Theoretical Physics 5 : WS 2024/2025
Exercise sheet 11

13.01.2025

Exercise 0.

How much time did it take to complete this exercise sheet?

Exercise 1. (75 [+25 bonus] points): Path integral with constant external field

Consider a particle in a constant external force field, with Lagrangian

$$L = \frac{1}{2}m\dot{x}^2 + Vx,$$

with V constant.

Remember how you worked with the free particle in class. Use the same approach to show that the transition amplitude of this particle from the state $(x_i,\, t_i)$ to the state $(x_f,\, t_f)$ is

$$K(x_f,\, t_f;\, x_i,\, t_i) = \left(\frac{m}{2\pi i\hbar\,(t_f - t_i)}\right)^{1/2} \times \exp\left\{\frac{i}{\hbar}\left[\frac{m(x_f - x_i)^2}{2(t_f - t_i)} + \frac{V(t_f - t_i)(x_f - x_i)}{2} - \frac{V^2(t_f - t_i)^3}{24m}\right]\right\}.$$

Hints:

- When summing over $\mathrm{d}x_k$, take the central value for x in each interval. That is, $V\,x \to V\,(x_{k+1} + x_k)/2$.

- In the free particle example we saw that after each $\mathrm{d}x_k$ integration the prefactor of Δt increased. This case works in the same way, but the term $\propto V^2$ has a more complicated relationship with this quantity. Perform the first few integrations to verify that its general form is

$$-\frac{V^2(N\Delta t)^3}{24m}\left(1-\frac{1}{N^2}\right),$$

  where N takes values 2, 3, 4, ... after successive $\mathrm{d}x_k$ integrations. Then proceed as in the free particle case.

Bonus: Use a classical particle with the same Lagrangian to verify the previous result. That is, show that its classical action is

$$S_{\mathrm{cl}} = \frac{mX^2}{2T} + \frac{VTX}{2} - \frac{V^2T^3}{24m},$$

where $T = t_f - t_i$ and $X = x_f - x_i$.
Hints:

- You can use translational invariance to set $x_i = t_i = 0$ and make the math a bit easier.
- A classical particle with this Lagrangian would obey

$$m\ddot{x} = V.$$

Exercise 2. (25 points): Transition amplitude & Schrödinger equation

Use the expression for the transition amplitude $K_0(x,t;0,0)$ for a free particle in one dimension as derived in the lecture. Show that K_0 satisfies a Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}K_0(x,t;0,0) = i\hbar\frac{\partial}{\partial t}K_0(x,t;0,0).$$