

IV

INTERACTING FIELDS AND FEYNMAN DIAGRAMS

- 1) S-MATRIX EXPANSION
- 2) WICK'S THEOREM
- 3) FEYNMAN DIAGRAMS & FEYNMAN RULES
FOR ϕ^4 THEORY
- 4) FEYNMAN DIAGRAMS & RULES FOR QED

1) S - MATRIX EXPANSION

⇒ TIME EVOLUTION : SCHRÖDINGER VS INTERACTION PICTURE

• SCHRÖDINGER PICTURE

$$\hat{H} |\Psi(t)\rangle_S = i \frac{d}{dt} |\Psi(t)\rangle_S \quad \hookrightarrow \text{SCHRÖDINGER}$$

IN SCHRÖDINGER PICTURE :

↳ ALL TIME DEPENDENCE IN STATE VECTOR $|\Psi(t)\rangle$

↳ OPERATORS (e.g. $\hat{H}$) : TIME INDEPENDENT

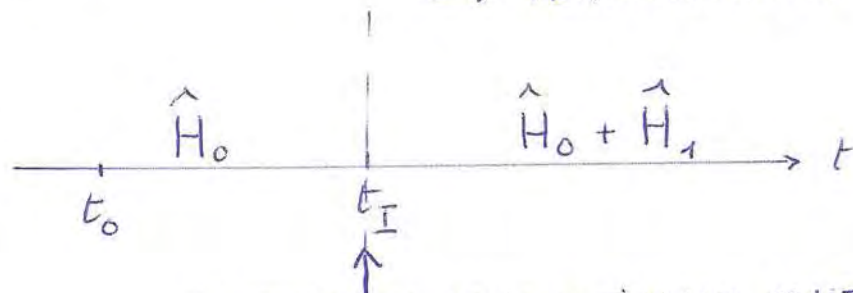
$$|\Psi(t)\rangle_S = e^{-i\hat{H}(t-t_0)} |\Psi(t_0)\rangle_S$$

$$= \underbrace{U(t, t_0)}_{\text{UNITARY TIME EVOLUTION OPERATOR}} |\Psi(t_0)\rangle_S$$

UNITARY TIME EVOLUTION OPERATOR

• INTERACTION PICTURE

↳ FOR $\hat{H} = \hat{H}_0 + \hat{H}_1$ $\hookrightarrow$ PERTURBATION



FOR $t < t_I$: $|\Psi(t)\rangle_S = e^{-i\hat{H}_0(t-t_0)} |\Psi(t_0)\rangle_S$

$\hookrightarrow$ WE CAN TAKE TIME DEPENDENCE DUE TO $\hat{H}_0$
 OUT OF STATE BY DEFINING A NEW STATE
 $\forall t$ (ALSO $t > t_I$) THROUGH A UNITARY TF.

$$\begin{aligned}
 |\Psi(t)\rangle_I &\equiv U_0^\dagger |\Psi(t)\rangle_S \\
 &\equiv e^{i\hat{H}_0(t-t_0)} |\Psi(t)\rangle_S
 \end{aligned}$$

WITH $|\Psi(t_0)\rangle_I = |\Psi(t_0)\rangle_S$
 $\hookrightarrow$ INTERACTION PICTURE

$\hookrightarrow$ NOTE: FOR $\hat{H} = \hat{H}_0$ (i.e. $\hat{H}_1 = 0$)

$$|\Psi(t)\rangle_I = |\Psi(t_0)\rangle_S = |\Psi(t_0)\rangle_I$$

i.e. IN ABSENCE OF INTERACTION

$|\Psi(t)\rangle_I$ IS CONSTANT IN TIME

- EQUIVALENCE OF MATRIX ELEMENTS

$${}_S \langle \Psi(t) | \hat{O}^S | \Psi(t) \rangle_S = {}_I \langle \Psi(t) | \hat{O}^I(t) | \Psi(t) \rangle_I$$

MATRIX ELEMENT SHOULD BE INVARIANT
UNDER UNITARY TRANSFORMATION

$\hat{O}^S$: TIME-INDEPENDENT OPERATOR IN SCHRÖDINGER
PICTURE

$\hat{O}^I(t)$: TIME-DEPENDENT OPERATOR IN INTERACTION PICTURE

USING : $|\Psi(t)\rangle_S = U_0 |\Psi(t)\rangle_I$

$${}_S \langle \Psi(t) | \hat{O}^S | \Psi(t) \rangle_S = {}_I \langle \Psi(t) | U_0^\dagger \hat{O}^S U_0 | \Psi(t) \rangle_I$$

$\Downarrow$

$$\hat{O}^I(t) \equiv U_0^\dagger \hat{O}^S U_0$$

- TIME DEPENDENCE OF OPERATORS IN INT. PICTURE

IN INTERACTION PICTURE: TIME DEP. DUE TO $\hat{H}_0$
IS TRANSFERED INTO OPERATOR $\hat{O}^I(t)$

$$i \frac{d}{dt} \hat{O}^I(t) = - \hat{H}_0 \hat{O}^I(t) + \hat{O}^I(t) \hat{H}_0$$

$$i \frac{d}{dt} \hat{O}^I(t) = [\hat{O}^I(t), \hat{H}_0]$$

TRIVIAL CASE $\hat{O}^I = \hat{H}_0 \rightarrow \hat{H}_0^S = \hat{H}_0^I = \hat{H}_0$ (CONSTANT)

• TIME DEPENDENCE OF STATE IN INT. PICTURE

$$i \frac{d}{dt} |\Psi(t)\rangle_I = i \frac{d}{dt} \left(\underbrace{e^{i\hat{H}_0(t-t_0)}}_{U_0^\dagger} |\Psi(t)\rangle_S \right)$$

$$= -\hat{H}_0 |\Psi(t)\rangle_I$$

$$+ U_0^\dagger (\hat{H}_0 + \hat{H}_1) \underbrace{|\Psi(t)\rangle_S}_{U_0 |\Psi(t)\rangle_I}$$

$$\downarrow U_0^\dagger \hat{H}_0 U_0 = \hat{H}_0 \quad U_0 |\Psi(t)\rangle_I$$

$$i \frac{d}{dt} |\Psi(t)\rangle_I = -\hat{H}_0 |\Psi(t)\rangle_I + (\hat{H}_0 + U_0^\dagger \hat{H}_1 U_0) |\Psi(t)\rangle_I$$

$\Downarrow$

$$i \frac{d}{dt} |\Psi(t)\rangle_I = \hat{H}_1^I(t) |\Psi(t)\rangle_I$$

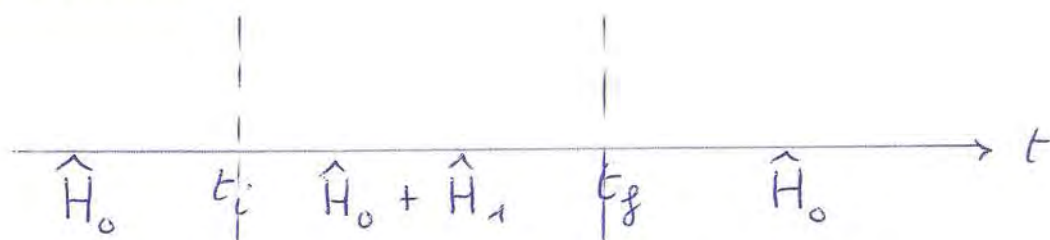
$$\text{with } \hat{H}_1^I(t) = U_0^\dagger \hat{H}_1 U_0$$

•• IN INTERACTION PICTURE :

TIME DEPENDENCE DUE TO $\hat{H}_0$ IS IN OPERATORS.

TIME DEPENDENCE DUE TO $\hat{H}_1$ IS IN STATES $|\Psi(t)\rangle_I$

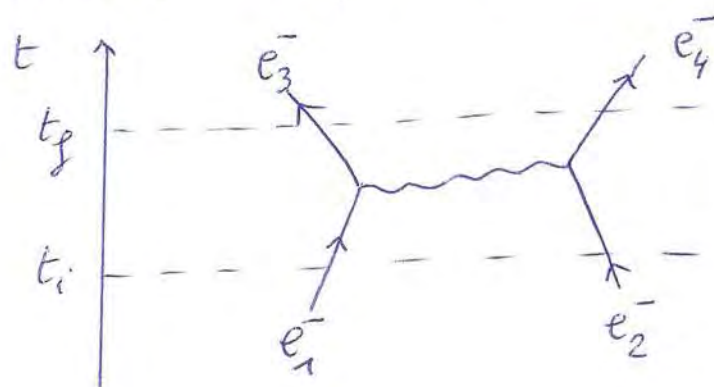
$\Rightarrow$ S - MATRIX



AT t_i : INTERACTION SWITCHED ON

AT t_f : " " OFF

BETWEEN t_i AND t_f : FIELDS INTERACT



$\hookrightarrow$ FOR $t \ll t_i$, i.e. $t = -\infty$

SYSTEM IS IN INITIAL STATE

$$|\Psi(t = -\infty)\rangle_I = |i\rangle \quad (\text{e.g. } |e_1^- e_2^-\rangle)$$

$\hookrightarrow$ DUE TO INTERACTION :

STATE EVOLVES ACCORDING TO A UNITARY TF.

$$|\Psi(t = +\infty)\rangle_I = \underset{\substack{\uparrow \\ \text{S-MATRIX}}}{S} |\Psi(t = -\infty)\rangle_I$$

UNITARY: $S^\dagger S = S S^\dagger = \mathbb{I}$

↳ TRANSITION PROBABILITY AMPLITUDE
TO A SPECIFIC FINAL STATE $|f\rangle$ (e.g. $|e_3^- e_4^-\rangle$)

$$\langle f | \Psi(t=+\infty) \rangle_I = \langle f | S | i \rangle \equiv S_{fi}$$

S-MATRIX
ELEMENT

• UNITARITY OF S-MATRIX

SUM OF PROBABILITIES OF SCATTERING
FROM $|i\rangle$ TO ALL POSSIBLE FINAL STATES $|f\rangle$
SHOULD BE 1

$$\sum_f |\langle f | S | i \rangle|^2 = \sum_f |S_{fi}|^2 = 1$$

PROOF: THIS FOLLOWS FROM UNITARITY OF S

$$S^\dagger S = 1$$

$$\langle i | S^\dagger S | i \rangle = \langle i | i \rangle = 1.$$

$$\langle i | S^\dagger \left(\sum_f |f\rangle \langle f| \right) S | i \rangle = 1$$

↓ COMPLETENESS

$$\sum_f \langle f | S | i \rangle^* \langle f | S | i \rangle = 1 \Rightarrow \sum_f |S_{fi}|^2 = 1$$

⇒ EXPANSION FOR S

- ITERATIVE SOLUTION FOR $|\Psi(t)\rangle_I$.

$$\hookrightarrow i \frac{d}{dt} |\Psi(t)\rangle_I = \hat{H}_1^I(t) |\Psi(t)\rangle_I$$

↓
INTEGRATE $\int_{-\infty}^t$

$$i (|\Psi(t)\rangle_I - \underbrace{|\Psi(-\infty)\rangle_I}_{|i\rangle}) = \int_{-\infty}^t dt_1 \hat{H}_1^I(t_1) |\Psi(t_1)\rangle_I$$

$$|\Psi(t)\rangle_I = |i\rangle - i \int_{-\infty}^t dt_1 \hat{H}_1^I(t_1) |\Psi(t_1)\rangle_I$$

- $\hookrightarrow$ INTEGRAL EQUATION CAN BE SOLVED ITERATIVELY
 IF $\hat{H}_1^I$ IS A PERTURBATION
 $\hookrightarrow$ CONTAINS SMALL EXPANSION PARAMETER

$$\begin{aligned} \hookrightarrow |\Psi(t)\rangle_I = & |i\rangle \\ & + (-i) \int_{-\infty}^t dt_1 \hat{H}_1^I(t_1) |i\rangle \\ & + (-i)^2 \int_{-\infty}^t dt_1 \hat{H}_1^I(t_1) \int_{-\infty}^{t_1} dt_2 \hat{H}_1^I(t_2) |\Psi(t_2)\rangle_I \end{aligned}$$

$$\begin{aligned}
\hookrightarrow |\Psi(t)\rangle_I &= |i\rangle \\
&+ (-i) \int_{-\infty}^t dt_1 \hat{H}_I^I(t_1) |i\rangle \\
&+ (-i)^2 \int_{-\infty}^t dt_1 \hat{H}_I^I(t_1) \int_{-\infty}^{t_1} dt_2 \hat{H}_I^I(t_2) |i\rangle \\
&+ (-i)^3 \int_{-\infty}^t dt_1 \hat{H}_I^I(t_1) \int_{-\infty}^{t_1} dt_2 \hat{H}_I^I(t_2) \int_{-\infty}^{t_2} dt_3 \hat{H}_I^I(t_3) |i\rangle \\
&+ \dots
\end{aligned}$$

$\hookrightarrow$ FOR $t \rightarrow +\infty$

$$|\Psi(t=+\infty)\rangle_I = S |i\rangle$$

THIS YIELDS THE S-MATRIX EXPANSION

$$\left\| S = \sum_{n=0}^{\infty} (-i)^n \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \dots \int_{-\infty}^{t_{n-1}} dt_n \right.$$

$$\left. \cdot \hat{H}_I^I(t_1) \hat{H}_I^I(t_2) \dots \hat{H}_I^I(t_n) \right.$$

NOTE: IN THIS PRODUCT, THE INTEGRATION RANGES IMPLY THAT

$$t_1 > t_2 > t_3 > \dots > t_{n-1} > t_n$$

- TIME - ORDERED PRODUCT

$$\hookrightarrow T \{ \hat{H}_1^I(t_1) \hat{H}_1^I(t_2) \dots \hat{H}_1^I(t_m) \}$$

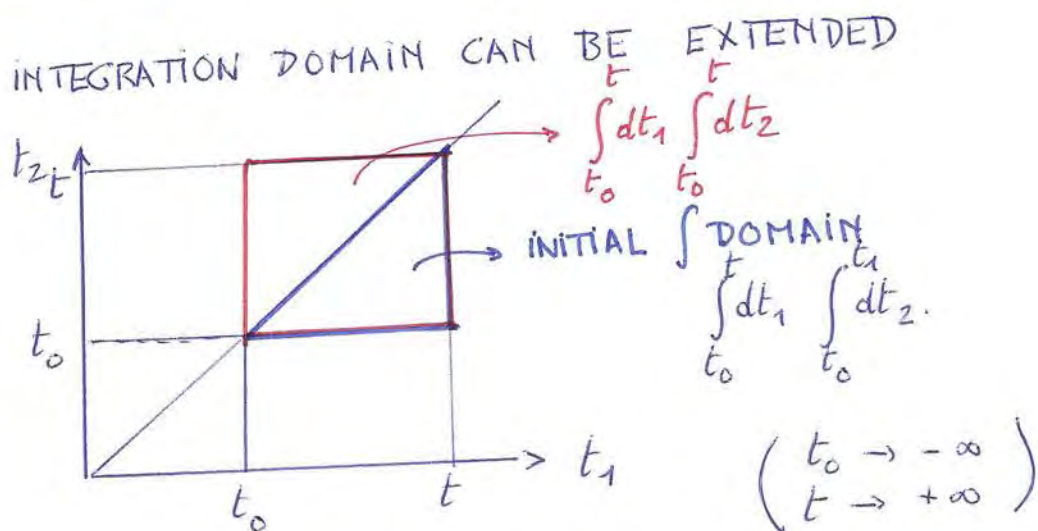
$$\equiv \hat{H}_1^I(t_1) \hat{H}_1^I(t_2) \dots \hat{H}_1^I(t_m)$$

$$\text{IF } t_1 > t_2 > \dots > t_m$$

$\hookrightarrow$ USING T-PRODUCT

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{+\infty} dt_1 \dots \int_{-\infty}^{+\infty} dt_n T \{ \hat{H}_1^I(t_1) \dots \hat{H}_1^I(t_n) \}$$

NOTE $\Rightarrow$ INTEGRATION DOMAIN CAN BE EXTENDED



RED DOMAIN : 2x LARGER THAN BLUE DOMAIN
 $\Rightarrow$ FACTOR 1/2.

$\rightarrow$ FOR m INTEGRATIONS

$m!$ PERMUTATIONS OF

$$\hat{H}_1^I(t_1) \dots \hat{H}_1^I(t_m)$$

$\Rightarrow$ FACTOR $1/m!$ WHEN EXTENDING
 INTEGRATION DOMAIN

- IN TERMS OF HAMILTONIAN DENSITY

$$\begin{aligned} \text{IN QFT} \quad \hat{H}_I(t) &= \int d^3\vec{x} \quad \mathcal{H}_I(t, \vec{x}) \\ &= \int d^3\vec{x} \quad \mathcal{H}_I(x) \end{aligned}$$

WITH $\mathcal{H}_I$: HAMILTONIAN DENSITY
IN INTERACTION PICTURE

✓ FROM NOW ONWARDS :
WE WILL DROP INDEX I (INTERACTION PICTURE)
WHICH IS ALWAYS UNDERSTOOD HOWEVER

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \dots \int d^4x_n T\{\mathcal{H}_I(x_1) \dots \mathcal{H}_I(x_n)\}$$

↳ DYSON EXPANSION OF S-MATRIX

⇒ EXAMPLE : ϕ^4 THEORY

↳ FREE SPIN 0 FIELD (KLEIN - GORDON)

$$\begin{aligned} \mathcal{L}_{\text{KG}} &= \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 \\ &\equiv \mathcal{L}_0 \end{aligned}$$

↳ INTERACTION BETWEEN FIELDS

$$\mathcal{L}_1 = - \frac{\lambda}{4!} \phi^4$$

λ DESCRIBES STRENGTH OF INTERACTION

$$\hookrightarrow \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$$

$$\hookrightarrow \pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}$$

$$\hookrightarrow \mathcal{H} = \pi \dot{\phi} - \mathcal{L}$$

$$= \underbrace{\frac{1}{2} (\pi^2 + (\nabla \phi)^2 + m^2 \phi^2)}_{\mathcal{H}_0} + \underbrace{\frac{\lambda}{4!} \phi^4}_{\mathcal{H}_1}$$

$$\hookrightarrow S = \sum_{n=0}^{\infty} (-i)^n \left(\frac{\lambda}{4!}\right)^n \frac{1}{n!} \int_{-\infty}^{+\infty} d^4x_1 \dots \int_{-\infty}^{+\infty} d^4x_n T\{\phi^4(x_1) \dots \phi^4(x_n)\}$$

$\hookrightarrow$ IF λ IS A SMALL PARAMETER



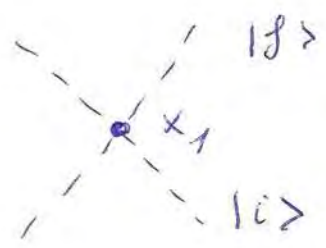
WE CAN MAKE A PERTURBATIVE EXPANSION IN λ

$\hookrightarrow$ 1^o ORDER

$$S^{(1)} = -i \frac{\lambda}{4!} \int d^4x_1 \phi^4(x_1)$$

GRAPHICALLY

$$\langle f | S^{(1)} | i \rangle$$



4 FIELDS INTERACT
IN ONE SPACE-TIME
POINT

$\hookrightarrow$ 2^o ORDER

$$S^{(2)} = (-i)^2 \frac{\lambda^2}{2!} \int d^4x_1 \int d^4x_2 T\{\phi^4(x_1) \phi^4(x_2)\}$$



WE HAVE TO WORK OUT THE T-PRODUCT



WICK'S THEOREM

⇒ EXAMPLE : QED

$$L, \mathcal{L}_1 = -e \bar{\psi} \gamma^\mu \psi A_\mu$$

$$\mathcal{H}_1 = -\mathcal{L}_1 = e \bar{\psi} \gamma^\mu \psi A_\mu$$

NORMAL ORDERING: ENSURES THAT $\langle 0 | \psi(\dots) | 0 \rangle = 0$

→ 1° ORDER $S^{(1)}$ FOR OBSERVABLES

$$S^{(1)} = -ie \int d^4 x_1 \bar{\psi}(x_1) \gamma^\mu \psi(x_1) A_\mu(x_1)$$



→ 2° ORDER $S^{(2)}$

$$S^{(2)} = (-i)^2 \frac{e^2}{2!} \int d^4 x_1 \int d^4 x_2$$

$$\cdot T \left\{ \bar{\psi}(x_1) \gamma^\mu \psi(x_1) A_\mu(x_1) \bar{\psi}(x_2) \gamma^\nu \psi(x_2) A_\nu(x_2) \right\}$$

2) WICK'S THEOREM

⇒ CONTRACTIONS (PROPAGATORS)

- NORMAL ORDERED PRODUCTS (CREATION OPERATOR LEFT OF ANNIHILATION)

$\mathcal{H}_1$ CAN BE WRITTEN AS $N(\dots)$

THIS ENSURES THAT $\langle 0 | N(\dots) | 0 \rangle = 0$

FOR OBSERVABLES (e.g. ENERGY)

- HIGHER ORDER TERMS IN S-MATRIX EXPANSION

$$T \{ A(x_1) B(x_2) \}$$

e.g. QED $A(x_1) = e N(\bar{\psi}(x_1) \gamma^\mu \psi(x_1) A_\mu(x_1))$

$$B(x_2) = e N(\bar{\psi}(x_2) \gamma^\mu \psi(x_2) A_\mu(x_2))$$

- CONTRACTION

$$\begin{aligned} & T \{ A(x_1) B(x_2) \} \\ &= N(A(x_1) B(x_2)) + \langle 0 | T \{ A(x_1) B(x_2) \} | 0 \rangle \end{aligned}$$

BECAUSE $\langle 0 | N(A(x_1) B(x_2)) | 0 \rangle = 0$

NOTATION : CONTRACTION

$$\boxed{\underbrace{A(x_1) B(x_2)} \equiv \langle 0 | T \{ A(x_1) B(x_2) \} | 0 \rangle}$$

$$\therefore T \{ A(x_1) B(x_2) \} = N(A(x_1) B(x_2)) + \underbrace{A(x_1) B(x_2)}$$

• PROPAGATORS

$$\boxed{\underbrace{\phi(x_1) \phi(x_2)} = \langle 0 | T \{ \phi(x_1) \phi(x_2) \} | 0 \rangle = i \Delta_F(x_1 - x_2)}$$

↑
FEYNMAN PROPAGATOR

$$\boxed{\underbrace{\psi_\alpha(x_1) \bar{\psi}_\beta(x_2)} = \langle 0 | T \{ \psi_\alpha(x_1) \bar{\psi}_\beta(x_2) \} | 0 \rangle = i S_F(x_1 - x_2)}$$

NOTE $\underbrace{\psi_\alpha(x_1) \bar{\psi}_\beta(x_2)} = - \underbrace{\bar{\psi}_\beta(x_2) \psi_\alpha(x_1)}$

$$\boxed{\underbrace{A^\mu(x_1) A^\nu(x_2)} = \langle 0 | T \{ A^\mu(x_1) A^\nu(x_2) \} | 0 \rangle = i D_F^{\mu\nu}(x_1 - x_2)}$$

$\Rightarrow$ WICK'S THEOREM

$$\begin{aligned}
 \hookrightarrow \quad & T \{ A B C \dots Y Z \} \\
 &= N (A B C \dots Y Z) \\
 &+ N (\underbrace{A B C} \dots Y Z) \\
 &+ N (\underbrace{A B C} \dots Y Z) \\
 &+ \dots N (\underbrace{A B C} \dots Y Z) \\
 &+ N (\underbrace{A B C D} \dots Y Z) + \dots
 \end{aligned}$$

T - PRODUCT = SUM OF N-ORDERED PRODUCTS
WITH ALL POSSIBLE CONTRACTIONS

$$\underbrace{A(x_1) B(x_2)}$$

$\hookrightarrow$ NOTE: PHASE FACTOR FOR FERMIONS

$$\begin{aligned}
 &N (A B C D E \dots J K L M \dots) \\
 &= (-1)^P \underbrace{A J} \underbrace{B C} \underbrace{D M} N (E \dots K L \dots)
 \end{aligned}$$

$\downarrow$
P : # INTERCHANGES OF FERMION FIELDS TO CHANGE ORDER

↳ EXAMPLE : 4 FIELDS (SCALAR)

USE SHORTHAND NOTATION $\phi_i \equiv \phi(x_i)$

$$\bullet \quad \mathcal{T} \{ \phi_1 \phi_2 \phi_3 \phi_4 \}$$

$$= \mathcal{N} (\phi_1 \phi_2 \phi_3 \phi_4$$

$$+ \underbrace{\phi_1 \phi_2}_{\text{}} \phi_3 \phi_4 + \underbrace{\phi_1 \phi_2 \phi_3}_{\text{}} \phi_4 + \underbrace{\phi_1 \phi_2 \phi_3 \phi_4}_{\text{}})$$

$$+ \underbrace{\phi_1 \phi_2 \phi_3}_{\text{}} \phi_4 + \underbrace{\phi_1 \phi_2 \phi_3 \phi_4}_{\text{}} + \underbrace{\phi_1 \phi_2 \phi_3}_{\text{}} \phi_4$$

$$+ \underbrace{\phi_1 \phi_2 \phi_3}_{\text{}} \phi_4 + \underbrace{\phi_1 \phi_2 \phi_3 \phi_4}_{\text{}} + \underbrace{\phi_1 \phi_2 \phi_3 \phi_4}_{\text{}})$$

↓

$$= \mathcal{N} (\phi_1 \phi_2 \phi_3 \phi_4)$$

$$+ i \Delta_F(x_1 - x_2) \mathcal{N} (\phi_3 \phi_4) + i \Delta_F(x_1 - x_3) \mathcal{N} (\phi_2 \phi_4)$$

$$+ i \Delta_F(x_1 - x_4) \mathcal{N} (\phi_2 \phi_3) + i \Delta_F(x_2 - x_3) \mathcal{N} (\phi_1 \phi_4)$$

$$+ i \Delta_F(x_2 - x_4) \mathcal{N} (\phi_1 \phi_4) + i \Delta_F(x_3 - x_4) \mathcal{N} (\phi_1 \phi_2)$$

$$- \Delta_F(x_1 - x_2) \Delta_F(x_3 - x_4) - \Delta_F(x_1 - x_3) \Delta_F(x_2 - x_4)$$

$$- \Delta_F(x_1 - x_4) \Delta_F(x_2 - x_3)$$

- $\langle 0 | T \{ \phi_1 \phi_2 \phi_3 \phi_4 \} | 0 \rangle$

$$= - \Delta_F(x_1 - x_2) \Delta_F(x_3 - x_4) - \Delta_F(x_1 - x_3) \Delta_F(x_2 - x_4) \\ - \Delta_F(x_1 - x_4) \Delta_F(x_2 - x_3)$$

⇒ PROOF OF WICK'S THEOREM

↳ FOR $n = 2$

$$\phi(x) = \phi^+(x) + \phi^-(x)$$

WITH $\phi^+(x) | 0 \rangle = 0$

ϕ^+ CONTAINS ANNIHILATION OP.

$$\langle 0 | \phi^-(x) = 0$$

ϕ^- " CREATION OP.

$$\phi^+(x) = \int \frac{d^3 \vec{k}}{(2\pi)^3 \sqrt{2E_{\vec{k}}}} a(\vec{k}) e^{-ik \cdot x}$$

$$\phi^-(x) = \int \frac{d^3 \vec{k}}{(2\pi)^3 \sqrt{2E_{\vec{k}}}} a^\dagger(\vec{k}) e^{+ik \cdot x}$$

$$T \{ \phi(x_1) \phi(x_2) \}$$

$$= (\phi^+(x_1) + \phi^-(x_1)) (\phi^+(x_2) + \phi^-(x_2))$$

FOR
 $x_1^0 > x_2^0$

$$= \phi^+(x_1) \phi^+(x_2) + \phi^-(x_1) \phi^+(x_2) + \underbrace{\phi^+(x_1) \phi^-(x_2)}_{\text{commutator}} + \phi^-(x_1) \phi^-(x_2)$$

$$= \phi^+(x_1) \phi^+(x_2) + \phi^-(x_1) \phi^+(x_2) + \phi^-(x_2) \phi^+(x_1) + \phi^-(x_1) \phi^-(x_2)$$

$$+ [\phi^+(x_1), \phi^-(x_2)]$$

$$\begin{aligned}
& T \{ \phi(x_1) \phi(x_2) \} \\
&= N(\phi(x_1) \phi(x_2)) + [\phi^+(x_1), \phi^-(x_2)]_- \\
&\quad x_1^0 > x_2^0 \\
&= N(\phi(x_1) \phi(x_2)) + [\phi^+(x_2), \phi^-(x_1)]_- \\
&\quad x_2^0 > x_1^0
\end{aligned}$$

$$\Downarrow$$

$$\begin{aligned}
& \langle 0 | T \{ \phi(x_1) \phi(x_2) \} | 0 \rangle \\
&= \begin{cases} [\phi^+(x_1), \phi^-(x_2)]_- & , \quad x_1^0 > x_2^0 \\ [\phi^+(x_2), \phi^-(x_1)]_- & , \quad x_2^0 > x_1^0 \end{cases} \\
&= i \Delta_F(x_1 - x_2) \\
&\equiv \underbrace{\phi(x_1) \phi(x_2)}
\end{aligned}$$

↳ PROOF FOR $n > 2$ BY INDUCTION

ASSUME IT HOLDS FOR $n-1$ FIELDS

WITHOUT LOSS OF GENERALITY WE CAN ASSUME $x_1^0 > x_2^0 > \dots > x_n^0$
(IF THIS IS NOT CASE : WE CAN RELABEL POINTS)

$$\begin{aligned}
& T \{ \phi_1 \dots \phi_n \} \\
&= \phi_1 \phi_2 \dots \phi_n
\end{aligned}$$

$$= \phi_1 N(\phi_2 \dots \phi_n + \text{ALL CONTRACTIONS NOT INVOLVING } \phi_1)$$

$$= (\phi_1^+ + \phi_1^-) N(\phi_2 \dots \phi_n + \text{ALL CONTRACTIONS NOT INVOLVING } \phi_1)$$

$$= N(\phi_1^- \phi_2 \dots \phi_n + \text{ALL CONTRACTIONS NOT INVOLVING } \phi_1)$$

$$+ \phi_1^+ N(\phi_2 \dots \phi_n + \text{ALL CONTRACTIONS NOT INVOLVING } \phi_1)$$

$$\hookrightarrow \phi_1^+ N(\phi_2 \dots \phi_n)$$

$$= N(\phi_2 \dots \phi_n) \phi_1^+ + [\phi_1^+, N(\phi_2 \dots \phi_n)]_-$$

$$= N(\phi_1^+ \phi_2 \dots \phi_n) + [\phi_1^+, N(\phi_2 \dots \phi_n)]_-$$

$$\hookrightarrow [\phi_1^+, N(\phi_2 \phi_3)]_-$$

$$= [\phi_1^+, \phi_2^+ \phi_3^+ + \phi_2^- \phi_3^+ + \phi_3^- \phi_2^+ + \phi_2^- \phi_3^-]$$

$$= \phi_1^+ (\phi_2^+ + \phi_2^-) \phi_3^+ - (\phi_2^+ + \phi_2^-) \phi_3^+ \phi_1^+$$

$$+ \phi_1^+ \phi_3^- \phi_2^+ - \phi_3^- \phi_2^+ \phi_1^+ + \phi_1^+ \phi_2^- \phi_3^- - \phi_2^- \phi_3^- \phi_1^+$$

$$= N([\phi_1^+, \phi_2^-] \phi_3) + N(\phi_2 [\phi_1^+, \phi_3^-])$$

$$= N(\underbrace{\phi_1 \phi_2}_{} \phi_3) + N(\phi_2 \underbrace{\phi_1 \phi_3}_{})$$

$$\hookrightarrow \phi_1^+ N(\phi_2 \dots \phi_n)$$

$$= N(\phi_1^+ \phi_2 \dots \phi_n)$$

$$+ N(\underbrace{\phi_1 \phi_2}_{\text{contract}} \phi_3 \dots \phi_n) + N(\phi_1 \underbrace{\phi_2 \phi_3}_{\text{contract}} \dots \phi_n)$$

$$+ \dots + N(\phi_1 \phi_2 \dots \underbrace{\phi_n}_{\text{contract}})$$

$$\hookrightarrow N(\phi_1^- \phi_2 \dots \phi_n) + \phi_1^+ N(\phi_2 \dots \phi_n)$$

$$= N(\phi_1 \phi_2 \dots \phi_n)$$

$$+ N(\underbrace{\phi_1 \phi_2}_{\text{contract}} \dots \phi_n) + \dots + N(\phi_1 \phi_2 \dots \underbrace{\phi_n}_{\text{contract}})$$

$\hookrightarrow$ ANALOGOUS FOR OTHER TERMS IN $N(\)$
WHICH INVOLVE CONTRACTIONS

$$\circ \circ \parallel T \{ \phi_1 \dots \phi_n \}$$

$$\parallel = N(\phi_1 \dots \phi_n) + \text{ALL CONTRACTIONS}$$

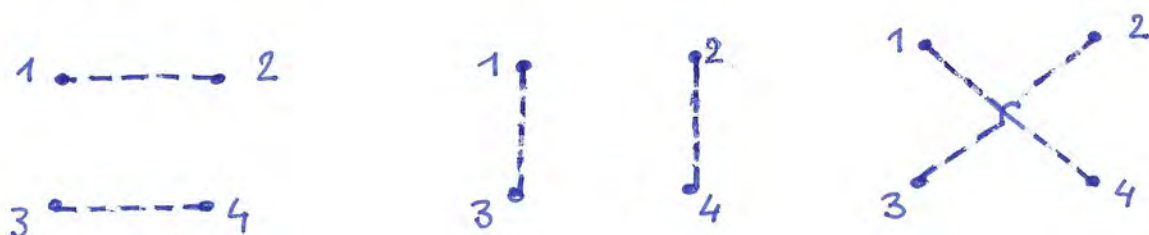
3) FEYNMAN DIAGRAMS & FEYNMAN RULES FOR ϕ^4 THEORY

$\Rightarrow$ EXAMPLE: VACUUM EXPECTATION VALUE

$$\langle 0 | T \{ \phi_1 \phi_2 \phi_3 \phi_4 \} | 0 \rangle$$

$\downarrow$ ONLY FULLY CONTRACTED TERMS
CONTRIBUTE TO $\langle 0 | \quad | 0 \rangle$

$$= \underbrace{\phi_1 \phi_2}_{\text{contract}} \underbrace{\phi_3 \phi_4}_{\text{contract}} + \underbrace{\phi_1 \phi_2 \phi_3 \phi_4}_{\text{contract}} + \underbrace{\phi_1 \phi_2 \phi_3 \phi_4}_{\text{contract}}$$



DIAGRAMMATIC INTERPRETATION: FEYNMAN DIAGRAMS

PROPAGATOR

$\underbrace{\phi(y) \phi(x)}$
 $= i \Delta_F(x-y)$

NOTE: WE WILL INDICATE SCALAR (SPINO) PROPAGATOR
BY DASHED LINE

$\Rightarrow$ EXAMPLE : $2 \rightarrow 2$ SCATTERING TO 1^0 ORDER

$\hookrightarrow$ S - MATRIX ELEMENT TO 1^0 ORDER

$$S^{(1)} = -i \frac{\lambda}{4!} \int d^4x \phi^4(x)$$

$\hookrightarrow$ INITIAL STATE : 2 SCALAR PARTICLES WITH MOMENTA $\vec{p}_1$ & $\vec{p}_2$

$$\begin{aligned} |i\rangle &= |\vec{p}_1, \vec{p}_2\rangle \\ &= \sqrt{2E_1} \sqrt{2E_2} a^\dagger(\vec{p}_1) a^\dagger(\vec{p}_2) |0\rangle \end{aligned}$$

$\hookrightarrow$ FINAL STATE : 2 SCALAR PARTICLES WITH MOMENTA $\vec{p}_3$ & $\vec{p}_4$

$$\begin{aligned} |f\rangle &= |\vec{p}_3, \vec{p}_4\rangle \\ &= \sqrt{2E_3} \sqrt{2E_4} a^\dagger(\vec{p}_3) a^\dagger(\vec{p}_4) |0\rangle \end{aligned}$$

$\hookrightarrow$ TRANSITION AMPLITUDE INDUCED BY $S^{(1)}$ BETWEEN $|i\rangle$ AND $|f\rangle$

$$\langle f | S^{(1)} | i \rangle$$

$$\rightarrow \phi^+(x) | \bar{p}_1 \rangle$$

$$= \int \frac{d^3 \bar{k}}{(2\pi)^3 \sqrt{2E_k}} a(\bar{k}) e^{-i\bar{k} \cdot x} \sqrt{2E_{\bar{p}_1}} a^+(\bar{p}_1) | 0 \rangle$$

$$= e^{-i\bar{p}_1 \cdot x} | 0 \rangle \Rightarrow \langle 0 | \phi(x) | \bar{p}_1 \rangle = e^{-i\bar{p}_1 \cdot x}$$

$$\rightarrow \langle \bar{p}_3 | \phi^-(x) = \langle 0 | e^{+i\bar{p}_3 \cdot x} \Rightarrow \langle \bar{p}_3 | \phi(x) | 0 \rangle = e^{+i\bar{p}_3 \cdot x}$$

$\rightarrow$ SYMMETRY FACTOR

TO ANNIHILATE PARTICLE $\bar{p}_1 \leadsto 4$ FIELDS POSSIBLE
 " " AFTERWARDS " $\bar{p}_2 \leadsto 3$ " "
 " CREATE " " $\bar{p}_3 \leadsto 2$ " "
 " " " " $\bar{p}_4 \leadsto 1$ FIELD

TOTAL 4! COMBINATIONS
 SYMMETRY FACTOR 4!

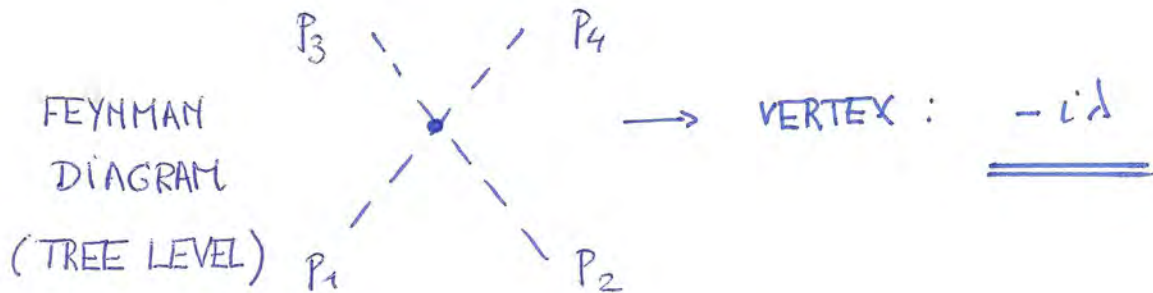
$$\rightarrow \langle f | S^{(4)} | i \rangle$$

$$= -i \frac{\lambda}{4!} \cdot \underset{\substack{\uparrow \\ \text{SYMMETRY} \\ \text{FACTOR}}}{(4!)} \int d^4 x e^{-i(p_1 + p_2 - p_3 - p_4) \cdot x}$$

$$= (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) (-i\lambda)$$

∴

$$\langle f | S^{(1)} | i \rangle = \underbrace{(2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)}_{\text{ENERGY-MOMENTUM CONSERVATION}} \cdot (-i\lambda)$$



⇒ EXAMPLE : $2 \rightarrow 2$ SCATTERING TO 2^{nd} ORDER

↳ S-MATRIX ELEMENT TO 2^{nd} ORDER

$$S^{(2)} = \frac{1}{2!} \left(-i\frac{\lambda}{4!} \right)^2 \int d^4x \int d^4y T \{ \phi^4(x) \phi^4(y) \}$$

↳ CALCULATE

$$\langle \bar{p}_3 \bar{p}_4 | S^{(2)} | \bar{p}_1 \bar{p}_2 \rangle$$

DIFFERENT CONTRIBUTIONS DUE TO
WICK'S THEOREM

WE NEED 4 ϕ 's TO ANNIHILATE $\bar{p}_1, \bar{p}_2$
& CREATE $\bar{p}_3, \bar{p}_4$

WE CAN CLASSIFY CONTRIBUTIONS THROUGH
FEYNMAN DIAGRAMS

$$\downarrow \int d^4 y \, e^{-i(p_1 + p_2 - k - \ell) \cdot y} = (2\pi)^4 \delta^4(p_1 + p_2 - k - \ell)$$

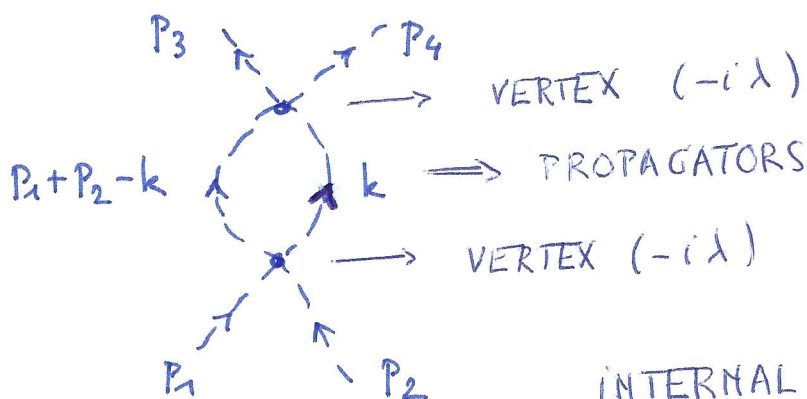
$$\langle f | S_a^{(2)} | i \rangle = \frac{(-i\lambda)^2}{2} \int \frac{d^4 k}{(2\pi)^4} i\Delta_F(k) i\Delta_F(p_1 + p_2 - k) \cdot \underbrace{\int d^4 x \, e^{+i(p_3 + p_4 - p_1 - p_2) \cdot x}}_{(2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)}$$

$$\langle f | S_a^{(2)} | i \rangle = (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \cdot \frac{(-i\lambda)^2}{2} \int \frac{d^4 k}{(2\pi)^4} i\Delta_F(k) i\Delta_F(p_1 + p_2 - k)$$

$$\text{WITH } i\Delta_F(k) = \frac{i}{k^2 - m^2 + i\epsilon}$$

FEYNMAN DIAGRAM IN MOMENTUM SPACE (1-LOOP DIAGRAM)

↳ QUANTUM CORRECTION

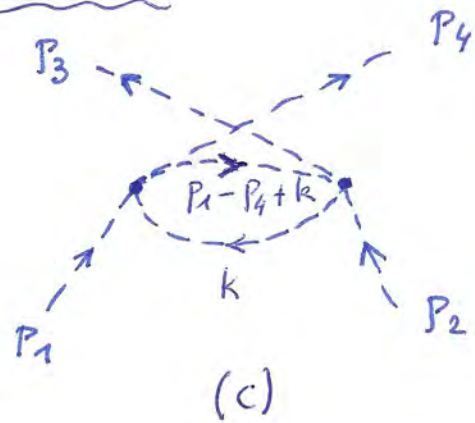
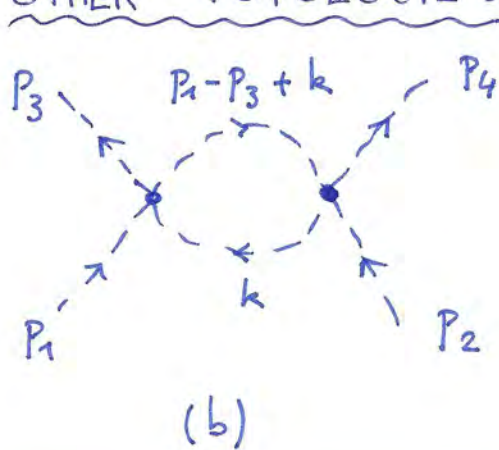


$i\Delta_F(k)$ AND $i\Delta_F(p_1 + p_2 - k)$

INTERNAL LOOP $\Rightarrow \int \frac{d^4 k}{(2\pi)^4} \dots$

$(2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)$: OVERALL ENERGY-MOMENTUM CONSERVATION

↳ OTHER TOPOLOGIES OF SAME KIND



$$\langle f | S_b^{(2)} | i \rangle = (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \cdot \frac{(-i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} i\Delta_F(k) i\Delta_F(p_1 - p_3 + k)$$

$$\langle f | S_c^{(2)} | i \rangle = (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \cdot \frac{(-i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} i\Delta_F(k) i\Delta_F(p_1 - p_4 + k)$$

↳ TOPOLOGIES WITH 2 FIELDS WHICH CONTRACT ON THEMSELVES

e.g. TERM

$$\langle f | N(\phi(x)\phi(x)\phi(x)\phi(x)\phi(y)\phi(y)\phi(y)\phi(y)) | i \rangle$$

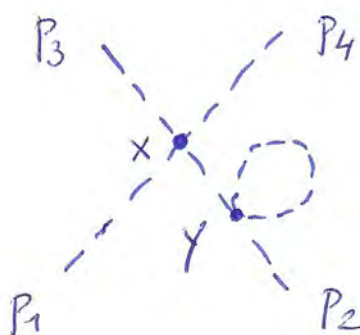
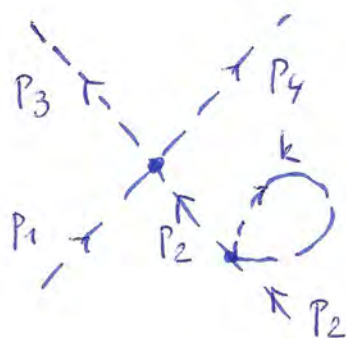


DIAGRAM (d)

$$\begin{aligned}
 \langle f | S_d^{(2)} | i \rangle &= \frac{(-i\lambda)}{4!} \frac{(-i\lambda)}{4!} \cdot \underset{\substack{\uparrow \\ \text{AT VERTEX}}} {4!} \underset{\substack{\nwarrow \\ \text{AT VERTEX } Y}} {(4.3)} \\
 &\cdot \int d^4x \int d^4y \, e^{i p_3 \cdot x} e^{i p_4 \cdot x} e^{-i p_1 \cdot x} e^{-i p_2 \cdot y} \\
 &\cdot i \Delta_F(x-y) \cdot i \Delta_F(0)
 \end{aligned}$$

↓ IN MOMENTUM SPACE

$$\begin{aligned}
 &= \frac{(-i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} i \Delta_F(k) \int \frac{d^4\ell}{(2\pi)^4} i \Delta_F(\ell) \\
 &\cdot \int d^4x \, e^{-i(p_1 - p_3 - p_4 + \ell) \cdot x} \\
 &\cdot \int d^4y \, e^{-i(p_2 - \ell) \cdot y} \\
 &\underbrace{\hspace{10em}}_{(2\pi)^4 \delta^4(\ell - p_2)}
 \end{aligned}$$



$$\begin{aligned}
 \langle f | S_d^{(2)} | i \rangle &= (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \\
 &\cdot i \Delta_F(p_2) \\
 &\cdot \frac{(-i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} i \Delta_F(k)
 \end{aligned}$$

BUT EXTERNAL PARTICLES ARE ON-SHELL

$$P_1^2 = P_2^2 = P_3^2 = P_4^2 = m^2$$

↓
PROBLEM $\Delta_F(P_2) = \frac{1}{P_2^2 - m^2 + i\epsilon} = \frac{1}{0 + i\epsilon}$ ↷

∴ WE NEED TO EXCLUDE DIAGRAMS WHERE
LOOPS ARE CONNECTED TO EXTERNAL LINES
(SEE FURTHER: THESE DIAGRAMS WILL
"RENORMALIZE" EXTERNAL LINES,
i.e. EXTERNAL LINES HAVE TO BE DEFINED
IN PRESENCE OF INTERACTION TO BE PHYSICAL)

↳ DISCONNECTED DIAGRAMS

e.g.



VACUUM BUBBLES

WHEN TURNING ON INTERACTION
TRIVIAL VACUUM $|0\rangle \Rightarrow$ PHYSICAL VACUUM $|\Omega\rangle$
WHICH CONTAINS
VACUUM GRAPHS

∴ WE WILL ABSORB THESE DISCONNECTED GRAPHS IN
DEFINITION OF PHYSICAL VACUUM & DEFINE 1-PARTICLE
STATES AS EXCITATIONS IN PHYSICAL VACUUM $|\Omega\rangle$
 $|\vec{p}\rangle \equiv \sqrt{2E_p} a^\dagger(\vec{p}) |\Omega\rangle$

⇒ FEYNMAN RULES FOR SCATTERING AMPLITUDES

↳ IN MOMENTUM SPACE : CONTRIBUTION TO

$$\langle f | S | i \rangle \rightarrow \text{SCATTERING AMPLITUDE}$$

CONSIDER ALL CONNECTED GRAPHS BETWEEN $|i\rangle$ AND $|f\rangle$
FOR SCALAR FIELDS:

1) FOR EACH PROPAGATOR (INTERNAL LINE)

$$\text{---} \xrightarrow{p} \text{---} \quad \frac{i}{p^2 - m^2 + i\epsilon}$$

2) FOR EACH VERTEX

$$\text{---} \times \text{---} \quad (-i\lambda)$$

3) FOR EACH EXTERNAL LINE $\text{---} \rightarrow \bullet$ $\frac{1}{(\text{NO LOOPS ATTACHED TO EXTERNAL LINES})}$

4) IMPOSE ENERGY-MOMENTUM CONSERVATION
AT EACH VERTEX

5) INTEGRATE OVER INDEPENDENT
LOOP-MOMENTA $\int \frac{d^4 k}{(2\pi)^4}$

6) DETERMINE SYMMETRY FACTOR & DIVIDE BY IT

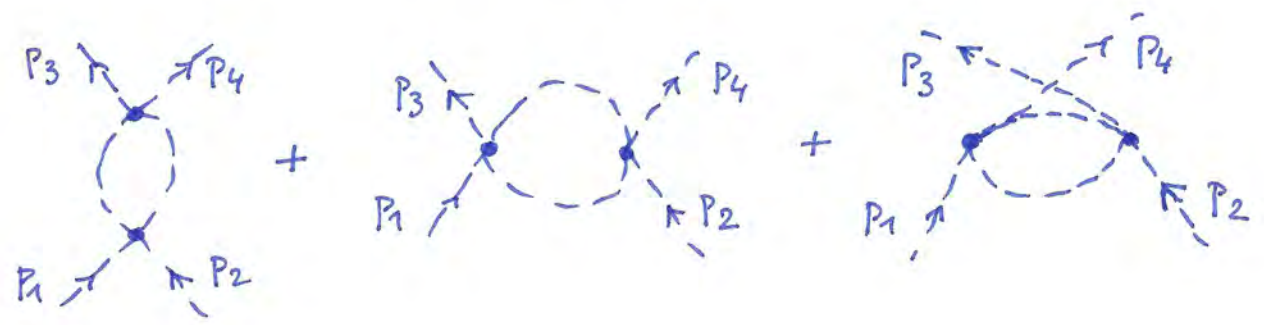
7) GLOBAL ENERGY MOMENTUM CONSERVATION

$$(2\pi)^4 \delta^4 \left(\sum_i p_i - \sum_f p_f \right)$$

↳ TO 1° ORDER : 1 DIAGRAM
(TREE LEVEL)



↳ TO 2° ORDER : 3 DIAGRAMS (1-LOOP)
 $\sim (-i\lambda)^2$

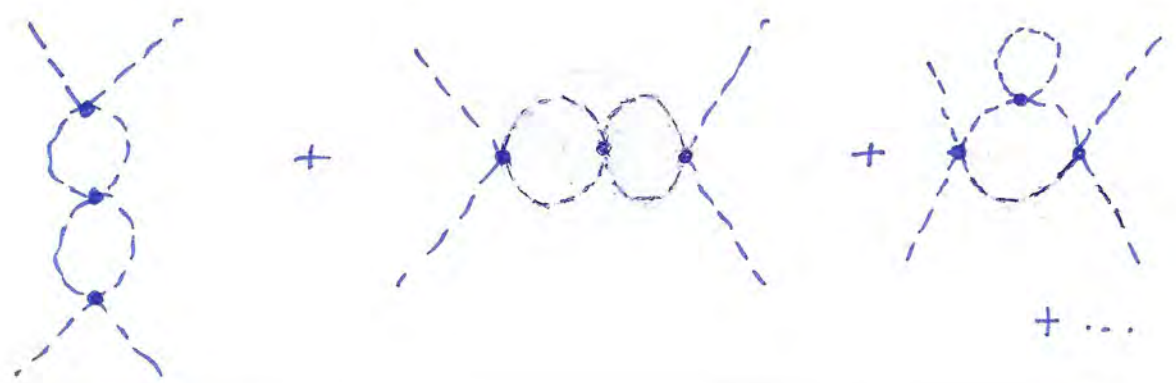


WE NEED TO PERFORM A LOOP INTEGRAL

$$\int \frac{d^4 k}{(2\pi)^4} \dots \Rightarrow \text{SEE LATER}$$

↳ TO 3° ORDER (2-LOOP)
 $\sim (-i\lambda)^3$

EXAMPLES OF 2-LOOP DIAGRAMS $\Rightarrow \int \frac{d^4 k}{(2\pi)^4} \int \frac{d^4 l}{(2\pi)^4} \dots$



4) FEYNMAN DIAGRAMS & RULES FOR QED

⇒ S-MATRIX TO 1st ORDER

$$\hookrightarrow \mathcal{L}_1 = -q \bar{\psi} \gamma^\mu \psi A_\mu$$

WITH q : ELECTRIC CHARGE

FOR e^- $q = -e$ ($e > 0$) WITH $\frac{e^2}{4\pi} \equiv \alpha \approx \frac{1}{137}$

↑
FINE-STRUCTURE
CONSTANT

WE WILL CONSIDER PROCESSES
WITH e^- IN FOLLOWING

$$\mathcal{L}_1 = e \bar{\psi} \gamma^\mu \psi A_\mu$$

$$\mathcal{H}_1 = -e \mathcal{N}(\bar{\psi} \gamma^\mu \psi A_\mu) \quad \text{NORMAL ORDERED SUCH THAT} \\ \langle 0 | \mathcal{H}_1 | 0 \rangle = 0$$

$$\hookrightarrow S^{(1)} = ie \int d^4x \mathcal{N}(\bar{\psi}(x) \gamma^\mu \psi(x) A_\mu(x))$$

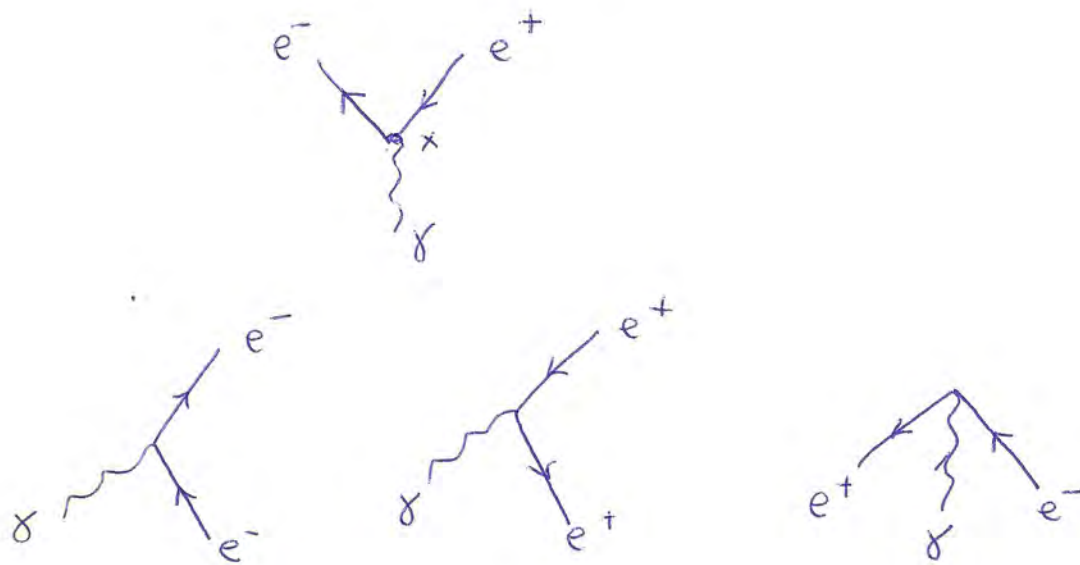
$$\psi(x) = \underbrace{\psi^+(x)}_{\substack{\uparrow \\ \text{ABSORBS } e^-}} + \underbrace{\psi^-(x)}_{\substack{\uparrow \\ \text{CREATES } e^+}}$$

$$\bar{\psi}(x) = \underbrace{\bar{\psi}^-(x)}_{\substack{\uparrow \\ \text{CREATES } e^-}} + \underbrace{\bar{\psi}^+(x)}_{\substack{\uparrow \\ \text{ABSORBS } e^+}}$$

$$A^\mu(x) = \underbrace{A^+(x)}_{\substack{\uparrow \\ \text{ANNIHILATES } \gamma}} + \underbrace{A^-(x)}_{\substack{\uparrow \\ \text{CREATES } \gamma}}$$

↳ TO 1⁰ ORDER: 8 PROCESSES

e.g. $N(\bar{\psi}(x) \gamma^\mu \psi(x) A_\mu^+(x))$

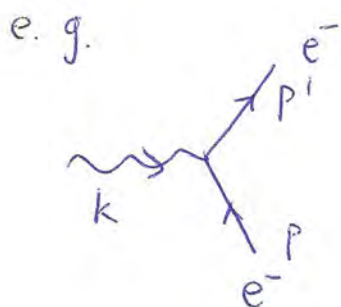


AND 4 MORE DIAGRAMS WHERE γ IS CREATED

↳ FOR PHYSICAL PROCESSES

$$\langle f | S^{(1)} | i \rangle$$

e^-, e^+	IN EITHER $ i\rangle$ OR $ f\rangle$	$P^2 = m^2$
γ	" " " " " "	$k^2 = 0$



$$p'^2 = (p + k)^2 = m^2$$

$$\Downarrow$$

$$\cancel{m^2} + 2k \cdot p + k^2 = \cancel{m^2}$$

$$\underline{k \cdot p = 0}$$

ONLY POSSIBLE
FOR $k^0 = 0$ ↙

∴ NO PHYSICAL PROCESSES AT 1⁰ ORDER

$\Rightarrow$ S - MATRIX TO 2⁰ ORDER

$$\hookrightarrow S^{(2)} = \sum_{i=A}^F S_i^{(2)}$$

$$S_A^{(2)} = - \frac{e^2}{2!} \int d^4x_1 \int d^4x_2 N \left\{ (\bar{\psi} \gamma^\mu A_\mu \psi)_{x_1} (\bar{\psi} \gamma^\nu A_\nu \psi)_{x_2} \right\}$$

$$S_B^{(2)} = - \frac{e^2}{2!} \int d^4x_1 \int d^4x_2 \left[N \left\{ \underbrace{(\bar{\psi} \gamma^\mu A_\mu \psi)_{x_1}}_{\text{}} (\bar{\psi} \gamma^\nu A_\nu \psi)_{x_2} \right\} + N \left\{ \underbrace{(\bar{\psi} \gamma^\mu A_\mu \psi)_{x_1}}_{\text{}} (\bar{\psi} \gamma^\nu A_\nu \psi)_{x_2} \right\} \right]$$

$$S_C^{(2)} = - \frac{e^2}{2!} \int d^4x_1 \int d^4x_2 N \left\{ \underbrace{(\bar{\psi} \gamma^\mu A_\mu \psi)_{x_1}}_{\text{}} (\bar{\psi} \gamma^\nu A_\nu \psi)_{x_2} \right\}$$

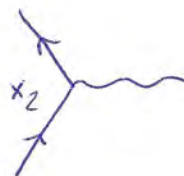
$$S_D^{(2)} = - \frac{e^2}{2!} \int d^4x_1 \int d^4x_2 \left[N \left\{ \underbrace{(\bar{\psi} \gamma^\mu A_\mu \psi)_{x_1}}_{\text{}} (\bar{\psi} \gamma^\nu A_\nu \psi)_{x_2} \right\} + N \left\{ \underbrace{(\bar{\psi} \gamma^\mu A_\mu \psi)_{x_1}}_{\text{}} (\bar{\psi} \gamma^\nu A_\nu \psi)_{x_2} \right\} \right]$$

$$S_E^{(2)} = - \frac{e^2}{2!} \int d^4x_1 \int d^4x_2 N \left\{ \underbrace{(\bar{\psi} \gamma^\mu A_\mu \psi)_{x_1}}_{\text{}} (\bar{\psi} \gamma^\nu A_\nu \psi)_{x_2} \right\}$$

$$S_F^{(2)} = - \frac{e^2}{2!} \int d^4x_1 \int d^4x_2 N \left\{ \underbrace{(\bar{\psi} \gamma^\mu A_\mu \psi)_{x_1}}_{\text{}} (\bar{\psi} \gamma^\nu A_\nu \psi)_{x_2} \right\}$$

↳

$$S_A^{(2)}$$



DISCONNECTED GRAPH $\Rightarrow$ DOES NOT CORRESPOND TO PHYSICAL PROCESS

↳

$$S_B^{(2)}$$

: 2 TERMS ARE IDENTICAL

$$x_1 \leftrightarrow x_2 \Rightarrow \text{FACTOR 2}$$

$$N \left\{ \underbrace{(\bar{\Psi} \gamma^\mu A_\mu \Psi)_{x_1}}_{\text{photon}} (\bar{\Psi} \gamma^\nu A_\nu \Psi)_{x_2} \right\}$$

CORRESPONDS e.g. TO COMPTON SCATTERING:

$$|i\rangle = |\gamma(\vec{k}_1, \lambda_1) e^-(\vec{p}_1, s_1)\rangle \quad \underbrace{\gamma + e^- \rightarrow \gamma + e^-}$$

$$= \sqrt{2|\vec{k}_1|} \sqrt{2E_{p_1}} \underbrace{a^\dagger(\vec{k}_1, \lambda_1)}_{\gamma} \underbrace{a^\dagger(\vec{p}_1, s_1)}_{e^-} |0\rangle$$

$$|f\rangle = |\gamma(\vec{k}_2, \lambda_2) e^-(\vec{p}_2, s_2)\rangle$$

$$= \sqrt{2|\vec{k}_2|} \sqrt{2E_{p_2}} a^\dagger(\vec{k}_2, \lambda_2) a^\dagger(\vec{p}_2, s_2) |0\rangle$$

$$\Psi^\dagger(x) |e^-(\vec{p}_1, s_1)\rangle = e^{-i\vec{p}_1 \cdot x} u(\vec{p}_1, s_1) |0\rangle$$

$$\langle e^-(\vec{p}_2, s_2) | \bar{\Psi}(x) = e^{+i\vec{p}_2 \cdot x} \bar{u}(\vec{p}_2, s_2) \langle 0|$$

$$A_{\nu}^{+}(x) |\gamma(\vec{k}_1, \lambda_1)\rangle = \varepsilon_{\nu}(\vec{k}_1, \lambda_1) e^{-ik_1 \cdot x} |0\rangle$$

$$\langle \gamma(\vec{k}_2, \lambda_2) | A_{\mu}^{-}(x) = \varepsilon_{\mu}^{*}(\vec{k}_2, \lambda_2) e^{+ik_2 \cdot x} \langle 0 |$$

°° FOR $\langle f | S^{(2)} | i \rangle \quad \gamma + e^{-} \rightarrow \gamma + e^{-}$

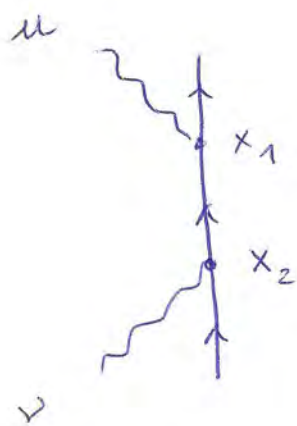
THERE ARE 2 CONTRIBUTIONS:

γ ABSORBED AT x_2 OR x_1

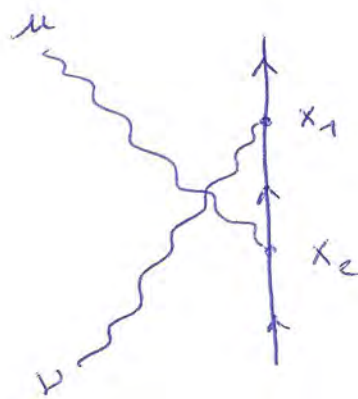
$$\langle f | S^{(2)} | i \rangle = \langle f | S_a + S_b | i \rangle$$

$$S_a = -e^2 \int d^4 x_1 \int d^4 x_2 \bar{\Psi}(x_1) \gamma^{\mu} i S_F(x_1 - x_2) \gamma^{\nu} \Psi^{\dagger}(x_2) \cdot A_{\mu}^{-}(x_1) A_{\nu}^{+}(x_2)$$

$$S_b = -e^2 \int d^4 x_1 \int d^4 x_2 \bar{\Psi}(x_1) \gamma^{\nu} i S_F(x_1 - x_2) \gamma^{\mu} \Psi^{\dagger}(x_2) \cdot A_{\nu}^{+}(x_1) A_{\mu}^{-}(x_2)$$



(a)



(b)

$$\leadsto \langle f | S_a | i \rangle = -e^2 \varepsilon_\nu(\bar{k}_1, \lambda_1) \varepsilon_\mu^*(\bar{k}_2, \lambda_2)$$

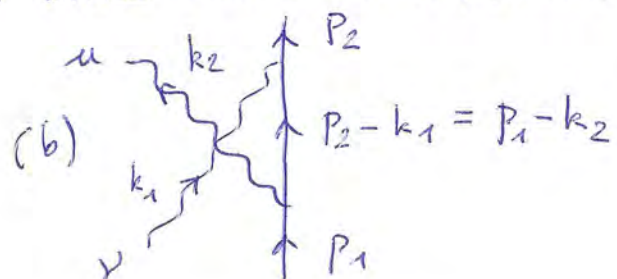
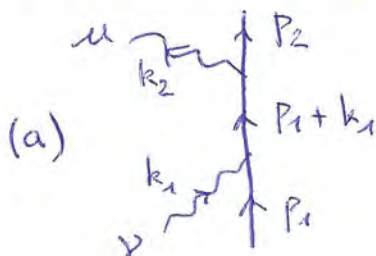
$$\begin{aligned} & \cdot \int \frac{d^4 k}{(2\pi)^4} \bar{U}(p_2, s_2) \gamma^\mu \frac{i(\not{k} + m)}{k^2 - m^2 + i\varepsilon} \gamma^\nu U(p_1, s_1) \\ & \cdot \int d^4 x_1 e^{-i(k - p_2 - k_2) \cdot x_1} \\ & \cdot \int d^4 x_2 e^{-i(p_1 + k_1 - k) \cdot x_2} \end{aligned}$$

$$\begin{aligned} \langle f | S_a | i \rangle &= (2\pi)^4 \delta^4(k_1 + p_1 - k_2 - p_2) \\ & \cdot \varepsilon_\nu(\bar{k}_1, \lambda_1) \cdot \varepsilon_\mu^*(\bar{k}_2, \lambda_2) \\ & \cdot \bar{U}(p_2, s_2) (ie \gamma^\mu) \frac{i(\not{p}_1 + \not{k}_1 + m)}{(p_1 + k_1)^2 - m^2 + i\varepsilon} (ie \gamma^\nu) U(p_1, s_1) \end{aligned}$$

$\leadsto$ ANALOGOUSLY

$$\begin{aligned} \langle f | S_b | i \rangle &= (2\pi)^4 \delta^4(k_1 + p_1 - k_2 - p_2) \\ & \cdot \varepsilon_\nu(\bar{k}_1, \lambda_1) \cdot \varepsilon_\mu^*(\bar{k}_2, \lambda_2) \\ & \cdot \bar{U}(p_2, s_2) (ie \gamma^\nu) \frac{i(\not{p}_1 - \not{k}_2 + m)}{(p_1 - k_2)^2 - m^2 + i\varepsilon} (ie \gamma^\mu) U(p_1, s_1) \end{aligned}$$

$\leadsto$ FEYNMAN DIAGRAMS IN MOMENTUM SPACE FOR COMPTON SCATT.



↳

$S_c^{(2)}$

$$N \left\{ \underbrace{(\bar{\Psi} \gamma^\mu A_\mu \Psi)_{x_1} (\bar{\Psi} \gamma^\nu A_\nu \Psi)_{x_2}} \right\}$$

4 UNCONTRACTED FERMION FIELDS :

THIS TERM CONTRIBUTES TO PHYSICAL PROCESSES AS

$$e^- + e^- \rightarrow e^- + e^- \quad (\text{MÖLLER SCATTERING})$$

$$e^+ + e^+ \rightarrow e^+ + e^+$$

$$e^- + e^+ \rightarrow e^- + e^+ \quad (\text{BHABHA SCATTERING})$$

↪ CONSIDER $e^-(p_1, s_1) + e^-(p_2, s_2) \rightarrow e^-(p'_1, s'_1) + e^-(p'_2, s'_2)$

MÖLLER SCATTERING

$$|i\rangle = a^\dagger(p_2, s_2) a^\dagger(p_1, s_1) \sqrt{2E_{p_1}} \sqrt{2E_{p_2}} |0\rangle$$

$$|f\rangle = a^\dagger(p'_2, s'_2) a^\dagger(p'_1, s'_1) \sqrt{2E_{p'_1}} \sqrt{2E_{p'_2}} |0\rangle$$

FERMION FIELDS $\Psi = \Psi^+ + \Psi^-$

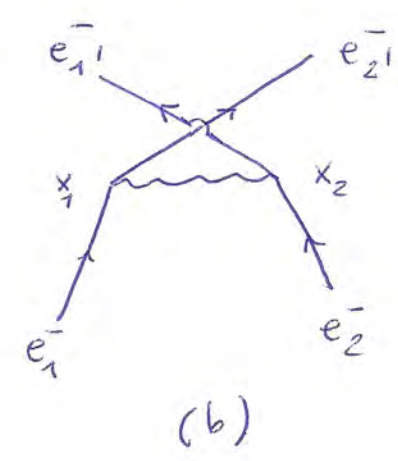
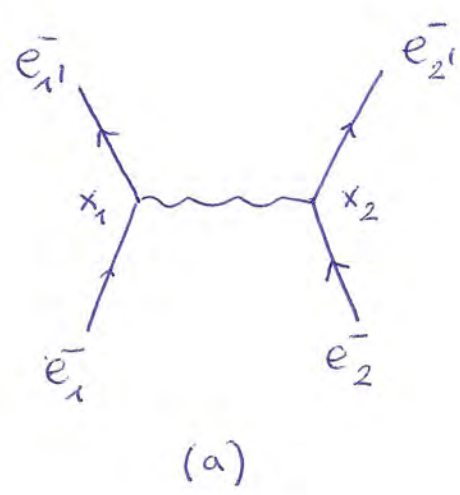
$$\bar{\Psi} = \bar{\Psi}^- + \bar{\Psi}^+$$

4 TERMS CONTRIBUTING

↳ 2 PAIRS WHICH ONLY DIFFER

BY INTERCHANGE $x_1 \leftrightarrow x_2 \Rightarrow \text{FACTOR 2}$

∴ IN GENERAL: THE FACTOR $\frac{1}{n!}$ IN $S^{(n)}$
CAN BE OMITTED BY CONSIDERING ONLY
TOPOLOGICALLY DIFFERENT FEYNMAN DIAGRAMS



$$\langle f | S^{(2)} | i \rangle = \langle f | S_a + S_b | i \rangle$$

$$\leadsto S_a = -e^2 \int d^4 x_1 \int d^4 x_2 N \left\{ \bar{\psi}^-(x_1) \gamma^\mu \psi^+(x_1) \cdot i D_{\mu\nu}(x_1 - x_2) \cdot \bar{\psi}^-(x_2) \gamma^\nu \psi^+(x_2) \right\}$$

$$\begin{aligned} & \langle f | S_a | i \rangle \\ &= \sqrt{2E_{p_1^-}} \sqrt{2E_{p_2^-}} \sqrt{2E_{p_1'^-}} \sqrt{2E_{p_2'^-}} \\ & \cdot (-e^2) \int d^4 x_1 \int d^4 x_2 \quad i D_{\mu\nu}(x_1 - x_2) \end{aligned}$$

$$\begin{aligned} & \langle 0 | \underline{a(p_1', s_1') a(p_2', s_2')} \cdot N \left\{ \underline{\bar{\psi}^-(x_1)} \gamma^\mu \underline{\psi^+(x_1)} \underline{\bar{\psi}^-(x_2)} \gamma^\nu \underline{\psi^+(x_2)} \right\} \\ & \cdot \underline{a^+(p_2, s_2) a^+(p_1, s_1)} | 0 \rangle \end{aligned}$$

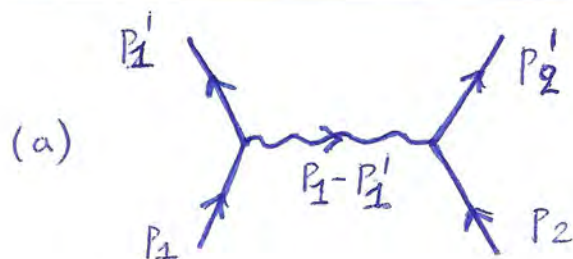
2 SIGN CHANGES NEEDED $\Rightarrow +1$

$$\langle f | S_a | i \rangle = (2\pi)^4 \delta^4(p_1 + p_2 - p'_1 - p'_2)$$

$$\cdot \frac{i (-g_{\mu\nu})}{(p_1 - p'_1)^2 + i\epsilon}$$

$$\cdot \bar{U}(p'_1, s'_1) (+ie \gamma^\mu) U(p_1, s_1)$$

$$\cdot \bar{U}(p'_2, s'_2) (+ie \gamma^\nu) U(p_2, s_2)$$



~> ANALOGOUSLY

$$\langle f | S_b | i \rangle = \sqrt{2E_{\vec{p}_1}} \sqrt{2E_{\vec{p}_2}} \sqrt{2E_{\vec{p}'_1}} \sqrt{2E_{\vec{p}'_2}}$$

$$\cdot (e^2) \int d^4x_1 \int d^4x_2 \quad i \mathcal{D}_{\mu\nu}(x_1 - x_2)$$

$$\cdot \langle 0 | \underline{a(p'_1, s'_1)} \underline{a(p'_2, s'_2)}$$

$$\cdot N \left\{ \underline{\bar{\Psi}(x_1)} \gamma^\mu \underline{\Psi^+(x_1)} \underline{\bar{\Psi}(x_2)} \gamma^\nu \underline{\Psi^+(x_2)} \right\}$$

$$\underline{a^+(p_2, s_2)} \underline{a^+(p_1, s_1)} | 0 \rangle$$

ODD # SIGN CHANGES NEEDED $\Rightarrow -1$

$\langle f | S_b | i \rangle$ OBTAINED FROM $\langle f | S_a | i \rangle$

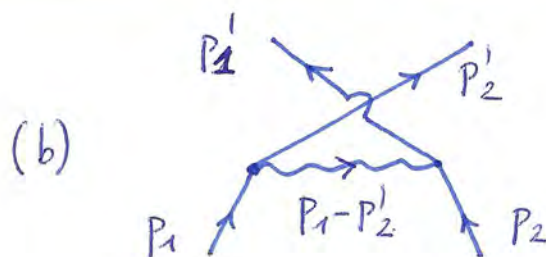
BY INTERCHANGING $p'_1, s'_1 \leftrightarrow p'_2, s'_2$

$$\langle f | S_b | i \rangle = (2\pi)^4 \delta^4(p_1 + p_2 - p'_1 - p'_2)$$

$$\cdot i \frac{(-g_{\mu\nu})}{(p_1 - p'_2)^2 + i\epsilon}$$

$$\cdot (-1) \bar{U}(p'_2, s'_2) (ie \gamma^\mu) U(p_1, s_1)$$

$$\cdot \bar{U}(p'_1, s'_1) (ie \gamma^\nu) U(p_2, s_2)$$



• • MØLLER SCATTERING (SUM OF (a) + (b))

$$\langle f | S^{(2)} | i \rangle = (2\pi)^4 \delta^4(p_1 + p_2 - p'_1 - p'_2)$$

$$\cdot \left\{ (-i) \frac{1}{(p_1 - p'_1)^2 + i\epsilon} \right.$$

$$\cdot \bar{U}(p'_1, s'_1) (ie \gamma^\mu) U(p_1, s_1)$$

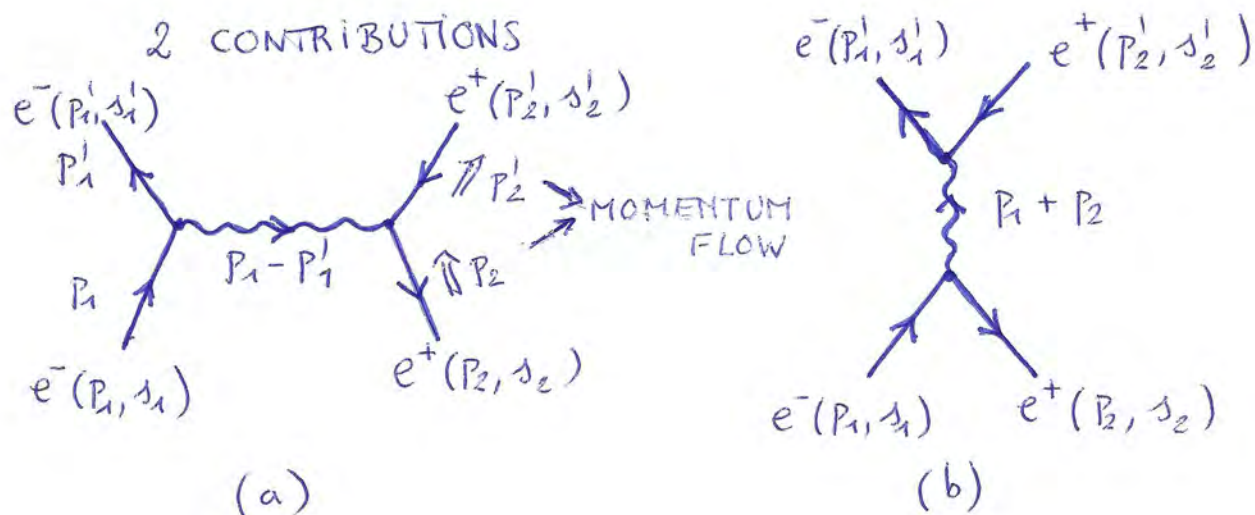
$$\cdot \bar{U}(p'_2, s'_2) (ie \gamma_\mu) U(p_2, s_2)$$

$$- (1' \leftrightarrow 2') \left. \right\}$$

↑
DUE TO ANTISYMMETRY OF e^-e^- STATE

→ BHABHA SCATTERING (EXERCISE)

$$e^-(p_1, s_1) + e^+(p_2, s_2) \rightarrow e^-(p'_1, s'_1) + e^+(p'_2, s'_2)$$



NOTE : FOR e^+ (ANTI-PARTICLE) :
CONVENTION IS TO DRAW FERMION LINE
OPPOSITE TO MOMENTUM FLOW

FOR e^- (PARTICLE) :
MOMENTUM FLOW ALONG FERMION LINE

e.g.

$$\begin{aligned} \langle f | S_b | i \rangle &= (2\pi)^4 \delta^4(p_1 + p_2 - p'_1 - p'_2) \\ &\cdot \frac{(-i g_{\mu\nu})}{(p_1 + p_2)^2 + i\epsilon} \\ &\cdot \bar{u}(p'_1, s'_1) (ie \gamma^\mu) v(p_2, s_2) \\ &\cdot \bar{v}(p_2, s_2) (ie \gamma^\nu) u(p_1, s_1) \end{aligned}$$

↳

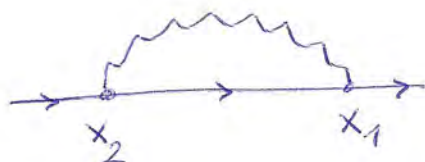
$$S_D^{(2)}$$

$$N \left\{ \underbrace{(\bar{\Psi} \gamma^\mu A_\mu \Psi)_{x_1} (\bar{\Psi} \gamma^\nu A_\nu \Psi)_{x_2}} \right\}$$

2 TERMS ARE EQUIVALENT $x_1 \leftrightarrow x_2 \Rightarrow$ FACTOR 2

$$S_D^{(2)}(e^- \rightarrow e^-) = -e^2 \int d^4 x_1 \int d^4 x_2 i D_{\mu\nu}(x_1 - x_2)$$

$$\bar{\Psi}(x_1) \gamma^\mu i S_F(x_1 - x_2) \gamma^\nu \Psi(x_2)$$



$$\text{FOR } |i\rangle = |e^-(p, s)\rangle = \sqrt{2E_{\vec{p}}} a^+(\vec{p}, s) |0\rangle$$

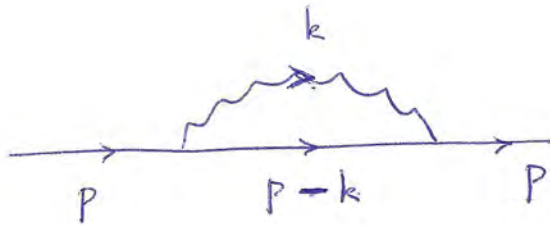
$$|f\rangle = |e^-(p', s')\rangle = \sqrt{2E_{\vec{p}'}} a^+(\vec{p}', s') |0\rangle$$

$$\langle f | S^{(2)} | i \rangle = -e^2 \int \frac{d^4 k}{(2\pi)^4} \frac{i(-g_{\mu\nu})}{k^2 + i\epsilon}$$

$$\int d^4 x_1 \int d^4 x_2 \bar{U}(p', s') \gamma^\mu i S_F(x_1 - x_2) \gamma^\nu U(p, s)$$

$$\underbrace{e^{-i(k-p') \cdot x_1} e^{-i(p-k) \cdot x_2}}$$

$$e^{-i(p-p') \cdot x_1} e^{+i(p-k) \cdot (x_1 - x_2)}$$



$$\int d^4 x_1 \int d^4 x_2 e^{-i(p-p') \cdot x_1} e^{i(p-k) \cdot (x_1 - x_2)} S_F(x_1 - x_2)$$

$$= (2\pi)^4 \delta^4(p-p') \tilde{S}_F(p-k)$$

∴

$$\langle f | S^{(2)} | i \rangle$$

$$= (2\pi)^4 \delta^4(p-p')$$

$$\cdot \bar{U}(p', s') \int \frac{d^4 k}{(2\pi)^4} (ie\gamma^\mu) \frac{i(\not{p}-\not{k}+m)}{(p-k)^2 - m^2 + i\epsilon} (ie\gamma^\nu)$$

$$\cdot \frac{i(-g_{\mu\nu})}{k^2 + i\epsilon} \cdot U(p, s)$$

1- LOOP INTEGRAL

" e^- SELF-ENERGY "

$$L \rightarrow S_E^{(2)}$$

$$N \left\{ \underbrace{(\bar{\Psi} \gamma^\mu A_\mu \Psi)_{x_1} (\bar{\Psi} \gamma^\nu A_\nu \Psi)_{x_2}} \right\}$$

EXTERNAL PHOTON



$$S_E^{(2)}(\gamma \rightarrow \gamma) = (-e^2) \int d^4 x_1 \int d^4 x_2 A_\mu^-(x_1) A_\nu^+(x_2)$$

$$\left(\gamma^\mu \right)_{\alpha\beta} \left(i S_F(x_1 - x_2) \right)_{\beta\gamma} \left(\gamma^\nu \right)_{\gamma\delta}$$

$$\cdot (-1) \left(i S_F(x_2 - x_1) \right)_{\delta\alpha}$$



ODD # PERMUTATIONS

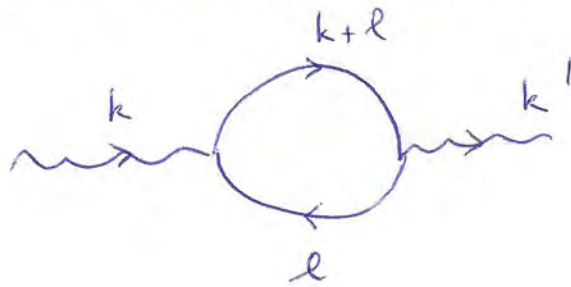
$$\left(\gamma^\mu \right)_{\alpha\beta} \left(S_F \right)_{\beta\gamma} \left(\gamma^\nu \right)_{\gamma\delta} \left(S_F \right)_{\delta\alpha}$$

$$= \text{Tr} \left\{ \gamma^\mu S_F \gamma^\nu S_F \right\}$$



TRACE IN DIRAC SPACE

~ IN MOMENTUM SPACE



$$|i\rangle = |\chi(\vec{k}, \lambda)\rangle$$

$$|f\rangle = |\chi(\vec{k}', \lambda')\rangle$$

$$\langle f | S^{(2)} | i \rangle$$

$$= (2\pi)^4 \delta^4(k - k')$$

$$\cdot \epsilon_\mu(\vec{k}, \lambda) \epsilon_\nu^*(\vec{k}', \lambda')$$

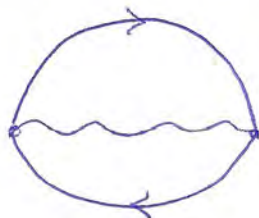
$$\cdot (-1) \int \frac{d^4 \ell}{(2\pi)^4} \text{Tr} \left\{ (ie\gamma^\nu) \frac{i(\not{k} + \not{\ell} + m)}{(k+\ell)^2 - m^2 + i\epsilon} (ie\gamma^\mu) \frac{i(\not{\ell} + m)}{\ell^2 - m^2 + i\epsilon} \right\}$$

1-LOOP INTEGRAL

χ -POLARIZATION (SELF-ENERGY)

$$\hookrightarrow S_F^{(2)} \quad N \left\{ \underbrace{(\bar{\Psi} \gamma^\mu A_\mu \Psi)_{x_1} (\bar{\Psi} \gamma^\nu A_\nu \Psi)_{x_2}} \right\}$$

FULLY CONTRACTED DIAGRAM



VACUUM DIAGRAM
NO EXTERNAL LINES.

$\hookrightarrow$ NO TRANSITIONS

⇒ FEYNMAN RULES FOR QED

↳ IN MOMENTUM SPACE

TO CALCULATE $\langle f | S^{(n)} | i \rangle$

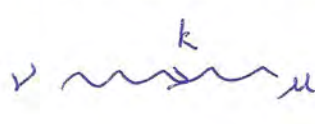
DRAW ALL CONNECTED, TOPOLOGICALLY DIFFERENT DIAGRAMS BETWEEN $|i\rangle$ AND $|f\rangle$ TO ORDER n WITH RULES

1) FOR EACH VERTEX



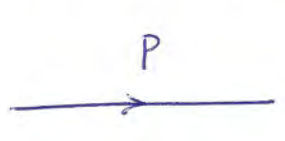
$$-iq\gamma^\mu \quad (\text{WITH } q = -e < 0 \text{ FOR } e^-)$$

2) FOR EACH INTERNAL PHOTON LINE



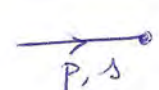
$$iD_F^{\mu\nu}(k) = \frac{i(-g^{\mu\nu})}{k^2 + i\epsilon}$$

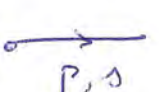
3) FOR EACH INTERNAL FERMION LINE

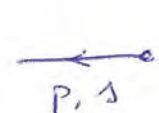


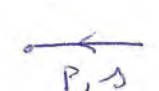
$$iS_F(p) = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$$

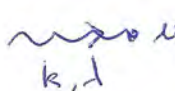
4) FOR EXTERNAL LINES :

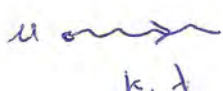
↳ INITIAL e^-  $U(\vec{p}, s)$

↳ FINAL e^-  $\bar{U}(\vec{p}, s)$

↳ INITIAL e^+  $\bar{v}(\vec{p}, s)$

↳ FINAL e^+  $v(\vec{p}, s)$

↳ INITIAL γ  $\epsilon_\mu(k, \lambda)$

↳ FINAL γ  $\epsilon_\mu^*(k, \lambda)$

5) FERMION LINE

SPINOR FACTORS ARE ORDERED SUCH THAT
WHEN READING FROM RIGHT TO LEFT
CORRESPONDS TO FOLLOWING FERMION LINE
ALONG ARROW

6) FOR CLOSED FERMION LOOP $\Rightarrow$ FACTOR (-1)

7) AT EACH VERTEX: 4-MOMENTUM CONSERVATION
FOR EACH 4-MOMENTUM WHICH IS NOT
FIXED BY ENERGY-MOMENTUM CONSERVATION

$$\int \frac{d^4 k}{(2\pi)^4}$$

8) GLOBAL ENERGY-MOMENTUM CONSERVATION

$$(2\pi)^4 \delta^4 \left(\sum_i p_i - \sum_f p_f \right)$$

9) PHASE FACTOR $\delta_p = \pm 1$

DEPENDING ON WHETHER ONE NEEDS EVEN (ODD)
NUMBER OF INTERCHANGES OF FERMION OPERATORS
TO BRING THEM IN NORMAL ORDER

e.g. e^-e^- SCATTERING, e^-e^+ SCATTERING
WHERE DIFFERENT DIAGRAMS CONTRIBUTE