Theoretical Physics 5 : WS 2024/2025
Exercise sheet 4

11.11.2024

Exercise 0.

How much time did it take to complete this exercise sheet?

Exercise 1. (50 points): Spacetime symmetries

In this exercise we will explore how symmetries of a Lagrangian lead to conserved currents. First, let's revisit the lecture notes and expand on them a little.

Suppose we have a Lagrangian $\mathcal{L}$. Let's look at a transformation where $\delta\mathcal{L}(x)$ is not necessarily equal to zero, but it is equal to a total divergence, $\delta\mathcal{L}(x) = \partial_\mu K^\mu(x)$ for some $K^\mu(x)$. By definition, the Noether current is given by

$$j^\mu = \frac{\partial\mathcal{L}(x)}{\partial(\partial_\mu\phi(x))}\delta\phi(x) - K^\mu(x). \tag{1}$$

If we consider an infinitesimal spacetime translation $\phi(x) \to \phi(x+a)$ we can only keep the first terms of the Taylor expansion of $\phi(x+a)$:

$$\phi(x+a) \simeq \phi(x) + a^\nu\partial_\nu\phi(x) \implies \delta\phi(x) = a^\nu\partial_\nu\phi(x), \tag{2}$$

and similarly

$$\mathcal{L}(x+a) \simeq \mathcal{L}(x) + a^\nu\partial_\nu\mathcal{L}(x) \implies \delta\mathcal{L}(x) = a^\nu\partial_\nu\mathcal{L}(x) = \partial_\nu\left(a^\nu\mathcal{L}(x)\right). \tag{3}$$

Thus, $K^\nu(x) = a^\nu\mathcal{L}(x)$ and the Noether current is given by

$$\begin{aligned} j^\mu(x) &= \frac{\partial\mathcal{L}(x)}{\partial(\partial_\mu\phi(x))}a^\nu\partial_\nu\phi(x) - a^\mu\mathcal{L}(x) \\ &= a_\nu T^{\mu\nu}, \end{aligned} \tag{4}$$

where the energy-momentum tensor is defined as

$$T^{\mu\nu} = \frac{\partial \mathcal{L}(x)}{\partial(\partial_\mu \phi)} \partial^\nu \phi(x) - g^{\mu\nu} \mathcal{L}(x). \tag{5}$$

Besides spacetime translation invariance, a quantum field theory should also be invariant under a Lorentz transformation (rotations and boosts). This is another symmetry which, according to Noether's theorem, will give rise to conserved currents. It is up to you to derive these currents.

a) (*15 p.*) The infinitesimal form of a Lorentz transformation is given by

$$\phi(x) \to \phi(x^\mu + \delta\omega^{\mu\nu} x_\nu), \tag{6}$$

where $\delta\omega_{\mu\nu}$ are constants and the tensor $\delta\omega$ is antisymmetric in its indices. For example, a rotation about a unit vector $\hat{\boldsymbol{n}}$ with an angle θ gives $\delta\omega_{ij} = -\varepsilon_{ijk}\hat{n}_k \delta\theta$, while a boost in the same direction with rapidity η gives $\delta\omega_{i0} = \hat{n}_i \delta\eta$.

Like we've done in the example, derive expressions for $\delta\phi(x)$, $\delta\mathcal{L}(x)$. Show that

$$K^\mu = \delta\omega^{\mu\nu} x_\nu \mathcal{L}. \tag{7}$$

In deriving this expression, it might appear as if you have to pull a derivative through x^μ. If you do this, explain why it is allowed.

b) (*15 p.*) Show that the Noether current can be written as

$$j^{\mu} = -\frac{1}{2}\mathcal{M}^{\mu\alpha\beta}\delta\omega_{\alpha\beta}, \tag{8}$$

where

$$\mathcal{M}^{\mu\alpha\beta} = x^{\alpha}T^{\mu\beta} - x^{\beta}T^{\mu\alpha}. \tag{9}$$

Side notes — you don't need these to solve the exercise:

- For any field which is not a scalar field, the expression above gets an extra term, $B^{\mu\nu\rho}$, which is in general quite complicated and made up from the fields and their derivatives. Moreover, the energy-momentum tensor ceases to be symmetric (which, for some deep reasons, is a rather bad thing). Therefore, using this new B tensor one can define another energy-momentum tensor, called the Belinfante tensor, which is symmetric in its indices. What's more, it is precisely this tensor which also shows up in the field equations of Einstein's theory of general relativity as the energy-momentum tensor!
- The conserved charges associated with this current are

  $$M^{\nu\rho} = \int \mathrm{d}^3x\, \mathcal{M}^{0\nu\rho}(x), \tag{10}$$

  and these are called the generators of the Lorentz group. Like how momentum was defined from the stress-energy tensor, we can define angular momentum as

  $$J_i := \frac{1}{2}\varepsilon_{ijk}M^{jk}. \tag{11}$$

  Moreover, it is possible to prove that then

  $$[J_i, J_j] = i\varepsilon_{ijk}J_k, \quad [J_i, P_j] = i\varepsilon_{ijk}P_k, \tag{12}$$

  which are the commutation relations for momentum and angular momentum.

c) (*20 p.*) Consider explicitly

$$\mathcal{L} = \partial_{\mu}\phi^{\dagger}\partial^{\mu}\phi - m^2\phi^{\dagger}\phi. \tag{13}$$

In the following, you should treat ϕ and $\phi^{\dagger}$ as independent variables.

Let us define two new real fields via,

$$\phi = \frac{1}{\sqrt{2}}\left[\phi_1 + i\phi_2\right], \quad \phi^{\dagger} = \frac{1}{\sqrt{2}}\left[\phi_1 - i\phi_2\right]. \tag{14}$$

Show that

$$\mathcal{L} = \frac{1}{2}\partial_{\mu}\phi_1\partial^{\mu}\phi_1 - \frac{1}{2}m^2\phi_1^2 + \frac{1}{2}\partial_{\mu}\phi_2\partial^{\mu}\phi_2 - \frac{1}{2}m^2\phi_2^2. \tag{15}$$

The Lagrangian with ϕ and $\phi^\dagger$ is invariant under

$$\phi \to e^{-i\beta}\phi, \quad \beta \in \mathbb{R}, \tag{16}$$

with the corresponding transformation for $\phi^\dagger$. This transformation belongs to the group U(1). Meanwhile, the Lagrangian with fields ϕ_1 and ϕ_2 is invariant under

$$\phi_1 \to \phi_1 \cos\beta + \phi_2 \sin\beta, \quad \phi_2 \to \phi_2 \cos\beta - \phi_1 \sin\beta, \tag{17}$$

which is a transformation belonging to SO(2).

Derive the Noether currents belonging to each transformation and show that they are proportional to each other.

In mathematical jargon, what you have just seen is a reflection of the fact that U(1) is isomorphic to SO(2).

Exercise 2. (50 points): Pionic atoms

A pionic atom is formed when a negative pion π^-, which is a spin-0 boson, is stopped in matter and is captured by an atom. The incident pion slows down by successive electromagnetic interactions with the electrons and nuclei. When the pion reaches the typical velocity of atomic electrons, the pion ejects a bound electron from its Bohr orbit and the pion is captured instead.

Let us approximate the potential between the nucleus and the pion by a square well:

$$V = \begin{cases} -V_0, & r \leq R \\ 0, & r > R \end{cases}, \tag{18}$$

where R is the nucleus radius.

a) *(20 p.)* Using the principle of minimal substitution, $p_\mu \to p_\mu - \frac{e}{c}A_\mu$ with $A_\mu = (V, \mathbf{0})$, show that the Klein-Gordon equation leads to the following radial equation for the field $u(r)$:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{l(l+1)}{r^2} + k^2\right] u(r) = 0, \tag{19}$$

where

$$k^2 = \frac{1}{\hbar^2 c^2}\left[(\epsilon - eV)^2 - m_\pi^2 c^4\right]$$

and ϵ is the energy of the pion.

Hint: How is the derivative operator defined?

Hint: Use the Klein-Gordon field in the following factorized form

$$\phi(\boldsymbol{x}, t) = u(r)\, Y_l^m(\Omega)\, e^{-\frac{i}{\hbar}\epsilon t} \tag{20}$$

and recall some properties of the spherical harmonics.

b) (*15 p.*) For a bound state we have $k^2 > 0$ for $r \leq R$ and $k^2 < 0$ for $r > R$. In both regions, solve the Eq. 19 for an s-state ($l = 0$).

Hint: Use the ansatz $u(r) = v(r)/r$.

c) (*15 p.*) Match the solutions in both regions by imposing equal logarithmic derivatives,

$$\left.\frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}r}\right|_{\text{interior solution},r=R} = \left.\frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}r}\right|_{\text{exterior solution},r=R},$$

and show that this matching amounts to solving the transcendental equation

$$k_i \cot(k_i R) = -k_o,$$

where $k_i^2 = \frac{1}{\hbar^2 c^2}\left[(\epsilon + eV_0)^2 - m_\pi^2 c^4\right]$ and $k_o^2 = \frac{1}{\hbar^2 c^2}\left(m_\pi^2 c^4 - \epsilon^2\right)$, with i standing for interior and o for exterior. You don't have to actually solve the equation.