Theoretical Physics 5 : WS 2024/2025
Exercise sheet 8

9.12.2024

Exercise 0.

How much time did it take to complete this exercise sheet?

Exercise 1. (50 points): Rotations around the x-axis

a) *(40 p.)* A finite rotation can be expressed as a series of N successive infinitesimal rotations where $N \to \infty$. For a rotation around the x-axis, this can be written as

$$x'^{\mu} = \lim_{N\to\infty} \left[\left(1 + \frac{\varphi}{N} J_x \right)^N \right]^{\mu}_{\ \nu} x^{\nu},$$

where

$$J_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

is the generator of an infinitesimal rotation around the x-axis.
Show that the finite rotation is given by

$$x'^{\mu} = \left[1 + J_x^2 + \cos\varphi(-J_x^2) + \sin\varphi J_x \right]^{\mu}_{\ \nu} x^{\nu}$$

and calculate the corresponding rotation matrix.
Hint: which function is defined by the limit $\lim_{N\to\infty} \left[\left(1 + \frac{x}{N} \right)^N \right]$?
Also, remember your Taylor series!

b) (*10 p.*) Show that

$$(\Lambda^\mu_{\ \nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\varphi & -\sin\varphi \\ 0 & 0 & \sin\varphi & \cos\varphi \end{pmatrix}$$

is a Lorentz transformation, i.e. that it satisfies

$$\Lambda^\mu_{\ \rho}\, g_{\mu\nu}\, \Lambda^\nu_{\ \sigma} = g_{\rho\sigma}.$$

Exercise 2. (25 points): Gamma matrices form a basis

Show that the most general 4×4 matrix can be written as a linear combination (with complex coefficients) of

$$1,\ \gamma^\mu,\ \sigma^{\mu\nu},\ \gamma^\mu\gamma_5,\ \gamma_5,$$

where 1 is the identity matrix and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$.

Hint: The standard inner product of two matrices is given by $\mathrm{tr}\left(A^\dagger B\right)$. Show that all matrices are linearly independent and then count the number of elements that you have.

Exercise 3. (25 points) : Dirac bilinears

Since a spinor gains a minus sign after rotating over 2π, physical quantities must be bilinears in the fields ψ. In this way physical quantities turn into themselves after a rotation over 2π. These bilinears have the general form $\bar{\psi}\Gamma\psi$. As you've seen in the previous exercise, there are 16 independent bilinears related to 16 complex 4×4 matrices:

- $\Gamma_S = 1$ (scalar),
- $\Gamma_P = \gamma_5$ (pseudoscalar),
- $\Gamma_V^\mu = \gamma^\mu$ (vector),
- $\Gamma_A^\mu = \gamma^\mu\gamma_5$ (axial vector),
- $\Gamma_T^{\mu\nu} = \sigma^{\mu\nu}$ (tensor).

For the following, don't use any explicit representation for the Γ matrices.

a) (*10 p.*) Show that $\Gamma_i^2 = \pm 1$ for $i = S, P, V, A, T$.
Note that $\Gamma_i^2 = (\Gamma_i^\mu)^2$ or $(\Gamma^{\mu\nu})^2$ without a sum over the Lorentz indices — you're taking the square of each of the 16 matrices, not contracting them!.

b) (*15 p.*) In the lecture, you saw that after a Lorentz transformation of the Dirac spinor ($\psi'(x') = S(a)\psi(x)$ with $x'^\mu = a^\mu_\nu x^\nu$), the scalar, pseudoscalar and vector bilinears transform properly. That is, the scalar transforms like a scalar, the vector like a vector and so on. In other words, you showed that:

- $\bar{\psi}' \psi' = \bar{\psi} \psi$
- $\bar{\psi}' \gamma_5 \psi' = \det(a) \bar{\psi} \gamma_5 \psi$
- $\bar{\psi}' \gamma^\mu \psi' = a^\mu_\nu \bar{\psi} \gamma^\nu \psi$

What remains is to check the same thing for the axial vector and the tensor bilinears. Show that:

- $\bar{\psi}' \gamma^\mu \gamma_5 \psi' = \det(a) a^\mu_\nu \bar{\psi} \gamma^\mu \gamma_5 \psi$
- $\bar{\psi}' \sigma^{\mu\nu} \psi' = a^\mu_\kappa a^\nu_\lambda \bar{\psi} \sigma^{\kappa\lambda} \psi$