Theoretical Physics 6a (QFT): SS 2025
Exercise sheet 3

28.04.2025

(0 points) How much time did it take you to solve this exercise sheet?

Exercise 1 (40 points): Dirac matrices

(a)(40 points) Without using an explicit representation of the gamma matrices, prove the following identities:

$$\begin{aligned}
\gamma_\mu \gamma^\mu &= 4, \\
\mathrm{Tr}[\gamma^\mu \gamma^\nu] &= 4 g^{\mu\nu}, \\
\mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^\sigma \gamma^\rho] &= 4(g^{\mu\nu} g^{\sigma\rho} - g^{\mu\sigma} g^{\nu\rho} + g^{\mu\rho} g^{\sigma\nu}), \\
\gamma_5 &= \frac{i}{4!} \epsilon_{\mu\nu\sigma\rho} \gamma^\mu \gamma^\nu \gamma^\sigma \gamma^\rho, \\
\gamma_5^2 &= 1, \\
[\gamma_5, \gamma^\mu]_+ &= 0, \\
\mathrm{Tr}[\gamma_5] &= 0, \\
\mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma_5] &= 0, \\
\mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^\sigma \gamma^\rho \gamma_5] &= 4i \epsilon^{\mu\nu\sigma\rho}, \\
\mathrm{Tr}[\gamma^{\mu_1} ... \gamma^{\mu_n}] &= 0, \text{ if } n \text{ is odd},
\end{aligned}$$

Where $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\epsilon_{0123} = 1$.

Exercise 2 (60 points): Klein-Gordon Green's function

Consider the differential equation for the Klein-Gordon Green's function

$$(\partial_\mu \partial^\mu + m^2) G(x - x') = -\delta^{(4)}(x - x') \,. \tag{1}$$

(a)(10 points) Show that, using a Fourier transformation, the solution of (1) is formally given as

$$G(x - x') = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{e^{-ik(x-x')}}{k^2 - m^2 \pm i\epsilon}. \tag{2}$$

Using the Green's function, show that

$$\phi(x) = -\int \mathrm{d}^4x'\, G(x-x')\rho(x')$$

is a solution of

$$(\partial_\mu \partial^\mu + m^2)\phi(x) = \rho(x)\,.$$

The expression of (2) can only be evaluated using a contour deformation for the integration over k^0, due to the poles on the real axes at

$$k^0 = \pm\sqrt{k^2+m^2}\,.$$

Consider the three contours in figures 1, 2 and 3.

(b)(10 points) Show, that the retarded Green's function given by

$$G_{\mathrm{ret}}(x-x') \equiv \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{e^{-ik(x-x')}}{(k_0+i\epsilon)^2 - \vec{k}^2 - m^2},$$

is zero for $t < t'$ and that the advanced Green's function given by

$$G_{\mathrm{adv}}(x-x') \equiv \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{e^{-ik(x-x')}}{(k_0-i\epsilon)^2 - \vec{k}^2 - m^2},$$

is zero for $t > t'$.

(c)(20 points) Show, that the Feynman Green's function

$$G_{\mathrm{F}}(x-x') \equiv \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{e^{-ik(x-x')}}{k^2 - m^2 + i\epsilon},$$

defines the integration contour of figure 3, i.e. has its poles at

$$k_0 = \pm(\sqrt{k^2+m^2} - i\epsilon)\,.$$

Show, that after evaluating the contour integration,

$$\begin{aligned} G_{\mathrm{F}}(x-x') &= (-i)\int \frac{\mathrm{d}^3k}{(2\pi)^3 2E_k} \Big[\Theta(t-t')e^{-ik(x-x')} + \Theta(t'-t)e^{ik(x-x')}\Big]_{k_0=E_k} \\ &= (-i)\langle 0|T\phi(x)\phi(x')|0\rangle\,. \end{aligned}$$

(d)(20 points) Evaluate the spacelike Klein-Gordon correlator, i.e. $\langle 0|\phi(x)\phi(y)|0\rangle$ with $(x-y)^2 < 0$, explicitly and express it in terms of a modified Bessel function of second kind

$$K_n(z) \equiv \frac{\sqrt{\pi}}{\Gamma(n+\frac{1}{2})}\left(\frac{z}{2}\right)^n \int_1^\infty \mathrm{d}\rho\, e^{-\rho z}(\rho^2-1)^{n-1/2}\,.$$

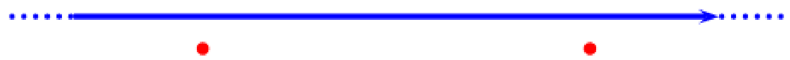

Figure 1: Contour for retarded Green's function with poles at $k_0 = \pm\sqrt{k^2 + m^2} - i\epsilon$

Figure 2: Contour for advanced Green's function with poles at $k_0 = \pm\sqrt{k^2 + m^2} + i\epsilon$

Figure 3: Contour for Feynman Green's function with poles at $k_0 = \pm(\sqrt{k^2 + m^2} - i\epsilon)$