Theoretical Physics 6a (QFT): SS 2025
Exercise sheet 2

21.04.2025

(0 points) How much time did it take you to solve this exercise sheet?

Exercise 1 (50 points): Scalar theory with $SO(2)$ invariance

Consider the following Lagrangian density of two real scalar fields $\phi_1(x)$, $\phi_2(x)$:

$$\mathcal{L} = \frac{1}{2}\left[(\partial\phi_1)^2 + (\partial\phi_2)^2\right] - \frac{m^2}{2}\left(\phi_1^2 + \phi_2^2\right) - \frac{\lambda}{4!}\left(\phi_1^2 + \phi_2^2\right)^2$$

(a)(10 points) Identify the corresponding equations of motion.

(b)(10 points) Show that the above Lagrangian is invariant under the transformations

$$\begin{aligned} \phi_1 \to \phi_1' &= \phi_1 \cos\theta - \phi_2 \sin\theta\,, \\ \phi_2 \to \phi_2' &= \phi_1 \sin\theta + \phi_2 \cos\theta\,. \end{aligned}$$

(c)(10 points) Calculate the Noether current j_μ and show explicitly that its divergence vanishes for fields ϕ_i which satisfy the equations of motion.

(d)(10 points) Show explicitly that the Noether charge Q is a conserved quantity, assuming the surface integral $\int \mathrm{d}S\, \vec{n} \cdot \vec{j}$ vanishes.

(e)(10 points) Construct the Hamiltonian density $\mathcal{H}$.

Exercise 2 (50 points): Real Klein-Gordon field

Using the normal mode expansion of the real Klein-Gordon field

$$\phi(\vec{x}, t) \quad = \quad \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{k}}}} \left[a(\vec{k})\, e^{-ik\cdot x} + a^\dagger(\vec{k})\, e^{ik\cdot x}\right]$$

with $k^0 = E_{\vec{k}} \equiv \sqrt{\vec{k}^{\,2} + m^2}$, and the equal-time commutation relations

$$\begin{aligned}
\left[\phi(\vec{x},t), \phi(\vec{x}',t)\right] &= 0, \\
\left[\dot{\phi}(\vec{x},t), \dot{\phi}(\vec{x}',t)\right] &= 0, \\
\left[\phi(\vec{x},t), \dot{\phi}(\vec{x}',t)\right] &= i\,\delta^{(3)}(\vec{x} - \vec{x}'),
\end{aligned}$$

Show that:

(a)(25 points) the Hamiltonian $H = \int d^3\vec{x}\,\frac{1}{2}\left[\dot{\phi}^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right]$ takes the form

$$H = \int \frac{d^3\vec{k}}{(2\pi)^3}\, E_{\vec{k}} \left[a^\dagger(\vec{k})a(\vec{k}) + \frac{1}{2}\,(2\pi)^3\,\delta^3(0)\right],$$

(b)(25 points) the momentum $\vec{P} = -\int d^3\vec{x}\,\dot{\phi}\,\vec{\nabla}\phi$ takes the form

$$\vec{P} = \int \frac{d^3\vec{k}}{(2\pi)^3} \vec{k}\, a^\dagger(\vec{k})a(\vec{k}).$$