Theoretical Physics 5 : WS 2024/2025
Exercise sheet 10

06.01.2025

Exercise 0.

How much time did it take to complete this exercise sheet?

Exercise 1. (40 points): Ground-state Coulomb Dirac wave functions

In the lecture notes you have looked at the hydrogen atom using Dirac theory. There, you expressed the radial Dirac Coulomb wave functions $F(\rho)$ and $G(\rho)$ as a power series,

$$F(\rho) = \sqrt{k_2} e^{-\rho/2} \sum_{m=0}^{n'} a_m \rho^{m+\gamma}$$
$$G(\rho) = \sqrt{k_1} e^{-\rho/2} \sum_{m=0}^{n'} b_m \rho^{m+\gamma},$$

where $n' = n - (j + 1/2)$, $\rho = 2\sqrt{k_1 k_2} r$ and $k_{1,2} = \frac{1}{\hbar c}(\pm E + m_0 c^2)$.

Consider the $1s_{1/2}$ state ($n = 1$, $j = 1/2$).

a) (*30 p.*) Start from the normalization condition (the integral is over r, not ρ!)

$$\int_0^\infty \mathrm{d}r \, \left[F^2 + G^2\right] = 1$$

and show that

$$F(\rho) = \left(\frac{m_0 c^2}{\hbar c}\right)^{\frac{1}{2}} \sqrt{\frac{Z\alpha(1-\gamma)}{\Gamma(2\gamma+1)}} \rho^\gamma e^{-\rho/2},$$

$$G(\rho) = -\left(\frac{m_0 c^2}{\hbar c}\right)^{\frac{1}{2}} \sqrt{\frac{Z\alpha(1+\gamma)}{\Gamma(2\gamma+1)}} \rho^\gamma e^{-\rho/2}.$$

b) (*10 p.*) When can we expect the equation for the energy spectrum,

$$E_{nj} = \frac{m_0 c^2}{\sqrt{1 + \left(\frac{Z\alpha}{n-(j+1/2)+\gamma}\right)^2}},$$

to break down? What happens, physically, in this limit? No calculations are necessary.

Hint: Take a look at the expression for γ in the lecture notes.

Exercise 2. (60 points): Dirac field anticommutators

Using the normal mode expansion of the Dirac field

$$\psi(\boldsymbol{x}, t) = \sum_{\boldsymbol{p}} \sum_{s_z} \left(\frac{m_0 c^2}{E_p V}\right)^{1/2} \left[b(\boldsymbol{p}, s_z)\, u(\boldsymbol{p}, s_z) e^{-\frac{i}{\hbar} p\cdot x} + d^\dagger(\boldsymbol{p}, s_z)\, v(\boldsymbol{p}, s_z) e^{\frac{i}{\hbar} p\cdot x}\right]$$

$$\bar{\psi}(\boldsymbol{x}, t) = \sum_{\boldsymbol{p}} \sum_{s_z} \left(\frac{m_0 c^2}{E_p V}\right)^{1/2} \left[b^\dagger(\boldsymbol{p}, s_z)\, \bar{u}(\boldsymbol{p}, s_z) e^{\frac{i}{\hbar} p\cdot x} + d(\boldsymbol{p}, s_z)\, \bar{v}(\boldsymbol{p}, s_z) e^{-\frac{i}{\hbar} p\cdot x}\right],$$

and the equal-time creation and annihilation operators anticommutation relations

$$\{b(\boldsymbol{p}, s_z), b^\dagger(\boldsymbol{p}', s_z')\} = \delta_{\boldsymbol{p},\boldsymbol{p}'} \delta_{s_z, s_z'},$$
$$\{d(\boldsymbol{p}, s_z), d^\dagger(\boldsymbol{p}', s_z')\} = \delta_{\boldsymbol{p},\boldsymbol{p}'} \delta_{s_z, s_z'},$$
$$\{b(\boldsymbol{p}, s_z), b(\boldsymbol{p}', s_z')\} = 0, \qquad \{d(\boldsymbol{p}, s_z), d(\boldsymbol{p}', s_z')\} = 0,$$
$$\{b(\boldsymbol{p}, s_z), d(\boldsymbol{p}', s_z')\} = 0, \qquad \{d(\boldsymbol{p}, s_z), b(\boldsymbol{p}', s_z')\} = 0,$$
$$\{b(\boldsymbol{p}, s_z), d^\dagger(\boldsymbol{p}', s_z')\} = 0, \qquad \{d(\boldsymbol{p}, s_z), b^\dagger(\boldsymbol{p}', s_z')\} = 0;$$

a) (*20 p.*) Express

$$H = \int \mathrm{d}^3 x \, N\left(\bar{\psi}(-i\hbar\gamma^i \partial_i + m_0 c)\psi\right),$$

in terms of creation and annihilation operators.

b) (*20 p.*) Express the momentum operator,

$$\boldsymbol{P} = -i\hbar \int \mathrm{d}^3x \, N\left(\psi^\dagger \boldsymbol{\nabla} \psi\right),$$

in terms of creation and annihilation operators.

c) (*20 p.*) Show that

$$[H, b^\dagger(\boldsymbol{p}, s_z) b(\boldsymbol{p}, s_z)] = 0.$$