Theoretical Physics 5 : WS 2024/2025
Exercise sheet 6

25.11.2024

Exercise 0.

How much time did it take to complete this exercise sheet?

Exercise 1. (100 points): Dirac particle in a scalar potential

Consider a Dirac particle traveling along the z-axis and subject to the scalar square-well potential

$$V(z) = \begin{cases} 0 & \text{when} \quad z < -a/2 & \text{(region I)} \\ V_0 & \text{when} \quad -a/2 \leq z \leq a/2 & \text{(region II)} \\ 0 & \text{when} \quad a/2 < z & \text{(region III)} \end{cases}$$

where $a > 0$ and $V_0 < 0$. In regions I and III, the Dirac equation takes the form

$$\left(\boldsymbol{\alpha} \cdot \hat{\boldsymbol{p}}\, c + \beta m_0 c^2\right) \psi = E\psi,$$

while in region II it takes the form

$$\left[\boldsymbol{\alpha} \cdot \hat{\boldsymbol{p}}\, c + \beta(m_0 c^2 + V_0)\right] \psi = E\psi.$$

In region II, due to the potential, the particle has an effective mass $m_{\text{eff}} = m_0 + V_0/c^2$.

a) (*30 p.*) Write down the general solution $\psi(z)$ in each of the three regions for a particle with the spin in the z-direction.
Hints: A plane-wave solution with momentum $\boldsymbol{p}$, mass m and spin label s can be written as

$$u(\vec{p}, s) = A \begin{pmatrix} \chi_s \\ \frac{c\ \boldsymbol{\sigma} \cdot \boldsymbol{p}}{E + mc^2}\, \chi_s \end{pmatrix} e^{\frac{i}{\hbar}(\boldsymbol{p} \cdot \boldsymbol{r} - Et)},$$

where A is some complex number and χ_s a two-component spinor. For the spin projection along the z-axis, χ_s is an eigenstate of the Pauli matrix σ_3. Do not forget that plane waves can travel in both directions. Remember that the matrices $\boldsymbol{\alpha}$ and β in standard representation are given by

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

b) (*35 p.*) Impose the continuity condition at $z = \pm a/2$. Express the coefficient of the incoming plane wave of region I in terms of the plane waves of region III. Define, for convenience, the dimensionless quantity

$$\gamma := \frac{k_1 c}{E + m_0 c^2} \frac{E + m_{\text{eff}} c^2}{k_2 c},$$

where k_1 is the momentum in regions I and III, and k_2 is the momentum in region II. *Hint:* You should find

$$A = \frac{1}{4\gamma} \Bigg\{ C \left[(\gamma+1)^2 e^{i(k_1-k_2)a/\hbar} - (\gamma-1)^2 e^{i(k_1+k_2)a/\hbar} \right] - 2iC'(1-\gamma^2) \sin\left(\frac{k_2 a}{\hbar}\right) \Bigg\},$$

where A, C, and C' are the coefficients of the region I incoming, region III incoming and region III outgoing plane wave respectively.

c) (*25 p.*) Consider the special case $|m_{\text{eff}} c^2| < |E| < m_0 c^2$, which corresponds to bound states. Show that these states satisfy

$$k_2 \cot\left(\frac{k_2 a}{\hbar}\right) = \frac{m_0 V_0}{\kappa_1} + \kappa_1,$$

where $\kappa_1 = -i k_1$.
Hints: Show that in the considered case there can be neither an incoming wave in region I nor an outgoing wave in region III. Then, show that continuity demands

$$\text{Im}\left(\frac{1+\gamma}{1-\gamma} e^{-\frac{i}{\hbar} k_2 a}\right) = 0,$$

and that $\gamma = i\Gamma$ is imaginary, which leads to

$$\cot\left(\frac{k_2 a}{\hbar}\right) = \frac{1-\Gamma^2}{2\Gamma}.$$

d) (*10 p.*) How does the solution change if the potential is the 0th component of a 4-vector - that is, $V^\mu = (V_0, 0, 0, 0)$ - instead of a scalar (which as we saw modifies the mass term)?