Theoretical Physics 5 : WS 2024/2025
Exercise sheet 5

18.11.2024

Exercise 0.

How much time did it take to complete this exercise sheet?

Exercise 1. (40 points) : Real Klein-Gordon field

Using the normal mode expansion of the real Klein-Gordon field

$$\phi(\boldsymbol{x},t) = \sum_{\boldsymbol{k}} \left(\frac{\hbar c^2}{2\omega_k L^3}\right)^{1/2} \left(a(\boldsymbol{k})\, e^{-ik\cdot x} + a^\dagger(\boldsymbol{k})\, e^{ik\cdot x}\right) \tag{1}$$

and the equal-time commutation relations

$$\begin{aligned} [\phi(\boldsymbol{x},t), \phi(\boldsymbol{x'},t)] &= 0, \\ [\dot\phi(\boldsymbol{x},t), \dot\phi(\boldsymbol{x'},t)] &= 0, \\ [\phi(\boldsymbol{x},t), \dot\phi(\boldsymbol{x'},t)] &= i\hbar\, c^2\, \delta^{(3)}(\boldsymbol{x}-\boldsymbol{x'}), \end{aligned} \tag{2}$$

Show that:

a) (*20 p.*) the creation and annihilation operators satisfy the following commutation relations

$$\begin{aligned} [a(\boldsymbol{k}), a(\boldsymbol{k'})] &= 0, \\ [a^\dagger(\boldsymbol{k}), a^\dagger(\boldsymbol{k'})] &= 0, \\ [a(\boldsymbol{k}), a^\dagger(\boldsymbol{k'})] &= \delta_{\boldsymbol{k},\boldsymbol{k'}} \end{aligned}$$

b) (*10 p.*) the Hamiltonian $\mathcal{H} = \int \mathrm{d}^3x\, \frac{1}{2}\left(\frac{1}{c^2}\,\dot\phi^2 + (\boldsymbol{\nabla}\phi)^2 + \mu^2\phi^2\right)$ takes the form

$$\mathcal{H} = \sum_{\boldsymbol{k}} \hbar\omega_k \left(a^\dagger(\boldsymbol{k})a(\boldsymbol{k}) + \frac{1}{2}\right) \tag{3}$$

c) (*10 p.*) the momentum $\boldsymbol{P} = -\int \mathrm{d}^3x \, \frac{1}{c^2} \, \dot{\phi} \, \boldsymbol{\nabla}\phi$ takes the form

$$\boldsymbol{P} = \sum_{\boldsymbol{k}} \hbar \boldsymbol{k} \; a^\dagger(\boldsymbol{k}) a(\boldsymbol{k}) \tag{4}$$

Exercise 2. (60 points) : Complex Klein-Gordon field

The complex Klein-Gordon field is used to describe charged bosons. Its Lagrangian is given by

$$\mathcal{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - \mu^2 \phi^\dagger \phi, \tag{5}$$

where the normal mode expansion of field ϕ is

$$\phi(\boldsymbol{x}, t) = \sum_{\boldsymbol{k}} \left(\frac{\hbar c^2}{2\omega_k L^3} \right)^{1/2} \left[a(\boldsymbol{k}) \, e^{-ik\cdot x} + b^\dagger(\boldsymbol{k}) \, e^{ik\cdot x} \right]. \tag{6}$$

ϕ satifies the equal-time commutation relations

$$[\phi(\boldsymbol{x}, t), \Pi_\phi(\boldsymbol{x'}, t)] = i\hbar \, \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x'}), \tag{7}$$

$$\left[\phi^\dagger(\boldsymbol{x}, t), \Pi_{\phi^\dagger}(\boldsymbol{x'}, t)\right] = i\hbar \, \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x'}), \tag{8}$$

with all other commutators vanishing. In the following, you can consider the fields ϕ and $\phi^\dagger$ as independent.

a) (*15 p.*) Show that (5) is equivalent to the Lagrangian of two independent real scalar fields with the same mass, which satisfy the standard equal-time commutation relations.
Hint: Decompose the complex field into real components: $\phi = \frac{1}{\sqrt{2}} (\phi_1 + i\phi_2)$.

b) (*15 p.*) Write down the conjugate momentum fields Π_ϕ and $\Pi_{\phi^\dagger}$ in terms of ϕ and $\phi^\dagger$ and derive the equal-time commutation relations of a, $a^\dagger$, b and $b^\dagger$.
Hint: Use the normal mode expansion of ϕ and its decomposition into $\phi_{1,2}$ to derive relationships between (a, $a^\dagger$, b, and $b^\dagger$) and (a_1, $a_1^\dagger$, a_2, and $a_2^\dagger$). Think about the commutation relations of the latter and use them to derive the commutation relations of the former.

c) (*15 p.*) Show that (5) is invariant under any global phase transformation of the field $\phi \to \phi' = e^{-i\alpha}\phi$ with α real. Write down the associated conserved Noether current j^μ and express the conserved charge $Q = \int \mathrm{d}^3x \, j^0$ in terms of creation and annihilation operators.

d) (*15 p.*) Compute the commutators $[Q, \phi]$ and $[Q, \phi^\dagger]$. Using these commutators and the eigenstates $|q\rangle$ of the charge operator Q, show that the field operators ϕ and $\phi^\dagger$ modify the charge of the system. How would you interpret the operators a, $a^\dagger$, b and $b^\dagger$?