

QUANTUM THEORY OF  
RADIATION  
AND  
INTERACTION WITH MATTER

- 1) ABELIAN GAUGE THEORY (QED)
- 2) QUANTIZATION OF PHOTON FIELD
- 3) APPLICATIONS OF INTERACTION OF E.M. FIELD WITH MATTER (NON - RELATIVISTIC)

# 1) ABELIAN GAUGE THEORY (QED)

## ⇒ LOCAL PHASE TRANSFORMATION

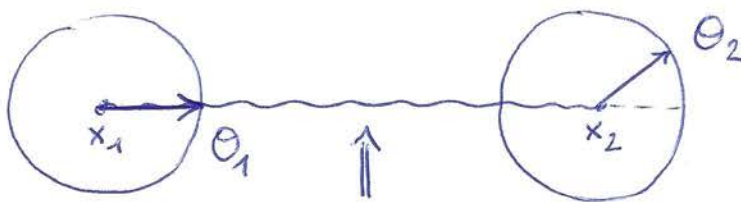
↳ FERMION (SPIN  $\frac{1}{2}$ )

$$\mathcal{L}_{\text{DIRAC}} = \bar{\Psi}(x) (i \gamma^\mu \partial_\mu - m) \Psi(x)$$

↳ LOCAL PHASE TF. :  $U(1)$  GROUP

$$\Psi(x) \xrightarrow{U(1)} e^{i\theta(x)} \Psi(x)$$

DEPENDS ON SPACE-TIME POINT



$$\Delta\theta = \theta_2 - \theta_1$$

SIGNAL WHICH COMMUNICATES PHASE DIFFERENCE ("PHOTON")

↳ TRANSFORMATION OF  $\mathcal{L}_{\text{DIRAC}}$  UNDER LOCAL PHASE TF.

$$\partial_\mu \Psi(x) \longrightarrow e^{i\theta(x)} \left[ \partial_\mu \Psi(x) + i(\partial_\mu \theta) \Psi \right]$$

$$\bar{\Psi} \gamma^\mu \partial_\mu \Psi \longrightarrow \bar{\Psi} \gamma^\mu \partial_\mu \Psi + i(\partial_\mu \theta) \bar{\Psi} \gamma^\mu \Psi$$

$$\bar{\Psi} \Psi \longrightarrow \bar{\Psi} \Psi$$

$$\mathcal{L}_{\text{DIRAC}} \longrightarrow \mathcal{L}_{\text{DIRAC}} - \underbrace{(\partial_\mu \theta) \bar{\Psi} \gamma^\mu \Psi}_{\text{VECTOR}}$$

$\hookrightarrow$  INVARIANCE UNDER LOCAL PHASE TF.

•  $\mathcal{L}_{\text{DIRAC}} \xrightarrow{U(1)} \mathcal{L}_{\text{DIRAC}} - \underbrace{(\partial_\mu \theta) \bar{\psi} \gamma^\mu \psi}_{\text{EXTRA TERM}} \sim (\partial_\mu \theta)$

DIFFERENCE IN PHASE BETWEEN DIFFERENT SPACE-TIME POINTS

- TO MAKE  $\mathcal{L}$  INVARIANT UNDER LOCAL PHASE TF  $\Rightarrow$  INTRODUCE VECTOR FIELD (GAUGE FIELD) WHICH COMPENSATES FOR DIFFERENT CHOICES OF PHASE BETWEEN DIFFERENT SPACE-TIME POINTS

INTRODUCE COVARIANT DERIVATIVE

$\partial_\mu \Rightarrow$  REPLACE  $\underline{\underline{D_\mu = \partial_\mu + ie A_\mu}}$

↑ VECTOR FIELD  
↑ ELECTRIC CHARGE  
↑ COUPLING OF VECTOR FIELD TO DIRAC FIELD

$A^\mu \xrightarrow{U(1)} A'^\mu$

CHOOSE  $A'^\mu$  SUCH THAT TOTAL  $\mathcal{L}$  IS INVARIANT UNDER LOCAL PHASE TF.

$$\bullet \quad D_\mu \psi = \partial_\mu \psi + ie A_\mu \psi$$

$$\xrightarrow{U(x)} e^{i\theta(x)} \left[ \partial_\mu \psi + i(\partial_\mu \theta) \psi + ie A'_\mu \psi \right]$$

$$= e^{i\theta(x)} \left[ \partial_\mu \psi + ie \left( A'_\mu + \frac{1}{e} \partial_\mu \theta \right) \psi \right]$$

$$\text{CHOOSE } A'_\mu = A_\mu - \frac{1}{e} \partial_\mu \theta$$

$$D_\mu \psi \xrightarrow{U(x)} e^{i\theta(x)} \left[ \partial_\mu \psi + ie A_\mu \psi \right]$$

$$\underline{\underline{D_\mu \psi \xrightarrow{U(x)} e^{i\theta(x)} D_\mu \psi}}$$

$$\bar{\psi} i \gamma^\mu D_\mu \psi \xrightarrow{U(x)} \bar{\psi} i \gamma^\mu D_\mu \psi$$

INVARIANT UNDER  
LOCAL  $U(1)$  TF !

∴

$$\text{LAGRANGIAN } \mathcal{L} = \bar{\psi} (i \gamma^\mu D_\mu - m) \psi$$

IS INVARIANT UNDER LOCAL PHASE TF :

$$\psi(x) \xrightarrow{U(x)} e^{i\theta(x)} \psi(x)$$

$$A^\mu \xrightarrow{U(x)} A^\mu - \frac{1}{e} \partial^\mu \theta$$



⇒ INTERACTION BETWEEN MATTER & GAUGE FIELD

↳ SYMMERY OF  $\mathcal{L} = \bar{\Psi} (i\gamma^\mu D_\mu - m) \Psi$  UNDER LOCAL  $U(1)$  IS CALLED ABELIAN GAUGE SYMMETRY

↓  
REFERS TO GAUGE GROUP  $U(1)$

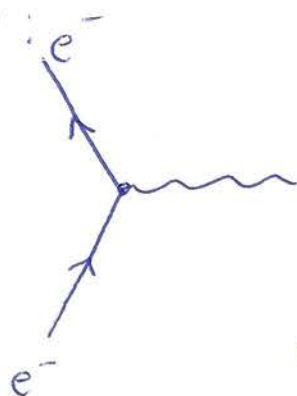
↳ COUPLING TO GAUGE (PHOTON) FIELD

$$\begin{aligned} \mathcal{L} &= \bar{\Psi} (i\gamma^\mu D_\mu - m) \Psi \\ &= \bar{\Psi} (i\gamma^\mu \partial_\mu - m) \Psi - e (\bar{\Psi} \gamma^\mu \Psi) A_\mu \\ &= \mathcal{L}_{\text{DIRAC}} + \mathcal{L}_{\text{INT}} \end{aligned}$$

↑  
INTERACTION LAGRANGIAN

$$\mathcal{L}_{\text{INT}} = -e (\bar{\Psi} \gamma^\mu \Psi) A_\mu$$

GRAPHICALLY:  $e^-$



$\gamma$  (DESCRIBED BY  $A_\mu$  FIELD).

$e$  IS STRENGTH OF COUPLING BETWEEN FIELDS

# ⇒ QUANTUM ELECTRODYNAMICS (QED)

↳ INTERACTION BETWEEN MATTER (SPIN  $1/2$ ) AND GAUGE FIELDS (PHOTONS) IS DESCRIBED BY

$$\mathcal{L}_{\text{QED}} = \bar{\Psi} (i\gamma^\mu \partial_\mu - m) \Psi - e \bar{\Psi} \gamma^\mu \Psi A_\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

- FREE MATTER FIELD  $\Psi$

$$\mathcal{L}_{\text{DIRAC}} = \bar{\Psi} (i\gamma^\mu \partial_\mu - m) \Psi$$

- INTERACTION BETWEEN MATTER & GAUGE FIELDS

$$\mathcal{L}_{\text{INT}} = - e \bar{\Psi} \gamma^\mu \Psi A_\mu$$

$$\equiv - \underbrace{J_{\text{em}}^\mu(x)} A_\mu$$

ELECTROMAGNETIC CURRENT

$$J_{\text{em}}^\mu = e \bar{\Psi} \gamma^\mu \Psi$$

- FREE PHOTON FIELD

FIELD TENSOR  $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$

UNDER  $U(1)$   $A^\mu \xrightarrow{U(1)} A^\mu - \frac{1}{e} \partial^\mu \Theta$

$F^{\mu\nu} \xrightarrow{U(1)} F^{\mu\nu}$

$$\mathcal{L}_{\text{em}} = - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

↳  $\mathcal{L}_{\text{QED}}$  IS INVARIANT UNDER LOCAL  $U(1)$  !

$$\left\| \begin{array}{l} \Psi(x) \xrightarrow{U(1)} e^{i\Theta(x)} \Psi(x) \\ A^\mu(x) \xrightarrow{U(1)} A^\mu(x) - \frac{1}{e} \partial^\mu \Theta \end{array} \right.$$

$U(1)$  GAUGE SYMMETRY

↳ FIELD EQUATIONS FOR  $A^\mu$

$$\frac{\partial \mathcal{L}}{\partial \Phi_r} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi_r)} = 0$$

↓ FOR  $\Phi_r = A_\nu$

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = - J_{em}^\nu$$

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu A_\nu)} = - \frac{1}{2} \cdot 2 \cdot F^{\mu\nu}$$

$$\boxed{\partial_\mu F^{\mu\nu} = J_{em}^\nu}$$

INHOMOGENEOUS  
MAXWELL EQ

↳ EM CURRENT : SOURCE TERM IN  
MAXWELL EQ.

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu (\partial_\mu A^\mu) = J_{em}^\nu$$

↳ GAUGE INVARIANCE

$$A^\mu \xrightarrow{U(1)} A^\mu - \frac{1}{e} \partial^\mu \theta$$

FREEDOM TO CONSTRAIN  $A^\mu$

POSSIBLE CHOICES OF GAUGE ARE

- $\partial_\mu A^\mu = 0$  : LORENZ GAUGE (COVARIANT)
- $\nabla \cdot \vec{A} = 0$  ( $A^0 = 0$ ) : COULOMB GAUGE (NON-COVARIANT)  
 ↳ USED FOR FREE FIELDS  
 ( $J_{em}^\nu = 0$ , ABSENCE OF SOURCES)
- $m_\mu A^\mu = 0$  : AXIAL GAUGE  
 WITH  $m_\mu m^\mu = -1$   
 e.g.  $m^\mu (0, 0, 0, 1)$

IN FOLLOWING WE WILL OFTEN USE

LORENZ GAUGE  $\partial_\mu A^\mu = 0$

$$\begin{array}{|l} \square A^\nu = J_{em}^\nu \\ \partial_\mu A^\mu = 0 \end{array} \quad \text{CONSTRAINT}$$

GAUGE INVARIANCE ENSURES THAT RESULTS FOR PHYSICAL OBSERVABLES ARE INVARIANT UNDER CHOICE OF GAUGE



## 2) QUANTIZATION OF PHOTON FIELD

⇒ PRESENCE OF CONSTRAINTS : ISSUES

$$\hookrightarrow \square A^\nu = J_{em}^\nu$$

$$\partial_\nu A^\nu = 0 \quad (\text{LORENZ GAUGE})$$

↙ HOW TO QUANTISE  $A^\nu$  IN PRESENCE OF CONSTRAINT  $\partial_\nu A^\nu = 0$ .

↙ ISSUE 1 : CANONICAL MOMENTA

$$A^\nu \rightarrow \pi^\nu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\nu} = -F^{0\nu}$$

$$\pi^0 = 0$$

$$\begin{aligned} \pi^i &= -F^{0i} = -\partial^0 A^i + \partial^i A^0 \\ &= \left( -\bar{\nabla} A^0 - \frac{\partial \bar{A}}{\partial t} \right)^i \\ &= \bar{E}^i \quad (\text{ELECTRIC FIELD}) \end{aligned}$$

FOR  $A^1, A^2, A^3$  OK

FOR  $A^0 \Rightarrow$  WE CANNOT IMPOSE COMMUTATION RELATIONS BECAUSE  $\pi^0 = 0$

## ⇒ GAUGE FIXING

↳ TO SOLVE ABOVE PROBLEMS WHEN QUANTIZING THE ABELIAN GAUGE THEORY  
CONSIDER ALTERNATIVE  $\mathcal{L}$

$$\mathcal{L}_{em} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad \text{DESCRIBES FREE PHOTON FIELD}$$

$$\mathcal{L}_{em} \rightarrow \boxed{\mathcal{L}'_{em} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial_\mu A^\mu)^2}$$

$\uparrow$   $\mathcal{L}_{em}$       +       $\uparrow$   $\mathcal{L}_{GF}$   
 (GAUGE FIXING)

↳ EQUATIONS OF MOTION

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu A_\nu)} = -F^{\mu\nu} - (\partial_\alpha A^\alpha) g^{\mu\nu}$$

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = 0$$

$$\partial_\mu F^{\mu\nu} + \partial^\nu (\partial_\alpha A^\alpha) = 0$$

$$\square A^\nu - \partial^\nu (\partial_\mu A^\mu) + \partial^\nu (\partial_\alpha A^\alpha) = 0$$

$\square A^\nu = 0$   $\rightarrow$  FIELD EQ. CORRESPONDING WITH  $\mathcal{L}'_{em}$   
CORRESPOND WITH CHOICE OF  
LORENZ GAUGE

↳ CANONICAL MOMENTA

$$A^\nu \rightarrow \pi^\nu = -F^{0\nu} - (\partial_\alpha A^\alpha) g^{0\nu}$$

$$\begin{cases} \pi^0 = -\partial_\alpha A^\alpha \\ \pi^i = -F^{0i} \end{cases}, \quad \begin{array}{l} \text{OK ISSUE 1} \\ \text{IF } \partial_\alpha A^\alpha \text{ CONSIDERED AS} \\ \text{OPERATOR} \end{array}$$

↳ EQUAL TIME COMMUTATION RELATIONS (ETCR)

TO QUANTIZE THE THEORY, WE IMPOSE

$$\left\{ \begin{array}{l} [A^\mu(\bar{x}, t), A^\nu(\bar{x}', t)]_- = 0 \\ [\pi^\mu(\bar{x}, t), \pi^\nu(\bar{x}', t)]_- = 0 \\ [A^\mu(\bar{x}, t), \pi^\nu(\bar{x}', t)]_- = i g^{\mu\nu} \delta^3(\bar{x} - \bar{x}') \end{array} \right.$$

WITH  $[A, B]_- = AB - BA$  COMMUTATOR

SPIN 1 FIELD DESCRIBES BOSONS (PHOTONS)  
 $A^\mu, \pi^\mu$  CONSIDERED AS OPERATORS

$$\rightsquigarrow [A^\mu(\bar{x}, t), \pi^0(\bar{x}', t)]_- = i g^{\mu 0} \delta^3(\bar{x} - \bar{x}')$$

$$= -[A^\mu(\bar{x}, t), \dot{A}^0(\bar{x}', t)]_- - \vec{\nabla}_{\bar{x}'} \cdot [A^\mu(\bar{x}, t), \vec{A}(\bar{x}', t)]_-$$

0

$$\rightsquigarrow [A^\mu(\bar{x}, t), \underbrace{\Pi^i(\bar{x}', t)}_{-\dot{A}^i + \partial^i A^0}]_- = i g^{\mu i} \delta^3(\bar{x} - \bar{x}')$$

$$= - [A^\mu(\bar{x}, t), \dot{A}^i(\bar{x}', t)]_- - \bar{v}_{x'}^i \cdot \underbrace{[A^\mu(\bar{x}, t), A^0(\bar{x}', t)]}_0$$

$$\therefore \parallel [A^\mu(\bar{x}, t), \dot{A}^\nu(\bar{x}', t)]_- = -i g^{\mu\nu} \delta^3(\bar{x} - \bar{x}')$$

$$\hookrightarrow \mu = 0: [A^\mu(\bar{x}, t), \dot{A}^0(\bar{x}', t)]_- = -i \delta^3(\bar{x} - \bar{x}')$$

$$\hookrightarrow \mu = i: [A^\mu(\bar{x}, t), \dot{A}^i(\bar{x}', t)]_- = +i \delta^3(\bar{x} - \bar{x}')$$

$\hookrightarrow$  PHOTON PROPAGATOR (IN LORENZ GAUGE)

$$\square A^\mu = 0$$

$$\underbrace{(-k^2 g^{\mu\nu})}_{\text{INVERSE EXISTS}} A_\nu = 0$$

$$(-k^2 g^{\mu\nu}) \cdot (A g_{\nu\lambda} + B k_\nu k_\lambda) = g^{\mu\lambda}$$

$$A = -\frac{1}{k^2}$$

$$B = 0$$

PHOTON PROPAGATOR

$$\Rightarrow \sim \frac{-g^{\mu\nu}}{k^2 + i\epsilon} \quad \begin{array}{c} k \\ \text{---} \\ \nu \quad \mu \end{array}$$



## ⇒ NORMAL MODE EXPANSION OF $A^\mu(\vec{x}, t)$

↳ UPON QUANTIZATION

CLASSICAL FIELDS  $A^\mu$

ARE RE-INTERPRETED AS FIELD OPERATORS  $\hat{A}^\mu$   
THAT SATISFY ETCR

(NOTE: FOR SIMPLICITY WE WILL DROP  $\hat{\phantom{A}}$  NOTATION  
IN QFT THE FIELDS ARE UNDERSTOOD  
AS OPERATORS)

↳ POLARIZATION VECTORS:

A NORMAL MODE SOLUTION IS CHARACTERIZED

BY  $\sim e^{ik \cdot x} \epsilon^\mu(\vec{k}, \lambda)$   $\lambda = 0, 1, 2, 3$

WITH  $k = (\omega_{\vec{k}}, \underbrace{0, 0, |\vec{k}|}_{\vec{k}})$   
 $\omega_{\vec{k}} = |\vec{k}|$

CONVENIENT CHOICE  
 $\vec{k} = |\vec{k}| \vec{e}_z$

$\epsilon^\mu(\vec{k}, \lambda=0) = (1, 0, 0, 0)$  SCALAR POL.

$\epsilon^\mu(\vec{k}, \lambda=1) = (0, 1, 0, 0)$

$\epsilon^\mu(\vec{k}, \lambda=2) = (0, 0, 1, 0)$  } TRANSVERSE POL.  
 $0 = \vec{k} \cdot \vec{\epsilon}(\vec{k}, \lambda = \frac{1}{2})$

$\epsilon^\mu(\vec{k}, \lambda=3) = (0, 0, 0, 1)$  LONGITUDINAL POL.

NORMALIZATION  $\left\| \epsilon^\mu(\vec{k}, \lambda) \epsilon_\mu^*(\vec{k}, \lambda') = -\sum_\lambda \delta_{\lambda\lambda'} \right.$   
 $(\sum_0 = -1, \sum_i = +1)$

↳ NORMAL MODE EXPANSION

$$A^\mu(\vec{x}, t) = \int \frac{d^3\vec{k}}{(2\pi)^3 \sqrt{2\omega_{\vec{k}}}} \sum_{\lambda=0}^3 \left\{ a(\vec{k}, \lambda) \varepsilon^\mu(\vec{k}, \lambda) e^{-i\vec{k}\cdot\vec{x}} + a^\dagger(\vec{k}, \lambda) \varepsilon^{\mu*}(\vec{k}, \lambda) e^{+i\vec{k}\cdot\vec{x}} \right\}$$

$$\omega_{\vec{k}} = |\vec{k}|$$

PARTICLE WITH MASS 0 (PHOTON)  
(→ MOVES WITH SPEED OF LIGHT)

$a(\vec{k}, \lambda)$  ANNIHILATES PHOTON WITH MOMENTUM  $\vec{k}$   
& POLARIZATION  $\lambda$

$a^\dagger(\vec{k}, \lambda)$  CREATES PHOTON " "

⇒ COMMUTATION RELATIONS FOR  $a, a^\dagger$

$$\hookrightarrow \int d^3\vec{x} e^{-i\vec{k}\cdot\vec{x}} \left\{ A^\mu(\vec{x}, t) + \frac{i}{\omega_{\vec{k}}} \dot{A}^\mu(\vec{x}, t) \right\}$$

$$= 2 \frac{1}{\sqrt{2\omega_{\vec{k}}}} \sum_{\lambda=0}^3 a(\vec{k}, \lambda) \varepsilon^\mu(\vec{k}, \lambda) e^{-i\omega_{\vec{k}}t}$$

$$\hookrightarrow \int d^3\vec{x} e^{+i\vec{k}\cdot\vec{x}} \left\{ A^\mu(\vec{x}, t) - \frac{i}{\omega_{\vec{k}}} \dot{A}^\mu(\vec{x}, t) \right\}$$

$$= 2 \frac{1}{\sqrt{2\omega_{\vec{k}}}} \sum_{\lambda=0}^3 a^\dagger(\vec{k}, \lambda) \varepsilon^{\mu*}(\vec{k}, \lambda) e^{i\omega_{\vec{k}}t}$$

$$\downarrow \text{ USING } \varepsilon^\mu(\vec{k}, \lambda) \varepsilon_\mu^*(\vec{k}, \lambda') = -\delta_{\lambda\lambda'}$$

$$a(\vec{k}, \lambda) = -\delta_{\lambda} \frac{1}{2} \sqrt{2\omega_{\vec{k}}} e^{+i\omega_{\vec{k}}t} \int d^3\vec{x} e^{-i\vec{k}\cdot\vec{x}} \varepsilon_\mu^*(\vec{k}, \lambda) \left\{ A^\mu + \frac{i}{\omega_{\vec{k}}} \dot{A}^\mu \right\}$$

$$a^\dagger(\vec{k}, \lambda) = -\delta_{\lambda} \frac{1}{2} \sqrt{2\omega_{\vec{k}}} e^{-i\omega_{\vec{k}}t} \int d^3\vec{x} e^{+i\vec{k}\cdot\vec{x}} \varepsilon_\mu(\vec{k}, \lambda) \left\{ A^\mu - \frac{i}{\omega_{\vec{k}}} \dot{A}^\mu \right\}$$

$$\hookrightarrow [a(\vec{k}, \lambda), a^\dagger(\vec{k}', \lambda')] ]$$

$$= \delta_{\lambda} \delta_{\lambda'} \frac{1}{4} (2\omega_{\vec{k}})^{1/2} (2\omega_{\vec{k}'})^{1/2} e^{i(\omega_{\vec{k}} - \omega_{\vec{k}'})t} \varepsilon_\mu^*(\vec{k}, \lambda) \varepsilon_\nu(\vec{k}', \lambda')$$

$$\int d^3\vec{x} \int d^3\vec{x}' e^{-i\vec{k}\cdot\vec{x}} e^{+i\vec{k}'\cdot\vec{x}'}$$

$$\cdot \left[ A^\mu(\vec{x}, t) + \frac{i}{\omega_{\vec{k}}} \dot{A}^\mu(\vec{x}, t), A^\nu(\vec{x}', t) - \frac{i}{\omega_{\vec{k}'}} \dot{A}^\nu(\vec{x}', t) \right]$$

$$= \frac{1}{\omega_{\vec{k}'}} g^{\mu\nu} \delta^3(\vec{x} - \vec{x}') - \frac{1}{\omega_{\vec{k}}} g^{\mu\nu} \delta^3(\vec{x} - \vec{x}')$$

$$= \delta_{\lambda} \delta_{\lambda'} \frac{1}{4} (2\omega_{\vec{k}})^{1/2} (2\omega_{\vec{k}'})^{1/2} (-1) \left( \frac{1}{\omega_{\vec{k}}} + \frac{1}{\omega_{\vec{k}'}} \right) e^{i(\omega_{\vec{k}} - \omega_{\vec{k}'})t}$$

$$\cdot \varepsilon_\mu^*(\vec{k}, \lambda) \varepsilon^\mu(\vec{k}', \lambda') \underbrace{\int d^3\vec{x} e^{-i(\vec{k} - \vec{k}')\cdot\vec{x}}}_{(2\pi)^3 \delta^3(\vec{k} - \vec{k}')}$$

$$[a(\vec{k}, \lambda), a^\dagger(\vec{k}', \lambda')]_-$$

$$= - \sum_{\lambda} \sum_{\lambda'} \frac{1}{4} (2\omega_{\vec{k}}) \left(\frac{2}{\omega_{\vec{k}}}\right) (2\pi)^3 \underbrace{\epsilon_{\mu}^{\lambda}(\vec{k}, \lambda) \epsilon^{\mu}(\vec{k}, \lambda')}_{-\sum_{\lambda} \delta_{\lambda\lambda'}}$$

$$\downarrow \sum_{\lambda}^2 = 1$$

$$= \sum_{\lambda} \delta_{\lambda\lambda'} (2\pi)^3 \delta^3(\vec{k} - \vec{k}')$$

∴

$$[a(\vec{k}, \lambda), a^\dagger(\vec{k}', \lambda')]_- = \sum_{\lambda} \delta_{\lambda\lambda'} (2\pi)^3 \delta^3(\vec{k} - \vec{k}')$$

ANALOGOUSLY

$$[a(\vec{k}, \lambda), a(\vec{k}', \lambda')]_- = 0$$

$$[a^\dagger(\vec{k}, \lambda), a^\dagger(\vec{k}', \lambda')]_- = 0$$

$$\zeta_0 = -1, \zeta_i = +1 \quad (i=1,2,3)$$

FOR  $\lambda = 1, 2, 3$

$$[a(\vec{k}, \lambda), a^\dagger(\vec{k}', \lambda)]_- = \delta_{\lambda\lambda'} (2\pi)^3 \delta^3(\vec{k} - \vec{k}')$$

FOR  $\lambda = 0$

$$[a(\vec{k}, 0), a^\dagger(\vec{k}', 0)]_- = - (2\pi)^3 \delta^3(\vec{k} - \vec{k}')$$

FOR  $\lambda = 1, 2, 3 \rightarrow$  STANDARD BOSON COMMUTATION RELATIONS

$\lambda = 0 \rightarrow$  SIGN CHANGE



↳ IF WE CONSIDER VACUUM STATE  
AS STATE WITHOUT PHOTONS

$$\text{i.e. } a(\vec{k}, \lambda) |0\rangle = 0 \quad \lambda = 0, 1, 2, 3$$

⇓

1-PHOTON STATE WITH POLARIZATION  $\lambda$

$$|1\lambda, \lambda\rangle = \int d^3\vec{k} f(\vec{k}) a^\dagger(\vec{k}, \lambda) |0\rangle$$

↳ WAVE PACKET

$$\int d^3\vec{k} |f(\vec{k})|^2 < \infty$$

NORMALIZATION OF  $1\lambda$  STATE

$$\langle 1\lambda, \lambda | 1\lambda, \lambda \rangle$$

$$= \int d^3\vec{k} \int d^3\vec{k}' f^*(\vec{k}) f(\vec{k}')$$

$$\cdot \underbrace{\langle 0 | a(\vec{k}, \lambda) a^\dagger(\vec{k}', \lambda) | 0 \rangle}$$

$$\sum_\lambda (2\pi)^3 \delta^3(\vec{k} - \vec{k}')$$

$$= \sum_\lambda (2\pi)^3 \int d^3\vec{k} |f(\vec{k})|^2$$

$$= \begin{cases} > 0 & \text{FOR } \lambda = 1, 2, 3 \\ < 0 & \text{FOR } \lambda = 0 \quad \nabla \quad \text{NEGATIVE NORM STATE} \end{cases}$$

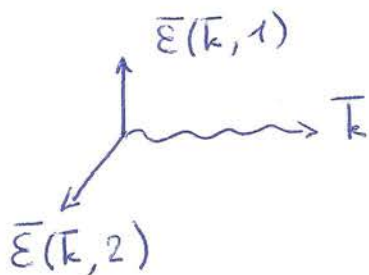
↳ HAMILTONIAN :

$$H = \int \frac{d^3 \vec{k}}{(2\pi)^3} \omega_{\vec{k}} \left\{ -a^\dagger(\vec{k}, 0) a(\vec{k}, 0) + \sum_{i=1}^3 a^\dagger(\vec{k}, i) a(\vec{k}, i) \right\}$$

STATES WITH  $\lambda=0$  LEAD TO NEGATIVE ENERGY !

↳ PHYSICAL STATES :

FREE MAXWELL FIELD HAS ONLY 2 TRANSVERSE COMPONENTS



↳  $\lambda=0, \lambda=3$  STATES SHOULD NOT APPEAR IN PHYSICAL STATES UPON QUANTIZATION

$|\Psi\rangle$  PHYSICAL VACUUM  
(HAS NO TRANSVERSE PHOTONS)

WE REQUIRE

$$\underline{\underline{\partial_\mu A^{\mu(+)} |\Psi\rangle = 0}} \quad (\text{GUPTA-BLEULER})$$

(+) STANDS FOR POS. FREQUENCY (ANNIHILATING) PART IN  $A^\mu$

NOTE: THIS IS A WEAKER CONDITION THAN CONSIDERING  $\partial_\mu \hat{A}^\mu = 0$  AS OPERATOR CONDITION

$$\partial_\mu A^{\mu(+)} |\underline{\Psi}\rangle = 0$$

↓

$$\partial_\mu \int \frac{d^3\vec{k}}{(2\pi)^3 \sqrt{2\omega_{\vec{k}}}} \sum_\lambda a(\vec{k}, \lambda) \epsilon^\mu(\vec{k}, \lambda) e^{-ik \cdot x} |\underline{\Psi}\rangle = 0$$

$$\int \frac{d^3\vec{k}}{(2\pi)^3 \sqrt{2\omega_{\vec{k}}}} \sum_\lambda (-i) a(\vec{k}, \lambda) e^{-ik \cdot x} k_\mu \epsilon^\mu(\vec{k}, \lambda) |\underline{\Psi}\rangle = 0$$

$$\begin{aligned} & \downarrow k_\mu \epsilon^\mu(\vec{k}, \lambda) \\ & = \omega_{\vec{k}} \epsilon^0(\vec{k}, \lambda) - |\vec{k}| \epsilon^3(\vec{k}, \lambda) \\ & = \omega_{\vec{k}} (\delta_{\lambda 0} - \delta_{\lambda 3}) \end{aligned}$$

$$\forall \vec{k} : \underline{\underline{(a(\vec{k}, 0) - a(\vec{k}, 3)) |\underline{\Psi}\rangle = 0}}$$

⇓

$$\langle \underline{\Psi} | a^\dagger(\vec{k}, 3) a(\vec{k}, 3) | \underline{\Psi} \rangle = \langle \underline{\Psi} | a^\dagger(\vec{k}, 0) a(\vec{k}, 0) | \underline{\Psi} \rangle$$

- PHYSICAL VACUUM STATE HAS EQUAL NUMBER OF SCALAR ( $\lambda=0$ ) AND LONGITUDINAL ( $\lambda=3$ ) PHOTONS

- THE COMBINED ENERGY OF SCALAR & LONGITUDINAL PHOTONS IS ZERO

$$\langle \Psi | H | \Psi \rangle = \int \frac{d^3 \vec{k}}{(2\pi)^3} \omega_{\vec{k}} \sum_{\lambda=1,2} \langle \Psi | a^\dagger(\vec{k}, \lambda) a(\vec{k}, \lambda) | \Psi \rangle$$

- PHYSICAL QUANTITIES ONLY INVOLVE TRANSVERSE PHOTONS ( $\lambda = 1, 2$ ).
- ALTERING THE ALLOWED ADMIXTURES OF SCALAR AND LONGITUDINAL PHOTONS IS EQUIVALENT TO A GAUGE TF. BETWEEN 2 POTENTIALS, BOTH OF WHICH ARE IN LORENZ GAUGE  
( $\rightarrow$  EXERCISE)



# 3) APPLICATIONS OF INTERACTION OF E.M. FIELD WITH MATTER (NON-RELATIVISTIC)

RAD 20

## HAMILTONIANS

↳ MANY-BODY SYSTEM (MATTER)

$$H_M = \sum_{i=1}^N \frac{\bar{p}_i^2}{2m_i} + V$$

↳ RADIATION FIELD

$$H_{EM} = \frac{1}{2} \int d^3\bar{x} (\bar{E} \cdot \bar{E}^* + \bar{B} \cdot \bar{B}^*)$$

↳ INTERACTION HAS TO RESPECT GAUGE-INVARIANCE

$$\psi(\bar{x}, t) \rightarrow e^{\frac{ie}{\hbar c} \chi(\bar{x}, t)} \psi(\bar{x}, t)$$

LOCAL PHASE TRANSFORMATION HAS TO LEAVE THEORY INVARIANT

$$\hat{p} \psi(\bar{x}, t) = -i\hbar \bar{\nabla} \psi(\bar{x}, t)$$

$$\rightarrow e^{\frac{ie}{\hbar c} \chi(\bar{x}, t)} \left[ -i\hbar \bar{\nabla} \psi + \frac{e}{c} (\bar{\nabla} \chi) \psi \right]$$

KINETIC ENERGY TERM CAN BE MADE INVARIANT BY MINIMAL SUBSTITUTION

$$\bar{p}_i \Rightarrow \bar{p}_i - \frac{e_i}{c} \bar{A}(x_i)$$

UNDER GAUGE TRANSFORMATION,  $\bar{A}$  TRANSFORMS AS

$$\bar{A} \rightarrow \bar{A} + \bar{\nabla} \chi$$

$$\therefore (\bar{p} - \frac{e}{c} \bar{A}) \psi(\bar{x}, t)$$

$$\rightarrow e^{\frac{ie}{\hbar c} \chi(\bar{x}, t)} \left\{ \begin{array}{l} -i\hbar \bar{\nabla} + \cancel{\frac{e}{c} (\bar{\nabla} \chi)} \\ - \frac{e}{c} \bar{A} - \cancel{\frac{e}{c} (\bar{\nabla} \chi)} \end{array} \right\} \psi$$

$$= e^{\frac{ie}{\hbar c} \chi(\bar{x}, t)} \underbrace{\left( -i\hbar \bar{\nabla} - \frac{e}{c} \bar{A} \right)}_{\bar{p}} \psi$$

$\therefore$  HAMILTONIAN IS GAUGE INVARIANT



EXPECTATION VALUES REMAIN UNCHANGED

e.g.  $\int d^3\bar{x} \psi^*(\bar{x}, t) \frac{1}{2m} \left( -i\hbar \bar{\nabla} - \frac{e}{c} \bar{A} \right)^2 \psi(\bar{x}, t)$

$$H_{\text{tot}} = \sum_{i=1}^N \frac{1}{2m_i} \left( \bar{p}_i - \frac{e_i}{c} \bar{A}_i \right)^2 + V + H_{\text{EM}}$$

$$= H_M + H_{\text{INT}} + H_{\text{EM}}$$

↑  
MATTER

↑  
INTERACTION  
RADIATION WITH  
MATTER

## ↳ INTERACTION HAMILTONIAN

$$H_{\text{INT}} = \sum_i \left\{ -\frac{e_i}{m_i c} \vec{p}_i \cdot \vec{A}(\vec{x}_i) + \frac{e_i^2}{2m_i c^2} \vec{A}^2(\vec{x}_i) \right\}$$

$$\equiv H_{\text{INT}}^{\text{I}} + H_{\text{INT}}^{\text{II}}$$

↑
↑  
 LINEAR IN  $\vec{A}$ 
↑
↑  
 QUADRATIC IN  $\vec{A}$

(NOTE IN  $H_{\text{INT}}^{\text{I}}$ :  $\vec{p}_i \cdot \vec{A} + \vec{A} \cdot \vec{p}_i = 2 \vec{p}_i \cdot \vec{A}$  (IN COULOMB GAUGE  $\vec{\nabla} \cdot \vec{A} = 0$ ))

## • INTERACTION HAMILTONIAN IN SECOND QUANTIZATION

↳ RECALL:  $\vec{A}(\vec{x}) = \sum_{\vec{k}} \sum_{\sigma=1,2} N_{\vec{k}} \epsilon_{\vec{k}\sigma} \left\{ \hat{a}_{\vec{k}\sigma} e^{i\vec{k} \cdot \vec{x}} + \hat{a}_{\vec{k}\sigma}^{\dagger} e^{-i\vec{k} \cdot \vec{x}} \right\}$

WITH  $N_{\vec{k}} = \left( \frac{\hbar c^2}{2\omega_{\vec{k}} L^3} \right)^{1/2}$

## ↳ INTERACTION OF 1 PARTICLE WITH RADIATION FIELD

$$\hat{H}_{\text{INT}} = -\frac{e}{mc} \sum_{\vec{k}} \sum_{\sigma} N_{\vec{k}} \hat{p} \cdot \epsilon_{\vec{k}\sigma} \left\{ \hat{a}_{\vec{k}\sigma} e^{i\vec{k} \cdot \vec{x}} + \hat{a}_{\vec{k}\sigma}^{\dagger} e^{-i\vec{k} \cdot \vec{x}} \right\}$$

$$\hat{H}_{INT}'' = \frac{e^2}{2mc^2} \sum_{\vec{k}\sigma} \sum_{\vec{k}'\sigma'} \left( \frac{\hbar c^2}{2L^3} \right) \frac{\vec{\epsilon}_{\vec{k}\sigma} \cdot \vec{\epsilon}_{\vec{k}'\sigma'}}{(\omega_k \omega_{k'})^{1/2}}$$

$$\cdot \left\{ \begin{aligned} & \hat{a}_{\vec{k}\sigma} \hat{a}_{\vec{k}'\sigma'} e^{i(\vec{k} + \vec{k}') \cdot \vec{x}} \\ & + \hat{a}_{\vec{k}\sigma} \hat{a}_{\vec{k}'\sigma'}^+ e^{i(\vec{k} - \vec{k}') \cdot \vec{x}} \\ & + \hat{a}_{\vec{k}\sigma}^+ \hat{a}_{\vec{k}'\sigma'} e^{i(-\vec{k} + \vec{k}') \cdot \vec{x}} \\ & + \hat{a}_{\vec{k}\sigma}^+ \hat{a}_{\vec{k}'\sigma'}^+ e^{-i(\vec{k} + \vec{k}') \cdot \vec{x}} \end{aligned} \right\}$$

↳ WE WILL CONSIDER  $H'_{INT} + H''_{INT}$  AS

PERTURBATION TO  $H_M + H_{EM}$

UNPERTURBED STATES

$$|\Psi_0\rangle = | \dots m_i \dots \rangle_M \otimes | \dots m_{\vec{k}\sigma} \dots \rangle_{EM}$$

↑  
 MATTER

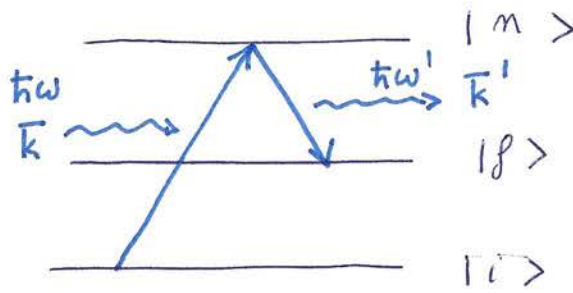
↑  
 RADIATION FIELD

$H'_{INT} + H''_{INT}$  WILL INDUCE TRANSITIONS BETWEEN THESE UNPERTURBED STATES.



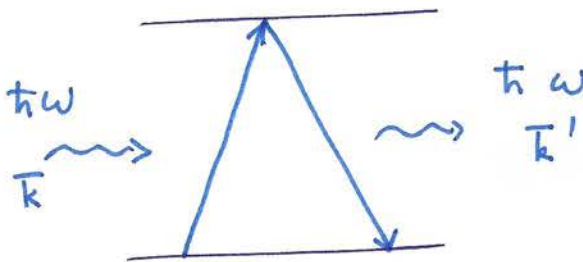
↳ EXAMPLES : PHOTONS ARE ABSORBED ( $\hat{a}_{\vec{k}\sigma}$ )  
AND/OR EMITTED ( $\hat{a}_{\vec{k}\sigma}^+$ )

1)



$$\omega' \neq \omega, \quad \vec{k}' \neq \vec{k}$$

INELASTIC  
PHOTON SCATTERING  
(RAMAN SCATTERING)

2) IF  $|i\rangle = |f\rangle$ 

$$\omega' = \omega, \quad |\vec{k}'| = |\vec{k}|$$

BUT DIRECTIONS.  
CAN BE DIFFERENT

ELASTIC  
PHOTON SCATTERING

● INTERLUDE : TIME - DEPENDENT PERTURBATION THEORY

↳ QUANTUM SYSTEM DESCRIBED BY  $\hat{H}_0$  (UNPERTURBED HAMILT.)

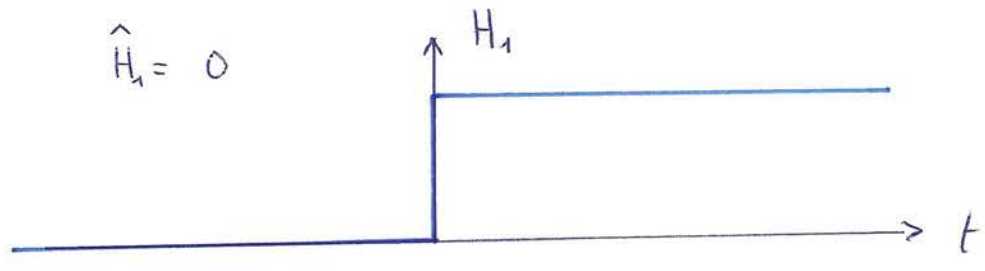
$$\hat{H}_0 |\Psi_0(t)\rangle = i\hbar \frac{\partial}{\partial t} |\Psi_0(t)\rangle$$

$$|\Psi_0(t)\rangle = \sum_m a_m e^{-\frac{i}{\hbar} E_m t} |\Psi_m\rangle$$

$$\hat{H}_0 |\Psi_m\rangle = E_m |\Psi_m\rangle$$

$\uparrow$                        $\uparrow$   
 EIGENSTATES        EIGENVALUES

↳ AT  $t=0$ , PERTURBATION  $\hat{H}_1$  IS SWITCHED ON



$t > 0$  :  $\hat{H} = \hat{H}_0 + \hat{H}_1$

$$(\hat{H}_0 + \hat{H}_1) |\Psi(t)\rangle = i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle$$

EXPAND  $|\Psi(t)\rangle$  IN EIGENSTATES OF  $\hat{H}_0$

$$|\Psi(t)\rangle = \sum_m a_m(t) e^{-\frac{i}{\hbar} E_m t} |\Psi_m\rangle$$

$a_m(t)$  VARIES NOW WITH TIME DUE TO  $\hat{H}_1$

$$i\hbar \frac{\partial}{\partial t} \sum_n a_n(t) e^{-\frac{i}{\hbar} E_n t} |\Psi_n\rangle$$

$$= (\hat{H}_0 + \hat{H}_1) \sum_n a_n(t) e^{-\frac{i}{\hbar} E_n t} |\Psi_n\rangle$$

⇓

$$\sum_n a_n(t) E_n e^{-\frac{i}{\hbar} E_n t} |\Psi_n\rangle + \sum_n \left( i\hbar \frac{\partial a_n}{\partial t} \right) e^{-\frac{i}{\hbar} E_n t} |\Psi_n\rangle$$

$$= \sum_n a_n(t) E_n e^{-\frac{i}{\hbar} E_n t} |\Psi_n\rangle + \sum_n a_n(t) e^{-\frac{i}{\hbar} E_n t} \hat{H}_1 |\Psi_n\rangle$$

↓

$\langle \Psi_m |$

$$i\hbar \left( \frac{d}{dt} a_m(t) \right) e^{-\frac{i}{\hbar} E_m t} = \sum_n a_n(t) e^{-\frac{i}{\hbar} E_n t} \langle \Psi_m | \hat{H}_1 | \Psi_n \rangle$$

$$i\hbar \frac{da_m}{dt} = \sum_n a_n(t) \langle \Psi_m | \hat{H}_1 | \Psi_n \rangle e^{+\frac{i}{\hbar} (E_m - E_n) t}$$

↳ ASSUME AT  $t \leq 0$  SYSTEM IS IN STATE  $|\Psi_m\rangle$

$$a_m(t \leq 0) = 1, \quad a_n(t \leq 0) = 0 \text{ FOR } m \neq n$$

FOR  $\hat{H}_1 \ll \hat{H}_0$  APPLY 1<sup>st</sup> ORDER PERTURBATION THEORY

i.e. ON RHS APPROXIMATE  $a_m(t > 0) \approx a_m(t=0) = 1$   
 $a_n(t > 0) \approx a_n(t=0) = 0, m \neq n$

$$\therefore i\hbar \frac{d}{dt} a_m \approx - \langle \Psi_m | \hat{H}_1 | \Psi_m \rangle e^{\frac{i}{\hbar}(E_m - E_m)t}$$

$$a_m(t) \approx - \frac{i}{\hbar} \int_{t'=0}^{t'=t} dt' \langle \Psi_m | \hat{H}_1 | \Psi_m \rangle e^{\frac{i}{\hbar}(E_m - E_m)t'}$$

↓

↳ IF  $\hat{H}_1$  DOES NOT DEPEND ON TIME

$$a_m(t) \approx - \frac{i}{\hbar} \langle \Psi_m | \hat{H}_1 | \Psi_m \rangle \int_{t'=0}^{t'=t} dt' e^{\frac{i}{\hbar}(E_m - E_m)t'}$$

$$a_m(t) \approx + \frac{i}{(E_n - E_m)} \langle \Psi_m | \hat{H}_1 | \Psi_m \rangle \left( e^{\frac{i}{\hbar}(E_m - E_m)t} - 1 \right)$$

TRANSITION PROBABILITY FROM  $m \rightarrow m$  IN 1<sup>0</sup> ORDER P.T.

$$P_{m \rightarrow m}(t) = |a_m(t)|^2 = \left| \langle \Psi_m | \hat{H}_1 | \Psi_m \rangle \right|^2 \frac{\sin^2 \left( \frac{E_m - E_m}{2\hbar} t \right)}{\left( \frac{E_m - E_m}{2} \right)^2}$$



# ↳ FERMI'S GOLDEN RULE

→ TAKE LONG TIME LIMIT  $t \rightarrow \infty$

USE  $\left\| \frac{\sin^2(xt)}{\pi t x^2} \right\|_{t \rightarrow \infty} = \delta(x)$

$$x \neq 0 \Rightarrow \frac{\sin^2(xt)}{\pi t x^2} \Big|_{t \rightarrow \infty} \rightarrow 0$$

$$\int_{-\infty}^{+\infty} dx \frac{\sin^2(xt)}{\pi t x^2} = \frac{2}{\pi} \int_0^{\infty} d(tx) \frac{\sin^2(xt)}{(xt)^2} = 1$$

$$\rightsquigarrow P_{n \rightarrow m}(t \gg) = \left| \langle \psi_m | \hat{H}_1 | \psi_n \rangle \right|^2 \frac{\pi t}{\hbar^2} \underbrace{\delta\left(\frac{E_m - E_n}{2\hbar}\right)}_{2\hbar \delta(E_m - E_n)}$$

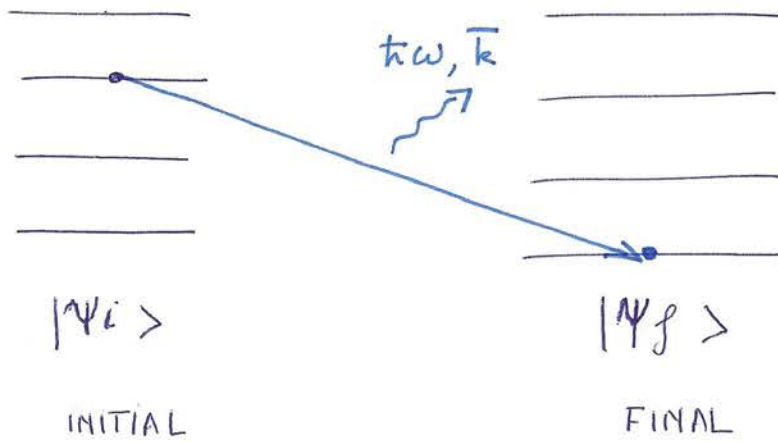
$$\frac{P_{n \rightarrow m}(t \gg)}{t} = \frac{2\pi}{\hbar} \delta(E_m - E_n) \left| \langle \psi_m | \hat{H}_1 | \psi_n \rangle \right|^2$$

↓  
TRANSITION PROBABILITY  
PER UNIT OF TIME

↑  
ENERGY CONSERVATION

THIS IS KNOWN AS FERMI'S GOLDEN RULE

• EMISSION OF RADIATION BY AN EXCITED ATOM



$$|\dots m_{\vec{k}\sigma} \dots\rangle$$

$$|\dots m_{\vec{k}\sigma} + 1 \dots\rangle$$

ONE EXTRA  
PHOTON WITH  $\vec{k}\sigma$

$$|i\rangle \equiv |\Psi_i\rangle \otimes |\dots m_{\vec{k}\sigma} \dots\rangle$$

↑  
MATTER  
STATE

(e.g.  $e^-$  IN 2<sup>o</sup> EXCITED STATE OF ATOM)

$$|f\rangle \equiv |\Psi_f\rangle \otimes |\dots m_{\vec{k}\sigma} + 1 \dots\rangle$$



TRANSITION PROBABILITY PER UNIT TIME  $\frac{dP}{dt}$

$$\frac{dP}{dt} = \frac{2\pi}{\hbar} |\mathcal{M}_{fi}|^2 \delta(E_i - E_f)$$

(FERMI'S GOLDEN RULE)

↳ TRANSITION AMPLITUDE  $\mathcal{M}_{fi}$   
 IN 1<sup>0</sup> ORDER PERTURBATION THEORY

$$\begin{aligned}
 \mathcal{M}_{fi} &= \langle f | H'_{\text{INT}} | i \rangle \\
 &= \langle \psi_f | \langle \dots m_{\vec{k}\sigma} + 1 \dots | \hat{H}'_{\text{INT}} | \dots m_{\vec{k}\sigma} \dots \rangle | \psi_i \rangle \\
 &= - \frac{e}{mc} \sum_{\vec{k}'\sigma'} N_{\vec{k}'} \langle \psi_f | \hat{P} \cdot \vec{\varepsilon}_{\vec{k}'\sigma'} \\
 &\quad \otimes \langle \dots m_{\vec{k}\sigma} + 1 \dots | \hat{a}_{\vec{k}'\sigma'} e^{i\vec{k}' \cdot \vec{x}} \\
 &\quad + \hat{a}_{\vec{k}'\sigma'}^\dagger e^{-i\vec{k}' \cdot \vec{x}} | \dots m_{\vec{k}\sigma} \dots \rangle \otimes | \psi_i \rangle \\
 &= - \frac{e}{mc} \left( \frac{\hbar c^2}{2\omega_{\vec{k}} L^3} \right)^{1/2} \langle \psi_f | \hat{P} \cdot \vec{\varepsilon}_{\vec{k}\sigma} \\
 &\quad \otimes \langle \dots m_{\vec{k}\sigma} + 1 \dots | \hat{a}_{\vec{k}\sigma}^\dagger e^{-i\vec{k} \cdot \vec{x}} | \dots m_{\vec{k}\sigma} \dots \rangle \\
 &\quad \otimes | \psi_i \rangle \\
 &\quad e^{-i\vec{k} \cdot \vec{x}} \sqrt{m_{\vec{k}\sigma} + 1}
 \end{aligned}$$

$$\therefore \mathcal{M}_{fi} = - \frac{e}{mc} \left( \frac{\hbar c^2}{2\omega_{\vec{k}} L^3} \right)^{1/2} \sqrt{m_{\vec{k}\sigma} + 1} \langle \psi_f | \hat{P} \cdot \vec{\varepsilon}_{\vec{k}\sigma} e^{-i\vec{k} \cdot \vec{x}} | \psi_i \rangle$$

$$\therefore \frac{dP}{dt} = \frac{2\pi}{\hbar} \delta(E_i - E_f) |\mathcal{M}_{fi}|^2$$

$$E_i = E_{M_i}$$

↑  
MATTER

$$E_f = E_{M_f} + \hbar\omega$$

$$\frac{dP}{dt} = \frac{2\pi}{\hbar} \left( \frac{e^2}{4\pi} \right) \cdot \frac{1}{(mc)^2} \cdot \left( \frac{2\pi\hbar c^2}{\omega_k L^3} \right) \cdot (n_{k\sigma} + 1)$$

$$\cdot \left| \langle N_f | \hat{p} \cdot \vec{\epsilon}_{k\sigma} e^{-i\vec{k}\cdot\vec{x}} | N_i \rangle \right|^2$$

$$\cdot \delta(E_{M_i} - E_{M_f} - \hbar\omega_k)$$

$$\Rightarrow \frac{dP}{dt} \sim (n_{k\sigma} + 1)$$

∴ THE MORE PHOTONS AVAILABLE WITH  $k\sigma$  THE MORE LIKELY THE EMISSION OF ANOTHER ONE (BOSONS!)

↳ STIMULATED EMISSION (→ LASER)

∴ FOR  $n_{k\sigma} = 0$ , NON-ZERO PROBABILITY

↳ SPONTANEOUS EMISSION.



LIFE TIME OF EXCITED STATE (SPONTANEOUS EMISSION)

↓  
τ

$$m_{k\sigma} = 0$$

$$\hookrightarrow \left( \frac{1}{\tau_{i \rightarrow f}} \right) = \frac{2\pi}{\hbar} \sum_{k\sigma} |\langle f | H'_{INT} | i \rangle|^2 \delta(E_{M_i} - E_{M_f} - \hbar\omega_k)$$

↓  $|\vec{k}| = \frac{\omega_k}{c}$

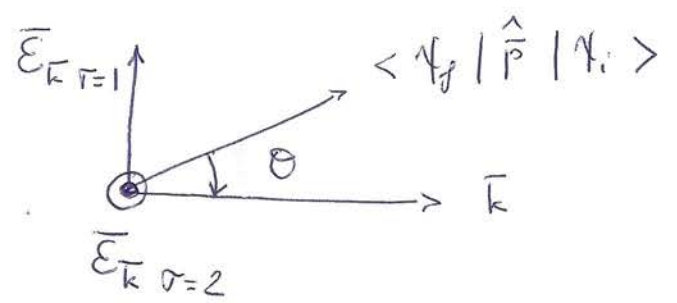
$$\frac{1}{\tau_{i \rightarrow f}} = \frac{2\pi}{\hbar} \cdot \frac{e^2}{4\pi} \cdot \frac{2\pi\hbar}{m^2 L^3}$$

$$\cdot \sum_{k\sigma} \frac{1}{\omega_k} \cdot |\langle \psi_f | \hat{p} \cdot \vec{\epsilon}_{k\sigma} e^{-i\vec{k} \cdot \vec{x}} | \psi_i \rangle|^2 \delta(E_{M_i} - E_{M_f} - \hbar\omega_k)$$

↓  $\sum_{\vec{k}} \dots = \frac{L^3}{(2\pi)^3} \int d^3\vec{k} \dots$

$$\frac{1}{\tau_{i \rightarrow f}} = \left( \frac{e^2}{4\pi} \right) \cdot \frac{1}{2\pi m^2} \int d^3\vec{k} \sum_{\sigma=1,2} \frac{1}{\omega_k} \delta(E_{M_i} - E_{M_f} - \hbar\omega_k) \cdot |\langle \psi_f | \hat{p} \cdot \vec{\epsilon}_{k\sigma} e^{-i\vec{k} \cdot \vec{x}} | \psi_i \rangle|^2$$

↓ CHOOSE  $\vec{\epsilon}_{\vec{k}\sigma=1}$  IN PLANE  $\vec{k}$ ,  $\langle \psi_f | \hat{p} | \psi_i \rangle$



$$\bar{\mathcal{E}}_{\vec{k}, \sigma=2} \cdot \langle \psi_f | \hat{p} e^{-i\vec{k} \cdot \vec{x}} | \psi_i \rangle = 0$$

↳ FOR ATOMS  $\hbar \omega_k \approx 10 \text{ eV}$  (TYPICALLY)

$$\vec{k} \cdot \vec{x} \approx \frac{2\pi}{\lambda} a_{\text{BOHR}}$$

$$\begin{aligned} \hbar c &\approx 1970 \text{ eV \AA} \\ (1 \text{ \AA} &= 10^{-10} \text{ m}) \end{aligned} \quad \begin{aligned} &= \frac{\hbar \omega_k}{\hbar c} a_{\text{BOHR}} \\ &\approx \frac{10 \text{ eV}}{1970 \text{ eV \AA}} \cdot \underbrace{a_{\text{BOHR}}}_{\approx 0.5 \text{ \AA}} \end{aligned}$$

$$\approx 2.5 \cdot 10^{-3} \ll 1$$

### LONG - WAVELENGTH APPROXIMATION

$$e^{-i\vec{k} \cdot \vec{x}} = 1 - i\vec{k} \cdot \vec{x} + \frac{1}{2} (i\vec{k} \cdot \vec{x})^2 + \dots$$

$$\approx 1 \quad (\vec{k} \cdot \vec{x} \ll 1)$$

$$\Downarrow$$

$$\bar{\mathcal{E}}_{\vec{k}, \sigma=1} \cdot \langle \psi_f | \hat{p} e^{-i\vec{k} \cdot \vec{x}} | \psi_i \rangle$$

$$\approx \bar{\mathcal{E}}_{\vec{k}, \sigma=1} \cdot \langle \psi_f | \hat{p} | \psi_i \rangle$$

$$= \sin \theta |\langle \psi_f | \hat{p} | \psi_i \rangle|$$

$$\hookrightarrow \frac{1}{\tau_{i \rightarrow f}} \approx \left( \frac{e^2}{4\pi} \right) \cdot \frac{1}{2\pi \cdot m^2}$$

$$\cdot \int d^3 \vec{k} \frac{1}{\omega_k} \delta(E_{M_i} - E_{M_f} - \hbar \omega_k) \cdot \sin^2 \theta \left| \langle \psi_f | \hat{P} | \psi_i \rangle \right|^2$$

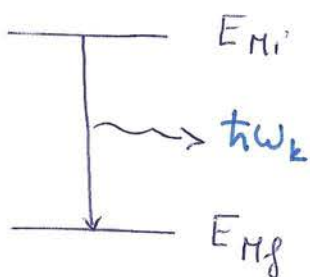
↓ CHOOSE z-AXIS ALONG  $\langle \psi_f | \hat{P} | \psi_i \rangle$

$$= \left( \frac{e^2}{4\pi} \right) \cdot \frac{1}{2\pi \cdot m^2} \cdot \left| \langle \psi_f | \hat{P} | \psi_i \rangle \right|^2$$

$$\cdot \underbrace{(2\pi) \int_{-1}^1 d\cos\theta (1 - \cos^2\theta)}_{\frac{4}{3}} \cdot \underbrace{\int_0^\infty d|\vec{k}| \frac{|\vec{k}|^2}{|\vec{k}|c} \delta(E_{M_i} - E_{M_f} - \hbar c|\vec{k}|)}_{\frac{1}{\hbar c} \delta(|\vec{k}| - \frac{E_{M_i} - E_{M_f}}{\hbar c})}$$

$$= \left( \frac{e^2}{4\pi} \right) \cdot \frac{1}{m^2} \frac{1}{\hbar c^2} \cdot \left( \frac{E_{M_i} - E_{M_f}}{\hbar c} \right) \cdot \frac{4}{3} \left| \langle \psi_f | \hat{P} | \psi_i \rangle \right|^2$$

$$\frac{1}{\tau_{i \rightarrow f}} = \frac{4}{3} \cdot \left( \frac{e^2}{4\pi} \right) \cdot \frac{1}{\hbar m^2 c^3} \left( \frac{E_{M_i} - E_{M_f}}{\hbar} \right) \cdot \left| \langle \psi_f | \hat{P} | \psi_i \rangle \right|^2$$



$$(E_{M_i} - E_{M_f}) \uparrow \Rightarrow \tau_{i \rightarrow f} \downarrow$$

$$\begin{aligned}
 & \langle \psi_f | \hat{p} | \psi_i \rangle \\
 &= m \frac{d}{dt} \langle \psi_f | \hat{x} | \psi_i \rangle \\
 &= -\frac{im}{\hbar} \langle \psi_f | [\hat{x}, \hat{H}_M] | \psi_i \rangle \\
 &= m \left(-\frac{i}{\hbar}\right) (E_{M_i} - E_{M_f}) \langle \psi_f | \hat{x} | \psi_i \rangle
 \end{aligned}$$

$$\therefore \frac{1}{\tau_{i \rightarrow f}} = \frac{4}{3} \left(\frac{e^2}{4\pi}\right) \frac{1}{\hbar c^3} \left(\frac{E_{M_i} - E_{M_f}}{\hbar}\right)^3 |\langle \psi_f | \hat{x} | \psi_i \rangle|^2$$

$e \hat{x}$  : ELECTRIC DIPOLE OPERATOR

$\therefore$  ELECTRIC DIPOLE APPROXIMATION.



## HYPERFINE SPLITTING IN HYDROGEN

## 21 cm LINE

$e^-$  WITH SPIN HAS  $\vec{\mu}$

$$\vec{\mu} = \frac{e}{2m} \frac{\hbar}{c} \vec{\sigma}$$

WHICH INTERACTS WITH  $\vec{B}$  AS

$$\hat{H}_{\text{INT}}^{\text{III}} = - \vec{\mu} \cdot \vec{B}$$

EXPRESS  $\hat{H}_{\text{INT}}^{\text{III}}$  THROUGH ITS NORMAL MODE EXPANSION.

$$\hat{H}_{\text{INT}}^{\text{III}} = - \frac{e\hbar}{2mc} \vec{\sigma} \cdot (\vec{\nabla} \times \hat{\vec{A}})$$

$$\hat{\vec{A}} = \sum_{\vec{k}\sigma} N_{\vec{k}} \vec{\epsilon}_{\vec{k}\sigma} \left\{ \hat{a}_{\vec{k}\sigma} e^{i\vec{k}\cdot\vec{x}} + \hat{a}_{\vec{k}\sigma}^\dagger e^{-i\vec{k}\cdot\vec{x}} \right\}$$

$$\vec{\nabla} \times \hat{\vec{A}} = i \sum_{\vec{k}\sigma} N_{\vec{k}} (\vec{k} \times \vec{\epsilon}_{\vec{k}\sigma}) \left\{ \hat{a}_{\vec{k}\sigma} e^{i\vec{k}\cdot\vec{x}} - \hat{a}_{\vec{k}\sigma}^\dagger e^{-i\vec{k}\cdot\vec{x}} \right\}$$

$$\hat{H}_{\text{INT}}^{\text{III}} = - \frac{ie\hbar}{2mc} \sum_{\vec{k}\sigma} N_{\vec{k}} \vec{\sigma} \cdot (\vec{k} \times \vec{\epsilon}_{\vec{k}\sigma}) \left\{ \hat{a}_{\vec{k}\sigma} e^{i\vec{k}\cdot\vec{x}} - \hat{a}_{\vec{k}\sigma}^\dagger e^{-i\vec{k}\cdot\vec{x}} \right\}$$

- CONSIDER  $e^-$  IN  $1s$  STATE OF H  
SPIN OF  $e^-$  INTERACTS WITH SPIN OF PROTON  
(BOTH ARE SPIN  $1/2$ )

$\Rightarrow$  RESULTING IN SPLITTING OF GROUND STATE  
(HYPERFINE SPLITTING)

$e^- \uparrow$  AND  $p \uparrow$  HAS SLIGHTLY HIGHER ENERGY

THAN  $e^- \uparrow$  AND  $p \downarrow$

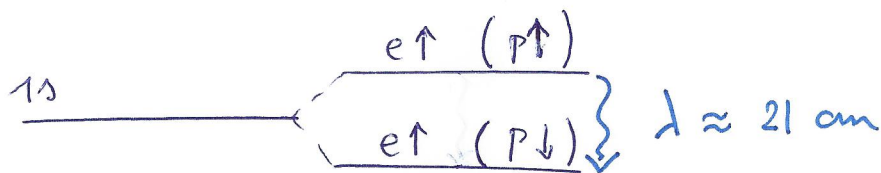
$\rightsquigarrow$  CONSIDER A TRANSITION BETWEEN

$e^-$  IN INITIAL STATE  $|\psi_i\rangle = |1s\rangle |\uparrow\rangle_e |\uparrow\rangle_p$

AND

$e^-$  IN FINAL STATE  $|\psi_f\rangle = |1s\rangle \frac{1}{\sqrt{2}} \left\{ |\uparrow\rangle_e |\downarrow\rangle_p - |\downarrow\rangle_e |\uparrow\rangle_p \right\}$

$\rightsquigarrow$  IN THIS TRANSITION, A PHOTON IS EMITTED  
WITH  $\lambda \approx 21 \text{ cm}$



$\rightsquigarrow$  QUESTION : CALCULATE LIFETIME OF TRANSITION

HELP : CALCULATE MATRIX ELEMENT

$$\langle f | H_{\text{INT}}''' | i \rangle \text{ IN DIPOLE APPROX.}$$

AND APPLY FERMI'S GOLDEN RULE

~> ANSWER

$$\langle f | \hat{H}_{\text{INT}}^{\text{III}} | i \rangle \quad \text{WITH} \quad |i\rangle = |N_i\rangle |0\rangle_{\text{PHOTONS}}$$

$$|f\rangle = |N_f\rangle |K\sigma\rangle_{\text{PHOTONS}}$$

$$\hookrightarrow \frac{1}{\tau_{i \rightarrow f}} = \frac{2\pi}{\hbar} \sum_{K\sigma} |\langle f | \hat{H}_{\text{INT}}^{\text{III}} | i \rangle|^2 \delta(E_{M_i} - E_{M_f} - \hbar\omega_k)$$

$$\hookrightarrow \langle f | \hat{H}_{\text{INT}}^{\text{III}} | i \rangle = \frac{ie\hbar}{2mc} \left( \frac{\hbar c^2}{2\omega_k L^3} \right)^{1/2}$$

$$\cdot \langle N_f | \vec{\sigma} \cdot (\vec{k} \times \vec{\epsilon}_{K\sigma}) | N_i \rangle$$

(DIPOLE APPROX  $e^{-i\vec{k}\cdot\vec{x}} \approx 1$ )

$$\hookrightarrow \frac{1}{\tau_{i \rightarrow f}} = \frac{2\pi}{\hbar} \frac{e^2 \hbar^2}{4m^2 c^2} \cdot \left( \frac{\hbar c^2}{2L^3} \right)$$

$$\cdot \frac{L^3}{(2\pi)^3} \int d^3k \sum_{\sigma} \frac{1}{(\hbar c)} \cdot \frac{1}{\hbar c} \delta(k - k_0)$$

$$\cdot |\langle N_f | \vec{\sigma} \cdot (\vec{k} \times \vec{\epsilon}_{K\sigma}) | N_i \rangle|^2$$

$$\text{WITH} \quad \left\| k_0 = \frac{i}{\hbar c} (E_{M_i} - E_{M_f}) = \frac{2\pi}{\lambda} \right\|$$

||  $\lambda$  ||  
|| 21 cm ||

$$\hookrightarrow \mathcal{M}_{fi} = \langle \Psi_f | \vec{\sigma} \cdot (\vec{k} \times \vec{\mathcal{E}}_{\vec{k}\sigma}) | \Psi_i \rangle$$

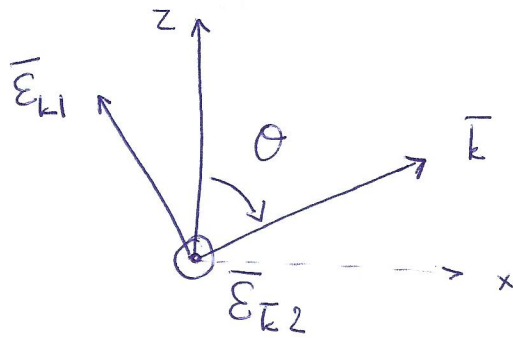
$$\text{IN } |\Psi_i\rangle \Rightarrow |\uparrow\rangle_p$$

IN  $|\Psi_f\rangle$  ONLY TERM WITH  $|\uparrow\rangle_p$  CONTRIBUTES

$$\mathcal{M}_{fi} = -\frac{1}{\sqrt{2}} \langle 1s, e\downarrow | \vec{\sigma} \cdot (\vec{k} \times \vec{\mathcal{E}}_{\vec{k}\sigma}) | 1s, e\uparrow \rangle$$

↓

Z-AXIS ALONG  $e^-$  SPIN



$$(\vec{k} \times \vec{\mathcal{E}}_{\vec{k}\sigma=1}) = -|\vec{k}| \hat{e}_y$$

$$(\vec{k} \times \vec{\mathcal{E}}_{\vec{k}\sigma=2}) = |\vec{k}| (\cos\theta \hat{e}_x - \sin\theta \hat{e}_z)$$

$$\textcircled{\sigma=1} \quad \mathcal{M}_{fi} = +\frac{1}{\sqrt{2}} |\vec{k}| i$$

$$\textcircled{\sigma=2} \quad \mathcal{M}_{fi} = -\frac{1}{\sqrt{2}} |\vec{k}| \cos\theta$$



$$L \rightarrow \frac{1}{\tau_{i \rightarrow f}} = \frac{e^2 \hbar^2}{8 m^2} \frac{1}{(2\pi)^2} \cdot \frac{1}{\hbar c^2} \cdot k_0^3 \cdot (2\pi) \cdot \int_{-1}^1 d\cos \theta \cdot \frac{1}{2} (1 + \cos^2 \theta)$$

$$\underbrace{\hspace{10em}}_{\frac{4}{3}}$$

$$\frac{1}{\tau_{i \rightarrow f}} = \left( \frac{e^2}{4\pi} \right) \cdot \frac{1}{3} \frac{\hbar}{m^2 c^2} k_0^3$$

$$= \left( \frac{e^2}{4\pi \hbar c} \right) \cdot \frac{1}{3} \frac{(\hbar c)^2}{(mc^2)^2} c k_0^3$$

$$\tau_{i \rightarrow f} = (137) \cdot 3 \cdot \frac{(0.5 \cdot 10^6 \text{ eV})^2}{(1.97 \cdot 10^{-7} \text{ eV}\cdot\text{m})^2} \cdot \frac{1}{(2\pi)^3} \frac{(21 \cdot 10^{-2} \text{ m})^3}{(3 \cdot 10^8 \text{ m/s})}$$

$$\approx 3.3 \cdot 10^{14} \text{ sec}$$

$$\approx 10^7 \text{ years}$$

$\Rightarrow$  RELEVANT IN RADIO ASTRONOMY  $\nabla$  (MICROWAVES)

CAN PENETRATE DUST CLOUDS THAT ARE OPAQUE TO VISIBLE WAVELENGTHS