Theoretical Physics 5 : WS 2024/2025
Exercise sheet 9

16.12.2024
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Exercise 0.

How much time did it take to complete this exercise sheet?

Exercise 1. (100 points): Dirac particle in a spherical potential well

Consider the Dirac equation,

$$\left[\boldsymbol{\alpha}\cdot\boldsymbol{p}\,c+\beta m_0c^2\right]\psi(\boldsymbol{r})=\left[E-V(r)\right]\psi(\boldsymbol{r}),$$

in a spherical potential well,

$$V(r)=\begin{cases}-V_0 & \text{for} \quad r\leq R,\\ 0 & \text{for} \quad r>R.\end{cases}$$

a) (*25 p.*) Show that

$$\boldsymbol{\alpha}\cdot\boldsymbol{p}=-i(\boldsymbol{\alpha}\cdot\hat{\boldsymbol{e}}_r)\left(\hbar\frac{\partial}{\partial r}+\frac{\hbar}{r}-\frac{\beta}{r}K\right),$$

with $K=\beta\left(\boldsymbol{\Sigma}\cdot\boldsymbol{L}+\hbar\right)$ and $\hat{\boldsymbol{e}}_r=\boldsymbol{r}/r$.
Hint: use $\boldsymbol{\nabla}=\hat{\boldsymbol{e}}_r(\hat{\boldsymbol{e}}_r\cdot\boldsymbol{\nabla})-\hat{\boldsymbol{e}}_r\times(\hat{\boldsymbol{e}}\times\boldsymbol{\nabla})$.

b) (*25 p.*) Starting from the Dirac equation, rewrite $\boldsymbol{\alpha} \cdot \boldsymbol{p}$ as in the previous question and then use the ansatz

$$\psi(\boldsymbol{r}) = \begin{pmatrix} g(r)\phi_A(\theta,\phi) \\ if(r)\phi_B(\theta,\phi) \end{pmatrix},$$

where $\phi_{A/B}$ are the eigenstates of the $\boldsymbol{\sigma} \cdot \boldsymbol{L}$ operator,

$$\begin{aligned} (\boldsymbol{\sigma} \cdot \boldsymbol{L})\, \phi_A &= -\hbar(\kappa+1)\phi_A, \\ (\boldsymbol{\sigma} \cdot \boldsymbol{L})\, \phi_B &= \hbar(\kappa-1)\phi_B, \end{aligned}$$

to find the differential equations for $G(r)$ and $F(r)$. They are related to $g(r)$ and $f(r)$ as

$$f(r) = \frac{F(r)}{r}, \qquad g(r) = \frac{G(r)}{r}.$$

c) (*25 p.*) For

$$k^2 = \frac{1}{(\hbar c)^2}\left[(E+V_0)^2 - m_0^2 c^4\right] > 0.$$

the general solution of the differential equation of question b is given by

$$\begin{aligned} G(r) &= r\left[a_1 j_{l_A}(kr) + a_2 y_{l_A}(kr)\right], \\ F(r) &= \frac{\kappa}{|\kappa|}\frac{\hbar c\, k\, r}{E+V_0+m_0c^2}\left[a_1 j_{l_B}(kr) + a_2 y_{l_B}(kr)\right], \end{aligned}$$

where j_l and y_l are the spherical Bessel functions of the first and second kind.

For

$$\tilde{k}^2 = -\frac{1}{(\hbar c)^2}\left[(E+V_0)^2 - m_0^2 c^4\right] > 0$$

the general solution is given by:

$$\begin{aligned} G(r) &= r\sqrt{\frac{2\tilde{k}r}{\pi}}\left[b_1 K_{l_A+1/2}(\tilde{k}r) + b_2 I_{l_A+1/2}(\tilde{k}r)\right], \\ F(r) &= \frac{\hbar c\, \tilde{k}\, r}{E+V_0+m_0c^2}\sqrt{\frac{2\tilde{k}r}{\pi}}\left[-b_1 K_{l_B+1/2}(\tilde{k}r) + b_2 I_{l_B+1/2}(\tilde{k}r)\right], \end{aligned}$$

where $K_{l+1/2}$ and $I_{l+1/2}$ are the modified Bessel functions.
In either case,

$$l_A = \begin{cases} j+\frac{1}{2} & \text{for} \quad \kappa = j+\frac{1}{2}, \\ j-\frac{1}{2} & \text{for} \quad \kappa = -(j+\frac{1}{2}), \end{cases}$$

and

$$l_B = \begin{cases} j-\frac{1}{2} & \text{for} \quad \kappa = j+\frac{1}{2}, \\ j+\frac{1}{2} & \text{for} \quad \kappa = -(j+\frac{1}{2}). \end{cases}$$

Show that the transcendental equation that gives the energy spectrum of bound states for which $E - m_0c^2 > -V_0$ and $-m_0c^2 < E < m_0c^2$ is

$$\frac{j_{l_A}(kR)}{j_{l_B}(kR)} = -\frac{\kappa}{|\kappa|}\frac{k}{\tilde{k}}\frac{E + m_0c^2}{E + V_0 + m_0c^2}\frac{K_{l_A+1/2}(\tilde{k}R)}{K_{l_B+1/2}(\tilde{k}R)}.$$

You do not need to solve this equation.

Hint: your solution must be normalizable. Consider limits of the special functions (Bessel, etc.) – do they exclude certain solutions?
You will end up with different expressions for F and G for $r < R$ and for $r > R$. You can match their ratio at $r = R$, ie $\frac{G^<(R)}{F^<(R)} = \frac{G^>(R)}{F^>(R)}$, to cancel out the constants and get an equation for the energy spectrum.

d) (*25 p.*) Starting from the equation of question c, derive the following relation for s-states ($l = 0$), which correspond to $j = \frac{1}{2}$ and $\kappa = -1$:

$$\tan\left(\frac{R}{\hbar c}\sqrt{(E+V_0)^2 - m_0^2c^4}\right)\sqrt{\frac{E + V_0 + m_0c^2}{E + V_0 - m_0c^2}} \\ \times\left\{\frac{\hbar c}{R}\left[\frac{1}{E + m_0c^2} - \frac{1}{E + V_0 + m_0c^2}\right] - \sqrt{\frac{m_0c^2 - E}{m_0c^2 + E}}\right\} = 1.$$

This equation relates the energy eigenvalues of the s-states to the properties of the spherical potential well. Remarkably, we can (still) find analytic solutions!

Hint: you can use the following Rayleigh formulas for (spherical) Bessel functions:

$$j_n(x) = (-1)^n x^n \left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(\frac{\sin(x)}{x}\right),$$
$$y_n(x) = -(-1)^n x^n \left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(\frac{\cos(x)}{x}\right),$$
$$i_n(x) = x^n \left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(\frac{\sinh(x)}{x}\right),$$
$$k_n(x) = (-1)^n x^n \left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(\frac{e^{-x}}{x}\right),$$

with

$$i_n(x) = \sqrt{\frac{\pi}{2x}}I_{n+1/2}(x) \qquad \text{and} \qquad k_n(x) = \sqrt{\frac{\pi}{2x}}K_{n+1/2}(x).$$