t, i), \dot{\phi}_{\kappa}(t, i))$$

$\uparrow$
 VOLUME ELEMENT

- CONJUGATE MOMENTA

$$p_{\kappa i}(t) = \frac{\partial L}{\partial \dot{q}_{\kappa i}} = \frac{\partial L}{\partial \dot{\phi}_{\kappa}(t, i)}$$

$$\equiv (\delta\vec{x}_i) \cdot \overline{\Pi}_{\kappa}(t, i)$$

$$\overline{\Pi}_{\kappa}(t, i) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}_{\kappa}(t, i)}$$

↳ HAMILTONIAN

$$H = \sum_r \sum_i p_{ri} \dot{q}_{ri} - L$$

$$= \sum_i (\delta \bar{x}_i) \left\{ \sum_r \pi_r(t, i) \dot{\phi}_r(t, i) - \mathcal{L} \right\}$$

↓ CONTINUUM

$$\equiv \int d^3 \bar{x} \quad \mathcal{H}(\phi_r(x), \pi_r(x))$$

FIELD THEORY : CONTINUUM LIMIT $\delta \bar{x}_i \rightarrow 0$

$$\pi_r(x) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}_r}$$

$$L(t) = \int d^3 \bar{x} \quad \mathcal{L}(\phi_r, \partial_\mu \phi_r)$$

$$H(t) = \int d^3 \bar{x} \quad \mathcal{H}(\phi_r, \pi_r)$$

$$\mathcal{H}(x) = \pi_r(x) \dot{\phi}_r(x) - \mathcal{L}$$

↳ HAMILTONIAN DENSITY

⇒ EXAMPLE : CLASSICAL KLEIN-GORDON FIELD

$$\hookrightarrow \underline{\underline{\mathcal{L}_{KG} = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2}}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi$$

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} = \partial^\mu \phi$$

$$\hookrightarrow \text{E-L EQUATIONS : } (\partial_\mu \partial^\mu + m^2) \phi = 0$$

KLEIN-GORDON EQ.

↳ CONJUGATE MOMENTUM

$$\underline{\underline{\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}}}$$

↳ HAMILTONIAN

$$\underline{\underline{\mathcal{H}_{KG} = \frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2}}$$

⇒ NOETHER'S THEOREM

↳ CONNECTION BETWEEN SYMMETRIES & CONSERVATION LAWS

CONSIDER CONTINUOUS TRANSFORMATION OF FIELDS

$$\Phi_{\alpha}(x) \rightarrow \Phi'_{\alpha}(x) = \Phi_{\alpha}(x) + \delta\Phi_{\alpha}(x)$$

THIS TF. INDUCES A CHANGE $\delta\mathcal{L}$ TO $\mathcal{L}$

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\Phi_{\alpha}} \delta\Phi_{\alpha} + \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\Phi_{\alpha})} \delta(\partial_{\mu}\Phi_{\alpha})$$

↓ E-L EQUATION

$$= \left(\partial_{\mu} \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\Phi_{\alpha})} \right) \delta\Phi_{\alpha} + \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\Phi_{\alpha})} \delta(\partial_{\mu}\Phi_{\alpha})$$

$$= \partial_{\mu} \left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\Phi_{\alpha})} \delta\Phi_{\alpha} \right)$$

THE ABOVE TF. IS SYMMETRY TRANSFORMATION

IF IT LEAVES ACTION INVARIANT $\delta S = 0$

⇕

LAGRANGIAN IS INVARIANT UP TO A 4-DIVERGENCE

$$\mathcal{L} \rightarrow \mathcal{L}' = \mathcal{L} + \partial_{\mu} \mathcal{F}^{\mu}$$

$$\therefore \partial_{\mu} \mathcal{F}^{\mu} = \partial_{\mu} \left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\Phi_{\alpha})} \delta\Phi_{\alpha} \right)$$



$$\underline{\underline{\partial_\mu j^\mu = 0}} \quad \text{FOR} \quad \underline{\underline{j^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_\kappa)} \delta \phi_\kappa - \mathcal{L}^\mu}}$$



CONSERVED CURRENT j^μ



CONSERVED CHARGE (CONSTANT IN TIME)

$$\underline{\underline{Q \equiv \int d^3 \bar{x} \quad j^0(t, \bar{x})}}$$

$$Q = \int d^3 \bar{x} \left\{ \Pi_\kappa(x) \delta \phi_\kappa(x) - \mathcal{L}^0 \right\}$$

$\hookrightarrow$ EXAMPLE : SYMMETRY UNDER SPACE-TIME TRANSLATION

$$x^\mu \rightarrow x'^\mu = x^\mu + a^\mu$$

$\uparrow$
 CONSTANT TRANSLATION

$$\phi'_\kappa(x') = \phi_\kappa(x)$$

IF. FIELD AT SHIFTED POSITION = ORIGINAL FIELD AT ORIGINAL POSITION

$$\delta \phi_\kappa(x) = \phi'_\kappa(x) - \phi_\kappa(x)$$

$$= \phi'_\kappa(x) - \phi'_\kappa(x+a)$$

$$= - (\partial_\mu \phi_\kappa) a^\mu$$

$\mathcal{L}$ IS SCALAR $\Rightarrow$ MUST TRANSFORM IN SAME WAY

$$\underline{\underline{\delta \mathcal{L} = -a^\mu (\partial_\mu \mathcal{L})}}$$

EXPLICIT PROOF: $\mathcal{L} = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2$

$$\phi \rightarrow \phi' = \phi - a^\nu (\partial_\nu \phi)$$

$$\partial_\mu \phi \rightarrow \partial_\mu \phi - a^\nu (\partial_\mu \partial_\nu \phi)$$

$$\mathcal{L} \rightarrow \mathcal{L} - a^\nu (\partial_\mu \partial_\nu \phi) (\partial^\mu \phi)$$

$$+ a^\nu m^2 (\partial_\nu \phi) \phi + O(a^2)$$

$$\delta \mathcal{L} \stackrel{!}{=} -a^\nu (\partial_\nu \mathcal{L})$$

$$\therefore \mathcal{J}^\mu = -a^\mu \mathcal{L}$$

$\hookrightarrow$ CONSERVED CURRENT

$$j^\mu \equiv -(\partial^\mu \phi_\kappa) a_\nu (\partial^\nu \phi_\kappa) + a_\nu g^{\mu\nu} \mathcal{L}$$

$$\begin{aligned} \partial_\mu j^\mu = 0 &\Leftrightarrow \partial_\mu [(\partial^\mu \phi_\kappa) (\partial^\nu \phi_\kappa) - g^{\mu\nu} \mathcal{L}] a_\nu \\ &= 0 \quad \underline{\underline{\forall a_\nu}} \end{aligned}$$



$$\partial_\mu T^{\mu\nu} = 0$$

WITH

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} (\partial^\nu \phi_n) - g^{\mu\nu} \mathcal{L}$$

↳ CONSERVED ENERGY - MOMENTUM TENSOR

$$T^{\mu\nu} = (\partial^\mu \phi_n) (\partial^\nu \phi_n) - g^{\mu\nu} \mathcal{L}$$

↳ CONSERVED CHARGE

4 - MOMENTUM

$$\begin{aligned} P^\nu &\equiv \int d^3\vec{x} \quad T^{0\nu} \\ &= \int d^3\vec{x} \left\{ \pi_n(x) \partial^\nu \phi_n(x) - g^{0\nu} \mathcal{L} \right\} \end{aligned}$$

$$\begin{aligned} P^0 &= \int d^3\vec{x} \left\{ \pi_n(x) \dot{\phi}_n(x) - \mathcal{L} \right\} \\ &= \int d^3\vec{x} \quad \mathcal{H}(x) = H \quad (\text{ENERGY}) \end{aligned}$$

$$\vec{P} = - \int d^3\vec{x} \quad \pi_n(x) \vec{\nabla} \phi_n(x) \quad \begin{array}{l} (3\text{-MOMENTUM}) \\ \text{CARRIED BY} \\ \text{FIELD} \end{array}$$

3) QUANTIZATION OF KLEIN - GORDON FIELD

⇒ CANONICAL COMMUTATION RELATIONS

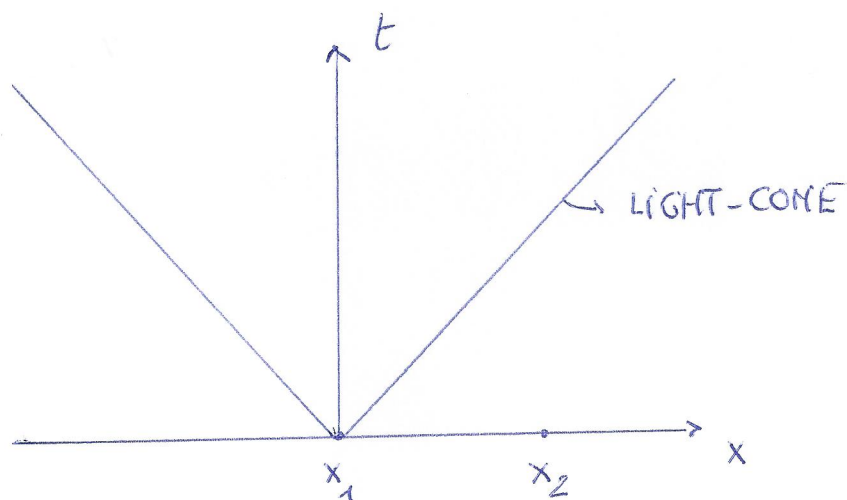
↳ IN QUANTUM THEORY :

WE POSTULATE EQUAL TIME COMMUTATION RELATIONS
(ETCR)
FOR FIELDS & CONJUGATE MOMENTA:

$$[\Phi_{\kappa}(\bar{x}, t), \Phi_{\lambda}(\bar{x}', t)] = 0$$

$$[\pi_{\kappa}(\bar{x}, t), \pi_{\lambda}(\bar{x}', t)] = 0$$

$$[\Phi_{\kappa}(\bar{x}, t), \pi_{\lambda}(\bar{x}', t)] = i\delta_{\kappa\lambda} \delta^3(\bar{x} - \bar{x}')$$



FIELDS AT SPACELIKE SEPARATED
POINTS $(x_1 - x_2)^2 < 0$

CANNOT INFLUENCE EACH OTHER

(OUTSIDE EACH OTHER'S LIGHT CONE)

↪ COMMUTATORS VANISH

↳ FOR REAL KLEIN-GORDON FIELD

(DESCRIBING NEUTRAL SPIN=0 PARTICLE)

$$\left\{ \begin{aligned} [\phi(\bar{x}, t), \phi(\bar{x}', t)] &= 0 \\ [\dot{\phi}(\bar{x}, t), \dot{\phi}(\bar{x}', t)] &= 0 \\ [\phi(\bar{x}, t), \dot{\phi}(\bar{x}', t)] &= i \delta^3(\bar{x} - \bar{x}') \end{aligned} \right.$$

⇒ NORMAL MODE EXPANSION FOR FIELD ϕ ($\phi = \phi^\dagger$)

$$\hookrightarrow \phi(\bar{x}, t) = \int \frac{d^3 \bar{k}}{(2\pi)^3} N(\bar{k}) \left\{ a(\bar{k}) e^{-ik \cdot x} + a^\dagger(\bar{k}) e^{+ik \cdot x} \right\}$$

NORMAL MODE $\bar{k}$ (CONTINUOUS MOMENTUM)

SOLUTIONS OF K-G EQUATION: $k^2 = m^2$

⇓

$$k^0 = \sqrt{\bar{k}^2 + m^2} \equiv E_{\bar{k}}$$

$a(\bar{k})$: ANNIHILATES A PARTICLE OF MOM. $\bar{k}$ & ENERGY $E_{\bar{k}}$

$a^\dagger(\bar{k})$: CREATES A PARTICLE OF MOM. $\bar{k}$ & ENERGY $E_{\bar{k}}$

$N(\bar{k})$: CONVENIENTLY CHOSEN NORMALIZATION FACTOR

$$\hookrightarrow i \dot{\Phi}(\vec{x}, t) = \int \frac{d^3 \vec{k}}{(2\pi)^3} N(\vec{k}) E_{\vec{k}} \left\{ a(\vec{k}) e^{-i\vec{k} \cdot \vec{x}} - a^\dagger(\vec{k}) e^{+i\vec{k} \cdot \vec{x}} \right\}$$

$$\begin{aligned} \hookrightarrow \int d^3 \vec{x} e^{-i\vec{k}' \cdot \vec{x}} \Phi(\vec{x}, t) \\ = \int \frac{d^3 \vec{k}}{(2\pi)^3} N(\vec{k}) \left\{ a(\vec{k}) e^{-iE_{\vec{k}} t} \underbrace{\int d^3 \vec{x} e^{i(\vec{k}-\vec{k}') \cdot \vec{x}}}_{(2\pi)^3 \delta^3(\vec{k}-\vec{k}')} \right. \\ \left. + a^\dagger(\vec{k}) e^{+iE_{\vec{k}} t} \underbrace{\int d^3 \vec{x} e^{-i(\vec{k}+\vec{k}') \cdot \vec{x}}}_{(2\pi)^3 \delta^3(\vec{k}+\vec{k}')} \right\} \end{aligned}$$

$$= N(\vec{k}') a(\vec{k}') e^{-iE_{\vec{k}'} t} + N(-\vec{k}') a^\dagger(-\vec{k}') e^{+iE_{-\vec{k}'} t}$$

↓

$$E_{-\vec{k}'} = E_{\vec{k}'}$$

CHOOSE NORMALIZATION

SUCH THAT $N(-\vec{k}') = N(\vec{k}')$

$$= N(\vec{k}') \left\{ a(\vec{k}') e^{-iE_{\vec{k}'} t} + a^\dagger(-\vec{k}') e^{+iE_{\vec{k}'} t} \right\}$$

↪ ANALOGOUSLY

$$\begin{aligned} \int d^3 \vec{x} e^{-i\vec{k}' \cdot \vec{x}} i \dot{\Phi}(\vec{x}, t) \\ = N(\vec{k}') E_{\vec{k}'} \left\{ a(\vec{k}') e^{-iE_{\vec{k}'} t} - a^\dagger(-\vec{k}') e^{+iE_{\vec{k}'} t} \right\} \end{aligned}$$

$$\hookrightarrow \int d^3 \vec{x} e^{-i \vec{k} \cdot \vec{x}} \left\{ E_{\vec{k}} \phi + i \dot{\phi} \right\}$$

$$= N(\vec{k}) 2 E_{\vec{k}} a(\vec{k}) e^{-i E_{\vec{k}} t}$$

$$\int d^3 \vec{x} e^{-i \vec{k} \cdot \vec{x}} \left\{ E_{\vec{k}} \phi - i \dot{\phi} \right\}$$

$$= N(\vec{k}) 2 E_{\vec{k}} a^+(-\vec{k}) e^{+i E_{\vec{k}} t}$$

$\Downarrow$ IN 2^o EQUATION WE CHANGE $\vec{k} \rightarrow -\vec{k}$

$$a(\vec{k}) = \frac{1}{2 E_{\vec{k}} N(\vec{k})} \int d^3 \vec{x} e^{i \vec{k} \cdot \vec{x}} \left\{ E_{\vec{k}} \phi + i \dot{\phi} \right\}$$

WITH $k^0 = E_{\vec{k}}$

$$a^+(\vec{k}) = \frac{1}{2 E_{\vec{k}} N(\vec{k})} \int d^3 \vec{x} e^{-i \vec{k} \cdot \vec{x}} \left\{ E_{\vec{k}} \phi - i \dot{\phi} \right\}$$

WITH $k^0 = E_{\vec{k}}$

$\Rightarrow$ COMMUTATION RELATIONS FOR a, a^+

$$\hookrightarrow [a(\vec{k}), a^+(\vec{k}')] = a(\vec{k})a^+(\vec{k}') - a^+(\vec{k}')a(\vec{k})$$

$$= \frac{1}{2E_{\vec{k}} N(\vec{k})} \frac{1}{2E_{\vec{k}'} N(\vec{k}')} \cdot \int d^3\vec{x} \int d^3\vec{x}' e^{i\vec{k}\cdot\vec{x}} e^{-i\vec{k}'\cdot\vec{x}'} \quad \text{WITH } \underline{x^0 = x^0}$$

$$\cdot \int d^3\vec{x} \int d^3\vec{x}' e^{i\vec{k}\cdot\vec{x}} e^{-i\vec{k}'\cdot\vec{x}'} \quad \text{WITH } \underline{x^0 = x^0}$$

$$\left[E_{\vec{k}} \phi(\vec{x}, t) + i\dot{\phi}(\vec{x}, t), \right. \\ \left. E_{\vec{k}'} \phi(\vec{x}', t) - i\dot{\phi}(\vec{x}', t) \right]$$

APPLY ETCR's

$$\text{ONLY } [\phi(\vec{x}, t), \dot{\phi}(\vec{x}', t)]$$

IS NON-VANISHING

$$E_{\vec{k}} \delta^3(\vec{x} - \vec{x}') + E_{\vec{k}'} \delta^3(\vec{x} - \vec{x}')$$

$$= \frac{1}{2E_{\vec{k}} N(\vec{k})} \frac{1}{2E_{\vec{k}'} N(\vec{k}')} \cdot \int d^3\vec{x} e^{i(E_{\vec{k}} - E_{\vec{k}'})t} \cdot (E_{\vec{k}} + E_{\vec{k}'}) e^{-i(\vec{k} - \vec{k}')\cdot\vec{x}}$$

$$\cdot \int d^3\vec{x} e^{i(E_{\vec{k}} - E_{\vec{k}'})t} \cdot (E_{\vec{k}} + E_{\vec{k}'}) e^{-i(\vec{k} - \vec{k}')\cdot\vec{x}}$$

$$\downarrow \\ (2\pi)^3 \delta^3(\vec{k} - \vec{k}')$$

$$[a(\vec{k}), a^\dagger(\vec{k}')] = \frac{1}{2E_{\vec{k}}(N(\vec{k}))^2} (2\pi)^3 \delta^3(\vec{k} - \vec{k}')$$

WE CAN CHOOSE (COVARIANT NORMALIZATION CONVENTION)

$$N(\vec{k}) = \frac{1}{\sqrt{2E_{\vec{k}}}}$$

$$[a(\vec{k}), a^\dagger(\vec{k}')] = (2\pi)^3 \delta^3(\vec{k} - \vec{k}')$$

$$[a(\vec{k}), a(\vec{k}')] = 0$$

$$[a^\dagger(\vec{k}), a^\dagger(\vec{k}')] = 0$$

ANALOGOUS RELATIONS
AS FOR HARMONIC OSCILLATOR

$\Rightarrow$ HAMILTONIAN IN TERMS OF a, a^+

$$\hookrightarrow \mathcal{H} = \frac{1}{2} \left\{ \dot{\phi}^2 + (\bar{\nabla} \phi)^2 + m^2 \phi^2 \right\}$$

$$H = \int d^3 \bar{x} \mathcal{H}(\bar{x}, t)$$

$\Downarrow$

SHOW THAT (EXERCISE!)

$$H = \int \frac{d^3 \bar{k}}{(2\pi)^3} E_{\bar{k}} \left(a^+(\bar{k}) a(\bar{k}) + \frac{\int d^3 \phi(0)}{2} \right)$$

$a^+(\bar{k}) a(\bar{k})$ is NUMBER OPERATOR

PARTICLES WITH MOMENTUM $\bar{k}$

$\Rightarrow$ MOMENTUM IN TERMS OF a, a^+

$$\bar{P} = - \int d^3 \bar{x} \phi (\bar{\nabla} \phi)$$

$\Downarrow$

SHOW THAT (EXERCISE!)

$$\bar{P} = \int \frac{d^3 \bar{k}}{(2\pi)^3} \bar{k} a^+(\bar{k}) a(\bar{k})$$



SINGLE PARTICLE STATES

↳ VACUUM $|0\rangle$

NO PARTICLES.

$$a(\vec{k})|0\rangle = 0$$

$$\langle 0|0\rangle = 1$$

NORMALIZATION

↳ SINGLE PARTICLE STATE

$$|\vec{k}\rangle \equiv \sqrt{2E_{\vec{k}}} a^{\dagger}(\vec{k}) |0\rangle$$

$$E_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$$

CREATES ONE PARTICLE
OF MOMENTUM $\vec{k}$
IN VACUUM

NORMALIZATION CHOSEN SUCH THAT

$$\langle \vec{k}' | \vec{k} \rangle = \sqrt{2E_{\vec{k}'}} \sqrt{2E_{\vec{k}}} \langle 0 | a(\vec{k}') a^{\dagger}(\vec{k}) | 0 \rangle$$

$$= \sqrt{2E_{\vec{k}'}} \sqrt{2E_{\vec{k}}} \langle 0 | [a(\vec{k}'), a^{\dagger}(\vec{k})] | 0 \rangle$$

BECAUSE $a(\vec{k}')|0\rangle = 0$

$$\underline{\underline{\langle \vec{k}' | \vec{k} \rangle = (2\pi)^3 2E_{\vec{k}} \delta^3(\vec{k} - \vec{k}')}}}$$

$$\hookrightarrow \phi(\vec{x}, t) |0\rangle$$

$$= \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{k}}}} \left\{ \cancel{a(\vec{k})} e^{-ik \cdot x} + a^\dagger(\vec{k}) e^{+ik \cdot x} \right\} |0\rangle$$

0

$$= \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{e^{ik \cdot x}}{2E_{\vec{k}}} |\vec{k}\rangle$$

CREATES ONE-PARTICLE STATE AT POSITION $\vec{x}$
AND TIME t FROM VACUUM

$\hookrightarrow$ WAVE FUNCTION INTERPRETATION AS IN QM
 $\langle \vec{x} | \vec{k} \rangle \sim e^{i\vec{k} \cdot \vec{x}}$

$$\therefore \left\{ \begin{array}{l} \langle \vec{k} | \phi(\vec{x}, t) | 0 \rangle = e^{ik \cdot x} \\ \langle 0 | \phi(\vec{x}, t) | \vec{k} \rangle = e^{-ik \cdot x} \end{array} \right.$$

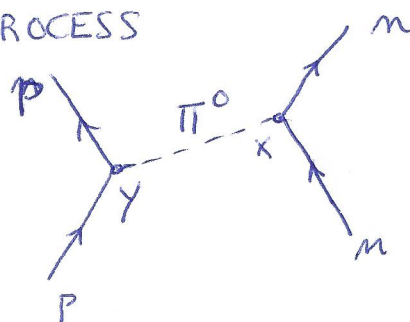
ANNIHILATES A PARTICLE
AT POSITION $\vec{x}$ & TIME t
FROM SINGLE PARTICLE STATE $|\vec{k}\rangle$

4) FEYNMAN PROPAGATOR FOR KLEIN-GORDON FIELD

⇒ 2-POINT CORRELATION FUNCTION

↳ AMPLITUDE FOR A PARTICLE TO BE CREATED AT POINT y AND ANNIHILATED AT POINT x AT DIFFERENT $x_0 \neq y_0$

e.g. IN PROCESS



π^0 : NEUTRAL PION (SPIN 0)

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle$$

ONLY $a^\dagger(\vec{k})$ CONTRIBUTES IN $\phi(y)$

↓ " $a(\vec{k})$ CONTRIBUTES IN $\phi(x)$

$$= \langle 0 | \int \frac{d^3 \vec{k}}{(2\pi)^3 \sqrt{2E_{\vec{k}}}} \int \frac{d^3 \vec{k}'}{(2\pi)^3 \sqrt{2E_{\vec{k}'}}} e^{-i\vec{k} \cdot \vec{x}} e^{+i\vec{k}' \cdot \vec{y}} a(\vec{k}) a^\dagger(\vec{k}') | 0 \rangle$$

$$\downarrow \quad \langle 0 | a(\vec{k}) a^\dagger(\vec{k}') | 0 \rangle = (2\pi)^3 \delta^3(\vec{k} - \vec{k}')$$

$$= \int \frac{d^3 \vec{k}}{(2\pi)^3 2E_{\vec{k}}} e^{-i\vec{k} \cdot (\vec{x} - \vec{y})} \equiv i\Delta(\vec{x} - \vec{y})$$

$$\text{WITH } E_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$$

↳ NOTE $\langle 0 | \phi(x) \phi(y) | 0 \rangle \equiv i \Delta(x-y)$

IS ONLY A FUNCTION OF DIFFERENCE $x-y$
(TRANSLATIONAL INVARIANCE)

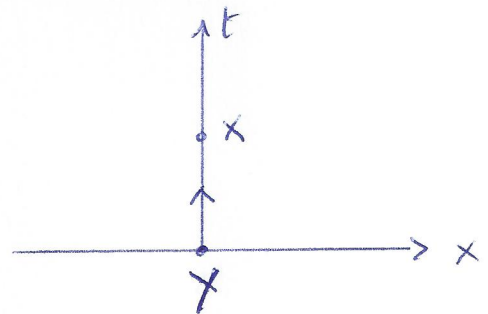
↳ SPECIAL CASES

① $x^0 - y^0 = t, \quad \vec{x} = \vec{y}$: PARTICLE AT REST

$$i \Delta(x-y)$$

$$= \int \frac{d^3 \vec{k}}{(2\pi)^3 2E_{\vec{k}}} e^{-iE_{\vec{k}} t}$$

$$= \frac{1}{4\pi^2} \int_0^\infty d|\vec{k}| \frac{|\vec{k}|^2}{E_{\vec{k}}} e^{-iE_{\vec{k}} t}$$



$$E_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$$

$$\frac{dE_{\vec{k}}}{d|\vec{k}|} = \frac{|\vec{k}|}{E_{\vec{k}}}$$

$$= \frac{1}{4\pi^2} \int_m^\infty dE \sqrt{E^2 - m^2} e^{-iEt}$$

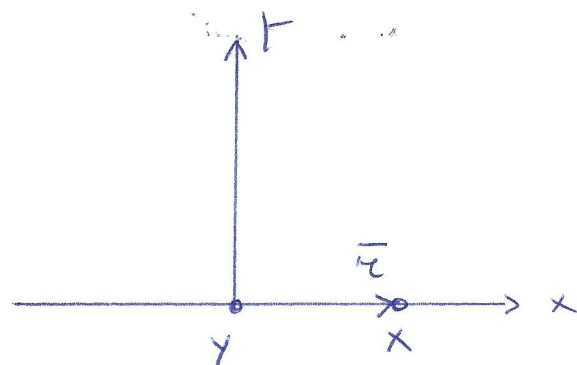
→
 $t \rightarrow \infty$

$$e^{-i m t}$$

⇒ PARTICLE AT REST WITH MASS m
PROPAGATING IN TIME

② $x^0 = y^0$, $\vec{x} - \vec{y} = \vec{r}$: PURE SPATIAL SEPARATION

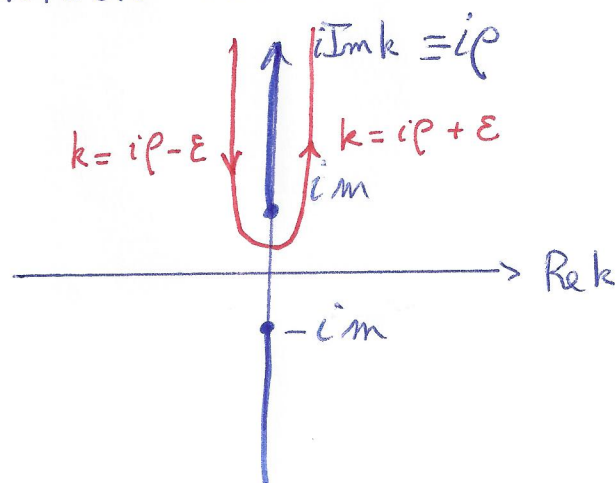
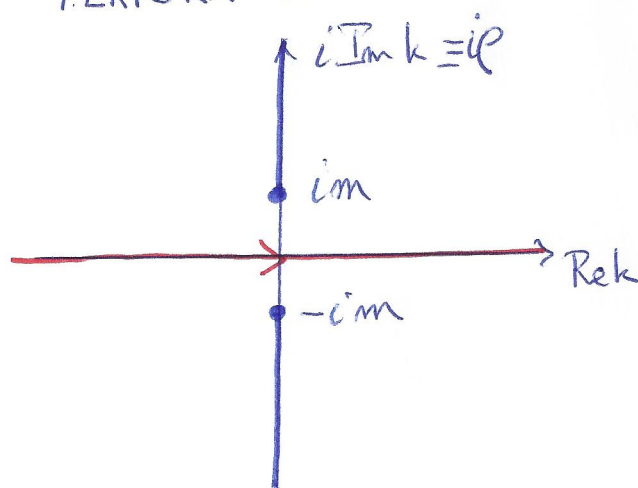
$$\begin{aligned}
 & i \Delta(x-y) \\
 &= \int \frac{d^3 \vec{k}}{(2\pi)^3 2E_k} e^{i\vec{k} \cdot \vec{r}} \\
 & \quad \downarrow k \equiv |\vec{k}| \\
 &= \frac{1}{4\pi^2} \int_0^\infty dk \frac{k^2}{2E_k} \int_{-1}^1 d\cos\theta e^{ikr\cos\theta} \\
 &= \frac{1}{4\pi^2} \frac{1}{i r} \int_0^\infty dk \frac{k}{2E_k} \left(e^{ikr} - e^{-ikr} \right) \\
 &= \frac{1}{8\pi^2 i r} \int_{-\infty}^{+\infty} dk \frac{k}{\sqrt{k^2 + m^2}} e^{ikr}
 \end{aligned}$$



$\frac{1}{\sqrt{k^2 + m^2}}$ HAS BRANCH CUTS IN COMPLEX k -PLANE

FOR $k > im$ AND $k < -im$

PERFORM INTEGRAL BY CONTOUR DEFORMATION



$$i \Delta(x-y) \Big|_{x^0=y^0, \bar{x}-\bar{y}=\bar{\tau}}$$

$$= \frac{1}{8\pi^2 i \tau} \left\{ i^2 \int_{-\infty}^m d\rho \frac{\rho e^{-\rho \tau}}{\sqrt{m^2 - \rho^2 - i\varepsilon \rho}} + i^2 \int_m^{\infty} d\rho \frac{\rho e^{-\rho \tau}}{\sqrt{m^2 - \rho^2 + i\varepsilon \rho}} \right\}$$

ε INFINITESIMAL

$$\sqrt{m^2 - \rho^2 - i\varepsilon \rho} = \sqrt{\rho^2 - m^2} \underbrace{e^{-i\frac{\pi}{2}}}_{-i}$$

$$\sqrt{m^2 - \rho^2 + i\varepsilon \rho} = \sqrt{\rho^2 - m^2} \underbrace{e^{+i\frac{\pi}{2}}}_{i}$$

$$= \frac{1}{4\pi^2 \tau} \int_m^{\infty} d\rho \frac{\rho e^{-\rho \tau}}{\sqrt{\rho^2 - m^2}} \xrightarrow{\tau \rightarrow \infty} \frac{e^{-m\tau}}{\tau}$$

∴ WE SEE THAT OUTSIDE LIGHT-CONE
CORRELATION IS EXPONENTIALLY VANISHING

$\Rightarrow$ TIME-ORDERED PRODUCT

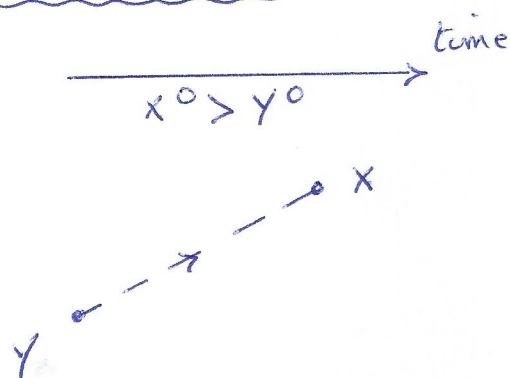
$$T \phi(x) \phi(y) \equiv \theta(x^0 - y^0) \phi(x) \phi(y) + \theta(y^0 - x^0) \phi(y) \phi(x)$$

$$\text{WITH } \theta(x) = \begin{cases} 1 & , x > 0 \\ 0 & , x < 0 \end{cases}$$

TIME-ORDERED PRODUCT :

FIELD WITH LARGEST TIME ARGUMENT STANDS ON LEFT

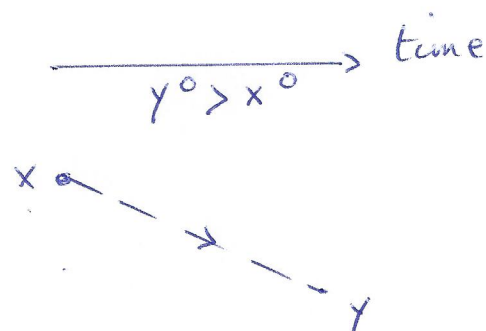
SCHEMATICALLY :



INTERPRETATION :

PARTICLE PROPAGATING
FROM y TO x

(FORWARD IN TIME)



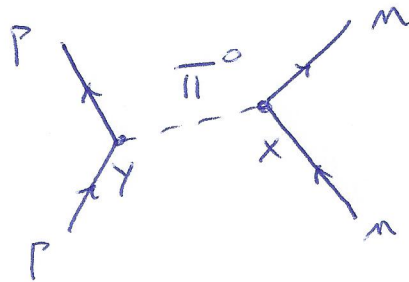
PARTICLE PROPAGATING
FROM x TO y

⇒ FEYNMAN PROPAGATOR FUNCTION

$$\hookrightarrow \langle 0 | T \phi(x) \phi(y) | 0 \rangle \equiv i \Delta_F(x-y)$$

THIS FUNCTION WILL ENTER PHYSICAL PROCESSES.

→ FEYNMAN DIAGRAMS



$$\begin{aligned} i \Delta_F(x-y) &= \theta(x^0 - y^0) \langle 0 | \phi(x) \phi(y) | 0 \rangle \\ &\quad + \theta(y^0 - x^0) \langle 0 | \phi(y) \phi(x) | 0 \rangle \\ &= \theta(x^0 - y^0) i \Delta(x-y) \\ &\quad + \theta(y^0 - x^0) i \Delta(y-x). \end{aligned}$$

$$i \Delta_F(x-y) = \int \frac{d^3 \vec{k}}{(2\pi)^3 2E_{\vec{k}}} \left\{ \theta(x^0 - y^0) e^{-ik \cdot (x-y)} + \theta(y^0 - x^0) e^{+ik \cdot (x-y)} \right\}$$

$$\text{WITH } k^0 = E_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$$

↳ PROPAGATOR IN MOMENTUM SPACE

$$\Delta_F(x) \equiv \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot x} \Delta_F(k^2)$$

WITH $k^2 = k_\mu k^\mu$

FOURIER TRANSFORM $\Delta_F(k^2)$ IS GIVEN BY

$$\Delta_F(k^2) = \frac{1}{k^2 - m^2 + i\varepsilon}$$

$\varepsilon > 0$
INFINITESIMAL

PROOF: BY CONTOUR DEFORMATION IN COMPLEX k^0 PLANE

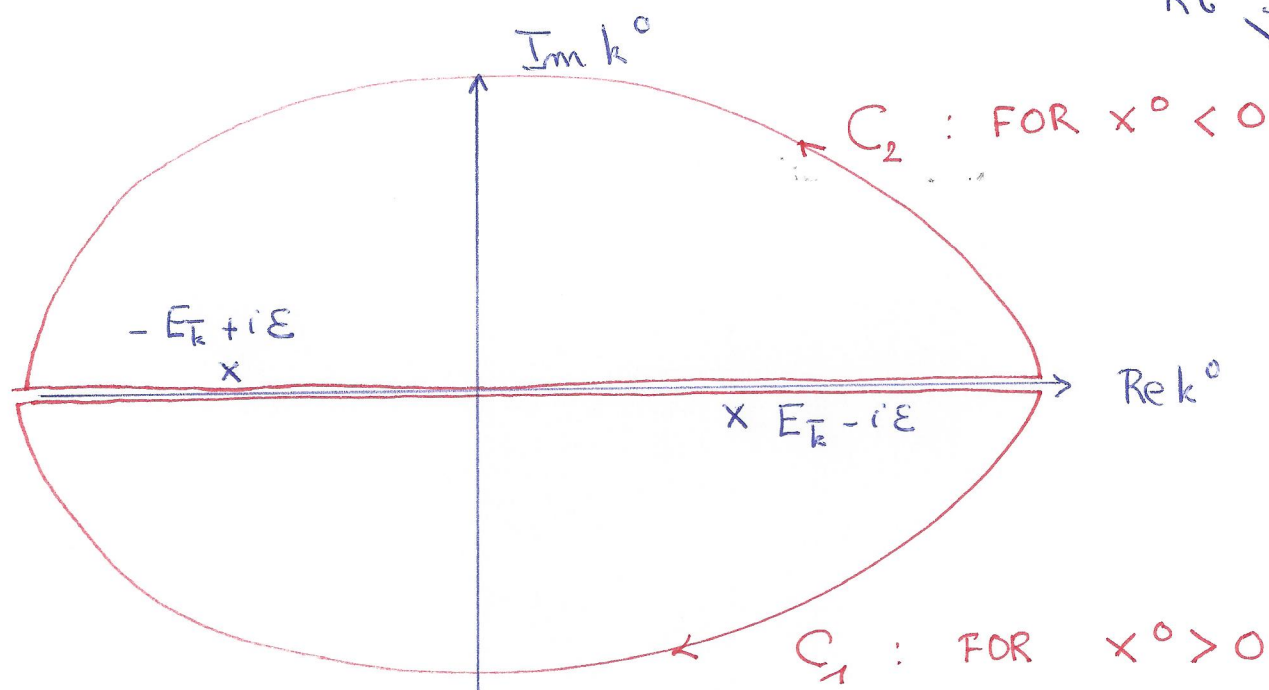
$$\begin{aligned} & k^2 - m^2 + i\varepsilon \\ &= (k^0)^2 - (\vec{k}^2 + m^2) + i\varepsilon \\ &= (k^0)^2 - E_{\vec{k}}^2 + i\varepsilon \end{aligned} \quad \begin{aligned} & \rightarrow E_{\vec{k}} = \sqrt{\vec{k}^2 + m^2} > 0 \\ & \nearrow \text{FOR } \varepsilon \text{ INFINITESIMAL} \end{aligned}$$

$$= (k^0 - E_{\vec{k}} + i\varepsilon)(k^0 + E_{\vec{k}} - i\varepsilon)$$

$\Delta_F(k^2)$ HAS 2 POLES IN COMPLEX k^0 PLANE

$$\rightarrow k^0 = E_{\vec{k}} - i\varepsilon$$

$$\rightarrow k^0 = -E_{\vec{k}} + i\varepsilon$$



- FOR $x^0 > 0$

CLOSE CONTOUR IN LOWER HALF-PLANE

$e^{-ik^0 x^0}$ REQUIRES $\text{Im } k^0 < 0$ IN ORDER TO NEGLECT SEMI-CIRCLE AT INFINITY

$$\underline{x^0 > 0}: \Delta_F(x) = \int \frac{d^3 \vec{k}}{(2\pi)^4} e^{i\vec{k} \cdot \vec{x}} \int_{-\infty}^{+\infty} dk^0 \frac{e^{-ik^0 x^0}}{(k^0 - E_{\vec{k}} + i\epsilon)(k^0 + E_{\vec{k}} - i\epsilon)}$$

$$= \int \frac{d^3 \vec{k}}{(2\pi)^4} e^{i\vec{k} \cdot \vec{x}} \int dk^0 \frac{e^{-ik^0 x^0}}{(k^0 - E_{\vec{k}} + i\epsilon)(k^0 + E_{\vec{k}} - i\epsilon)}$$

$\curvearrowright C_1$

RESIDUE THEOREM ↓

$$- 2\pi i \frac{1}{2E_{\vec{k}}} e^{-iE_{\vec{k}} x^0}$$

$$\underline{x^0 > 0} : \Delta_F(x) = -i \int \frac{d^3 \vec{k}}{(2\pi)^3 2E_{\vec{k}}} e^{-ik \cdot x} \Big|_{k^0 = E_{\vec{k}}}$$

• FOR $x^0 < 0$

CLOSE CONTOUR IN UPPER HALF-PLANE

$$e^{-ik^0 x^0} \quad \text{Im } k^0 > 0$$

$$x^0 < 0 : \Delta_F(x) = \int \frac{d^3 \vec{k}}{(2\pi)^4} e^{i\vec{k} \cdot \vec{x}} \int dk^0 \frac{e^{-ik^0 x^0}}{(k^0 - E_{\vec{k}} + i\varepsilon)(k^0 + E_{\vec{k}} - i\varepsilon)}$$



$$+ 2\pi i \frac{1}{(-2E_{\vec{k}})} e^{iE_{\vec{k}} x^0}$$

$$= -i \int \frac{d^3 \vec{k}}{(2\pi)^3 2E_{\vec{k}}} e^{iE_{\vec{k}} x^0 + i\vec{k} \cdot \vec{x}}$$

↓ CHANGE INTEGRATION VARIABLE
 $\vec{k} \rightarrow -\vec{k}$

$$= -i \int \frac{d^3 \vec{k}}{(2\pi)^3 2E_{\vec{k}}} e^{iE_{\vec{k}} x^0 - i\vec{k} \cdot \vec{x}}$$

$$\underline{x^0 < 0} : \Delta_F(x) = -i \int \frac{d^3 \vec{k}}{(2\pi)^3 2E_{\vec{k}}} e^{ik \cdot x} \Big|_{k^0 = E_{\vec{k}}}$$

• IN TOTAL

$$i \Delta_F(x) = \int \frac{d^3 \vec{k}}{(2\pi)^3 2E_{\vec{k}}} \left\{ \begin{array}{l} \theta(x^0) e^{-i k \cdot x} \\ + \theta(-x^0) e^{+i k \cdot x} \end{array} \right\}_{k^0 = E_{\vec{k}}}$$

THIS AGREES WITH EXPRESSION OBTAINED ABOVE

WHICH COMPLETES PROOF THAT

4D FOURIER TF IS GIVEN BY $\Delta_F(k^2) = \frac{1}{k^2 - m^2 + i\epsilon}$

↳ FEYNMAN PROPAGATOR AS GREEN'S FUNCTION

$$\Delta_F(x) = \int \frac{d^4 k}{(2\pi)^4} e^{-i k \cdot x} \frac{1}{k^2 - m^2 + i\epsilon}$$

$$(\square_x + m^2) \Delta_F(x) = \int \frac{d^4 k}{(2\pi)^4} e^{-i k \cdot x} \frac{(-k^2 + m^2)}{k^2 - m^2 + i\epsilon}$$

$$= - \int \frac{d^4 k}{(2\pi)^4} e^{-i k \cdot x}$$

$$= - \delta^4(x)$$

$$\boxed{(\square_x + m^2) \Delta_F(x) = - \delta^4(x)}$$

↓
GREEN'S FUNCTION OF KG EQUATION

↓
SOLUTION OF FIELD EQUATION
WITH $-\delta^4(x)$ SOURCE TERM