Theoretical Physics 5 : WS 2024/2025
Practice Exam

29.01.2025
(optional hand in: 05.02.2025)

Exercise 1. (25 [+ 10 bonus] points): Dark Matter model

Assume that the total luminous (i.e. not dark) mass of a galaxy has a value M_L and the dark matter is distributed uniformly spherically with radius R around the luminous mass. The dark matter is assumed to consist of degenerate massive fermions with total mass $M = Nm$.

a) (*5 p.*) Consider all luminous matter as a point, and evaluate the gravitational energy of the dark fermion gas interaction with the luminous matter.

b) (*20 p.*) Evaluate the radius of equilibrium for dark matter, for which the total energy is minimal. Find the amount of fermions at which the equilibrium radius is maximum for an individual galaxy with fixed luminous mass and fixed mass of dark matter fermions m.

c) (*10 p.*) **Bonus**: Now, assume a more realistic model of the distribution of dark matter, where $\rho(r) = \rho_0 \frac{1}{r}$. Calculate ρ_0 in terms of M and R. Calculate the total energy for this distribution.

Exercise 2. (25 points): Spontaneous symmetry breaking

Consider the following Lagrangian for a complex scalar field $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$,

$$\mathcal{L} = (\partial^\mu \phi^*)(\partial_\mu \phi) + \mu^2 \phi^* \phi - \frac{\lambda}{2} (\phi^* \phi)^2,$$

where $\lambda > 0$ and $\mu^2 > 0$.

$\mathcal{L}$ is invariant under a phase transformation $\phi \to e^{i\alpha}\phi$, where $\alpha \in \mathbb{R}$.

a) (*5 p.*) Treat ϕ and ϕ^* as independent variables, and show that the corresponding Hamiltonian is given by

$$H = \int d^3 x \left(\frac{1}{c^2} \dot{\phi}^* \dot{\phi} + \nabla \phi^* \cdot \nabla \phi - \mu^2 \phi^* \phi + \frac{\lambda}{2} (\phi^* \phi)^2 \right).$$

b) (*10 p.*) The Hamiltonian can be split into a kinetic part and a potential part. The potential part is given by

$$V(|\phi|) = -\mu^2 |\phi|^2 + \frac{\lambda}{2} |\phi|^4,$$

where $|\phi|^2 = \phi^* \phi$.

Parametrize the classical field configurations that minimize the energy and show that there are an infinite set of such configurations.

Hint: minimize the potential and solve the resulting equation for $|\phi|$. Given $|\phi|$, what is the most general form of ϕ? Make sure you are at a minimum, and not a maximum!

c) (*10 p.*) Suppose that the system is near the minimum $\phi_0 = \mu/\sqrt{\lambda}$. Then it is convenient to define

$$\phi(x) = \phi_0 + \frac{1}{\sqrt{2}} \left[\sigma(x) + i\theta(x) \right],$$

where the fields $\sigma(x)$ and $\theta(x)$ describe (real) fluctuations around the minimum. Rewrite $\mathcal{L}$ in terms of these fluctuations. What are the masses of the σ and θ fields?

Hint: Remember that the mass can be read off from the coefficient quadratic in the field, i.e. for a real scalar field Φ with mass m_0 we have the quadratic term $-\frac{1}{2} \frac{m_0^2 c^2}{\hbar^2} \Phi^2$.

Exercise 3. (25 points): The Dirac equation in the Weyl basis

In the standard Dirac representation, Dirac matrices have the form

$$\gamma_D^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \boldsymbol{\gamma}_D = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}$$

while in the so-called *Weyl representation*, they have the form

$$\gamma_W^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

with $\sigma^\mu = (1, \boldsymbol{\sigma})$ and $\bar{\sigma}^\mu = (1, -\boldsymbol{\sigma})$.

a) *(5 p.)* Derive the unitary matrix S connecting both representations $\gamma_D^\mu = S\gamma_W^\mu S^{-1}$.

b) *(7.5 p.)* In this question we work exclusively in the Weyl basis and so, in the following, we drop the subscript 'W'.

The Dirac equation is given by

$$(i\gamma^\mu \partial_\mu - m)\,\Psi = 0,$$

where in this exercise we set $\hbar = c = 1$. Ψ consists of particles and anti-particles, which have plane-wave solutions

$$\psi_s(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} u_s(p) e^{-ipx}, \quad \chi_s(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} v_s(p) e^{ipx},$$

where s denotes the spinor index.

Show that in the rest frame of a Dirac particle, where $p^\mu = (m, 0, 0, 0)$, the equations of motions lead to constant spinors,

$$u_s = \begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix}, \quad v_s = \begin{pmatrix} \eta_s \\ -\eta_s \end{pmatrix},$$

where η_s and ξ_s are any two-component spinors normalized to unity.

c *(12.5 p.)* Repeat exercise b), but for a particle in a boosted frame, where $p^\mu = (E, 0, 0, p_z)$. You should find

$$u_s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi_s \end{pmatrix}, \quad v_s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta_s \\ -\sqrt{p\cdot\bar{\sigma}}\,\eta_s, \end{pmatrix}$$

where

$$\sqrt{p\cdot\sigma} = \begin{pmatrix} \sqrt{E-p_z} & 0 \\ 0 & \sqrt{E+p_z} \end{pmatrix}, \quad \sqrt{p\cdot\bar{\sigma}} = \begin{pmatrix} \sqrt{E+p_z} & 0 \\ 0 & \sqrt{E-p_z} \end{pmatrix}.$$

Notice how the above form of the solution is manifestly Lorentz invariant – it holds in any frame! Check this, by seeing if the solution reduces to the solution you found in b).

Hint: It is helpful (though not necessary) to let $a = \sqrt{E-p_z}$ and $b = \sqrt{E+p_z}$, so that $m^2 = (E-p_z)(E+p_z) = a^2b^2$.

Question 4. (25 points): Hyperfine splitting in hydrogen and the 21 cm line

The interaction between an electron and a magnetic field is given by

$$H_{\text{int}} = -\boldsymbol{\mu} \cdot \boldsymbol{B},$$

where $\boldsymbol{\mu}$ is the magnetic moment,

$$\boldsymbol{\mu} = \frac{e}{2m}\frac{\hbar}{c}\boldsymbol{\sigma},$$

of the electron.

a) (*10 p.*) Show that H_{int} is given by

$$H_{\text{int}} = -\frac{ie\hbar}{2mc}\sum_{\boldsymbol{k},\sigma} N_k \boldsymbol{\sigma} \cdot (\boldsymbol{k} \times \boldsymbol{\epsilon}_{\boldsymbol{k},\sigma}) \left\{ a_{\boldsymbol{k},\sigma} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} - a^\dagger_{\boldsymbol{k},\sigma} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \right\},$$

where

$$N_k = \left(\frac{\hbar c^2}{2\omega_k L^3}\right)^{1/2}.$$

Hint: Use that

$$\boldsymbol{A} = \sum_{\boldsymbol{k},\sigma} N_k \left\{ a_{\boldsymbol{k},\sigma} \boldsymbol{\epsilon}_{\boldsymbol{k},\sigma} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + a^\dagger_{\boldsymbol{k},\sigma} \boldsymbol{\epsilon}^*_{\boldsymbol{k},\sigma} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \right\},$$

where $\boldsymbol{\epsilon}_{\boldsymbol{k},\sigma}$ is the photon polarization vector (here taken to be real, i.e. the linear polarizations) and that $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$.

b) (*15 p.*) The 1S-state with $f = 1$ has a slightly higher energy than the 1S-state with $f = 0$, where f is the total spin-1/2 electron and spin-1/2 nucleon angular momentum. In the transition between the initial state

$$|\psi_i\rangle = |1S\rangle \, |\uparrow\rangle_e \, |\uparrow\rangle_p ,$$

and the final state (with total electron + proton spin equal to zero)

$$|\psi_f\rangle = |1S\rangle \, \frac{1}{\sqrt{2}} \left\{ |\uparrow\rangle_e \, |\downarrow\rangle_p - |\downarrow\rangle_e \, |\uparrow\rangle_p \right\},$$

a photon with a wavelength of $\lambda \approx 21$ cm is emitted.

Derive the lifetime of this transition and estimate its numerical value in years (an order of magnitude is enough). You may use that, numerically,

$$\alpha = \frac{e^2}{4\pi\hbar c} \approx \frac{1}{137}, \quad \hbar c \approx 1.97\times10^{-7} \text{ eVm}, \quad mc^2 \approx 0.5\times10^6 \text{ eV}, \quad c \approx 3\times10^8 \text{ m/s}.$$

Hint: Use the dipole approximation ($e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \approx 1$) and apply Fermi's golden rule,

$$\frac{1}{\tau_{i\to f}} = \frac{2\pi}{\hbar}\sum_{\boldsymbol{k},\sigma} |\, \langle f|\, H_{\text{int}} \,|i\rangle \,|^2 \delta\left(E_{M_i} - E_{M_f} - \hbar\omega_k\right).$$