Theoretical Physics 5 : WS 2024/2025
Exercise sheet 2

28.10.2024

Exercise 0.

How much time did it take to complete this exercise sheet?

Exercise 1. (30 points) : Boson wave functions

Explicity write the following N-boson wave functions in terms of single-boson wave functions $\psi_i(x_j)$.

a) *(5 p.)* $\Phi_{4000\cdots0}(x_1, x_2, x_3, x_4)$

b) *(5 p.)* $\Phi_{0111\cdots0}(x_1, x_2, x_3)$

c) *(5 p.)* $\Phi_{10021\cdots0}(x_1, x_2, x_3, x_4)$

d) *(5 p.)* $\Phi_{3010\cdots0}(x_1, x_2, x_3, x_4)$

How many terms will the following wave functions contain? Don't write them out explicitly!

e) *(5 p.)* $\Phi_{1131\cdots0}(x_1, x_2, x_3, x_4)$

f) *(5 p.)* $\Phi_{4502\cdots0}(x_1, x_2, x_3, x_4)$

Exercise 2. (45 Points) : Number of Bosons

Consider the particle number operator $N = \sum_m C_m^\dagger C_m$ for a system of bosons.

a) (*15 p.*) Calculate $[N, C_i C_j]$ and $\left[N, C_i^\dagger C_j^\dagger\right]$.

b) (*10 p.*) Calculate $[N, (C_i)^n]$.

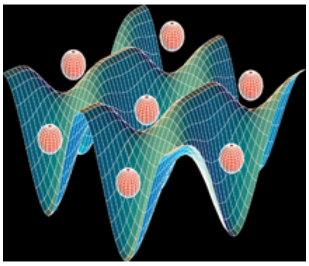
Hint: use induction. That is, see how the expression behaves for small values of n, extrapolate its general behavior for any n, and then show that if this behavior holds for $n-1$, it still holds for n.

c) (*20 p.*) Show that N commutes with the Hamiltonian

$$H = \sum_{i,j} \langle i|H_0|j\rangle \, C_i^\dagger C_j + \frac{1}{2} \sum_{i,j,k,l} \langle i,j|V|k,l\rangle \, C_i^\dagger C_j^\dagger C_k C_l.$$

What is the physical meaning of this commutation relation?

Exercise 3. (25 Points) : Bose-Hubbard model



The Bose-Hubbard model gives an approximate description of the physics of interacting bosons on a lattice. It can be used to study systems such as bosonic atoms on an optical lattice, *i.e.* a periodic trap formed by the interference of counter-propagating laser beams. This system resembles a crystal in the sense that the atoms are in a periodic potential.

The Hamiltonian of this model is given by

$$H = -t \sum_{\langle i,j\rangle} \left(C_i^\dagger C_j + C_j^\dagger C_i\right) + \frac{U}{2} \sum_i C_i^\dagger C_i \left(C_i^\dagger C_i - 1\right),$$

where Latin indices refer to lattice sites, $\langle i,j\rangle$ means that the sum is restricted over first neighbors only, and $U > 0$.

a) (*10 p.*) Provide an interpretation of each term of this Hamiltonian.

b) (*10 p.*) Show that in this model the number of particles is conserved.

c) (*5 p.*) Qualitatively discuss the behavior of this model in the limits $t \ll U$ and $t \gg U$.