Theoretical Physics 5 : WS 2024/2025
Exercise sheet 7

2.12.2024

Exercise 0.

How much time did it take to complete this exercise sheet?

Exercise 1. (15 points): Tensor algebra refresher

a) (*5 p.*) Simplify the expressions below by contracting the tensors with the metric $g^{\mu\nu}$ and carrying out the derivatives:

1. $g_{\mu\nu} A^\nu$
2. $g^\mu_\nu A^{\nu\rho}$
3. $A^{\mu\nu} B_\rho \, g^\rho_\sigma \, g^\sigma_\nu$
4. $A^\mu_\nu \, g^{\rho\sigma} g^\nu_\rho$
5. $(\partial_\mu x_\nu) A^\nu$

Hint:

$$\partial_\mu := \frac{\partial}{\partial x^\mu} \qquad \text{and} \qquad \partial_\mu x^\nu = \frac{\partial x^\nu}{\partial x^\mu} = \delta^\nu_\mu$$

b) (*3 p.*) Show that $v^\mu w_\mu = v_\nu w^\nu$ for any v, w.

c) (*2 p.*) Use an equality to express that $A^{\mu\nu}$ is symmetric with respect to μ, ν. Do the same for B^μ_ν, which is antisymmetric w.r.t μ, ν.

d) (*5 p.*) Are the following expressions valid in terms of index notation? If not, what's wrong with them?

1. $A^{\mu\nu}B^{\sigma}_{\mu}g^{\sigma\rho}$
2. $A^{\nu}_{\mu}B_{\sigma\rho}g^{\rho}_{\nu}$
3. $A^{\rho}B_{\mu}C^{\mu\nu}_{\rho} = D^{\nu}_{\alpha}E^{\alpha}$
4. $A^{\mu}_{\rho}B^{\rho} = C^{\nu}_{\sigma}B^{\sigma}$
5. $A^{\mu\nu}_{\sigma} = g_{\sigma\rho}B^{\mu}C^{\rho}D^{\nu}_{\alpha}E^{\alpha}$

Exercise 2. (85 points): Dirac matrix gymnastics

Without using an explicit representation for the Dirac matrices, show that:

a) (*5 p.*) $\gamma_{\mu}\gamma^{\mu} = 4\mathbb{I}_4$
Hint: use the anticommutation relations and the fact that the metric is symmetric.

b) (*5 p.*) $\gamma^{\mu}\gamma^{\nu}\gamma_{\mu} = -2\gamma^{\nu}$

c) (*10 p.*) $\mathrm{tr}[\gamma^{\nu}] = 0$
Hint: try to show that $\mathrm{tr}[\gamma^{\nu}] = -\mathrm{tr}[\gamma^{\nu}]$ by inserting an identity matrix in the trace and replacing it using the anticommutation relation for $\mu = \nu$. Keep in mind the cyclic property of the trace, $\mathrm{tr}[ABC] = \mathrm{tr}[BCA] = \mathrm{tr}[CAB]$.

d) (*5 p.*) $\mathrm{tr}[\gamma^{\mu}\gamma^{\nu}] = 4g^{\mu\nu}$

e) (*10 p.*) $\mathrm{tr}[\not{a}\,\not{b}\,\not{c}\,\not{d}] = 4\,(a_{\mu}b^{\mu}c_{\nu}d^{\nu} - a_{\mu}c^{\mu}b_{\nu}d^{\nu} + a_{\mu}d^{\mu}b_{\nu}c^{\nu})$, where $\not{a} := \gamma^{\mu}a_{\mu}$
Hint: $\mathrm{tr}[\not{a}\,\not{b}\,\not{c}\,\not{d}] = a_{\mu}b_{\nu}c_{\rho}d_{\sigma}\,\mathrm{tr}[\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}]$. Try moving γ^{σ} to the left using anticommutation relations and then apply the cyclic property to finally express this trace in terms of combinations of the metric.

f) (*10 p.*) $(\gamma_5)^2 = 1$, with $\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$

g) (*10 p.*) $\{\gamma_5, \gamma^{\mu}\} = 0$

h) (*5 p.*) $\mathrm{tr}[\gamma_5] = 0$

i) (*10 p.*) $\mathrm{tr}[\gamma^{\mu}\gamma^{\nu}\gamma_5] = 0$

j) (*10 p.*) $\mathrm{tr}[\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\gamma_5] = 4i\,\varepsilon^{\mu\nu\rho\sigma}$, where $\varepsilon^{0123} = -\varepsilon_{0123} = -1$
Hint: consider what happens if at least two of the four indices are equal, and then consider the case $\mu \neq \nu \neq \rho \neq \sigma$ separately.

k) (*5 p.*) $\mathrm{tr}[\gamma^{\mu_1}\cdots\gamma^{\mu_n}] = 0$ if n is odd
Hint: try to insert the identity $(\gamma_5)^2 = 1$ in the trace and manipulate as usual.