I. REMINDER

During the previous classes we learned about the symmetry breaking and Higgs mechanism. The idea behind it is a fact that any system seeks to minimize its potential energy and fall into a ground state. Symmetry considerations often imply multiple equivalent ground states with the same energy, differing by some parameter. However, only one of these minima can be chosen, resulting in the spontaneous symmetry breaking.

If the symmetry is discrete, that is all about it. When a global continuous symmetries break spontaneously, new massless particles called Goldstone bosons emerge. The number of Goldstone bosons corresponds to the number of spontaneously broken symmetry generators.

If the symmetry is local, the situation is more complex. Minima are distinguished by gauge transformations, and choosing a specific minimum is equivalent to selecting a gauge. However, the Lagrangian expressed in a specific gauge is no longer explicitly gauge invariant. For example, examining the Lagrangian of a gauge field in the Lorenz or Coulomb gauge reveals this lack of invariance. This doesn't imply symmetry violation; it's merely a consequence of the chosen configuration. This brings us to the Elitzur's theorem, which states:

$$\textbf{Local gauge symmetry cannot be spontaneously broken} \tag{1}$$

In the context of the electroweak interaction the words "spontaneous breaking of local symmetry" are often used, which is kinda jargon. In this case there are no Goldstone bosons, gauge fields consume them and obtain an additional polarization, resulting in a mass term.

Exactly this happens in the electroweak sector. We introduce a Higgs particle with the potential of the form:

$$V \propto \left(\phi^{+}\phi - v^{2}\right)^{2} \tag{2}$$

Where the sign of the mass term is inverted, i.e. this can be treated as a tachyon field. Expansion around the chosen minimum restores the correct sign of the mass, the gauge bosons acquire mass, as well as the matter fields:

$$\phi\bar{\psi}\psi \to v\bar{\psi}\psi \tag{3}$$

Summarizing, we would also like to emplhasize that this is a completely classical mechanism, i.e. it originates only from Lagrangian properties - no quantum mechanics is involved at any level of this reasoning.

II. QUANTUM HIGGS MECHANISM

Another option for the mass generation is the so-called quantum Higgs mechanism, when no actual Higgs particle is required.

Consider chiral QCD as an illustrative example. Quarks, initially massless in the chiral limit, create a strong attractive potential. Consequently, the creation of quark-antiquark pairs becomes energetically favorable, leading to a ground state flooded with such a condensate:

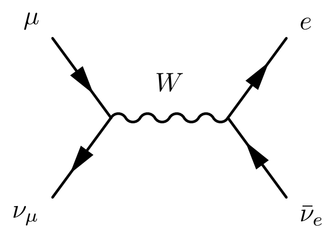
$$\langle|\bar{Q}Q|\rangle = \langle|\bar{Q}_R Q_L + \bar{Q}_L Q_R|\rangle \neq 0 \tag{4}$$

The product $\bar{Q}Q$ resembles a mass term, implying that quarks acquire effective masses due to quantum effects. A similar phenomenon occurs with electrons at low energies, known as superconductivity.

Curiously, this suggests that the Higgs boson might not be essentially needed in the Standard Model. Instead of introducing new particles, we could obtain masses solely through the quantum effects described above. Unfortunately, the Technicolor model, which explores this idea, has been completely ruled out by various experiments.

III. DETERMINATION OF HIGGS PARAMETER

A typical effect which is induced by the electroweak interaction is the $\mu \to e\bar{\nu}_e\nu_\mu$ decay:



Let's calculate the corresponding decay rate and for simplicity treat electron and neutrino to be massless. We also denote:

$$\mu = 1, \quad \nu_\mu = 2 \tag{5}$$
$$e = 3, \quad \bar{\nu}_e = 4 \tag{6}$$

At low energies ($Q_W^2 \ll M_Z^2$) the W-boson propagator can be simplified:

$$\frac{1}{Q_W^2 - M_W^2} \approx -\frac{1}{M_W^2} \tag{7}$$

In this case the matrix element reads:

$$\mathcal{M} = \frac{i}{M_W^2}\frac{1}{4}\left(\frac{ie}{\sqrt{2}}\sin\theta_w\right)^2 \left[\bar{u}_2\gamma^\alpha\left(1-\gamma^5\right)u_1\right]\left[\bar{u}_3\gamma_\alpha\left(1-\gamma^5\right)v_4\right] \tag{8}$$

As the electroweak unification was understood, we know that the gauge boson masses are:

$$M_W = \frac{v}{2}\frac{e}{\sin\theta_w} \tag{9}$$
$$M_Z = \frac{M_W}{\cos\theta_w} \tag{10}$$

Where θ_w is the mixing angle, a free parameter of the Standard Model, and v is the Higgs vacuum expectation value. Thus we can combine couplings into:

$$\frac{e^2}{2M_W^2\sin^2\theta_w} = \frac{g^2}{2M_W^2} = \frac{2}{v^2} \equiv \frac{4G_F}{\sqrt{2}} \tag{11}$$

Where we introduced Fermi constant G_F and thus:

$$\mathcal{M} = -\frac{G_F}{\sqrt{2}}\left[\bar{u}_2\gamma^\alpha\left(1-\gamma^5\right)u_1\right]\left[\bar{u}_3\gamma_\alpha\left(1-\gamma^5\right)v_4\right] \tag{12}$$

After squaring and summing/averaging over polarizations:

$$|\mathcal{M}|^2 = \frac{G_F^2}{4}\cdot \mathrm{Tr}\left\{\gamma^\alpha\left(1-\gamma^5\right)\left(\hat{p}_1+m_1\right)\gamma^\beta\left(1-\gamma^5\right)\hat{p}_2\right\}\cdot \mathrm{Tr}\left\{\gamma_\alpha\left(1-\gamma^5\right)\hat{p}_4\gamma_\beta\left(1-\gamma^5\right)\hat{p}_3\right\} \tag{13}$$

Using the standard identities for gamma matrices, we obtain:

$$\mathrm{Tr}\left\{\gamma^\alpha\left(1-\gamma^5\right)\left(\hat{p}_1+m_1\right)\gamma^\beta\left(1-\gamma^5\right)\hat{p}_2\right\} = 8\left[-g^{\alpha\beta}\left(p_1p_2\right)+i\varepsilon^{\alpha\beta\sigma\rho}p_{1,\sigma}p_{2,\rho}+p_1^\alpha p_2^\beta+p_1^\beta p_2^\alpha\right] \tag{14}$$
$$\mathrm{Tr}\left\{\gamma_\alpha\left(1-\gamma^5\right)\hat{p}_4\gamma_\beta\left(1-\gamma^5\right)\hat{p}_3\right\} = 8\left[-g_{\alpha\beta}\left(p_3p_4\right)-i\varepsilon_{\alpha\beta\sigma\rho}p_3^\sigma p_4^\rho+p_{3,\alpha}p_{4,\beta}+p_{3,\beta}p_{4,\alpha}\right] \tag{15}$$

After all the contractions are performed, this obtains quite a nice form:

$$|\mathcal{M}|^2 = 64 G_F^2 \left(p_1 p_4\right)\left(p_2 p_3\right) \tag{16}$$

In the center-of-momentum frame we obviously have $p_1 = (m_1, 0)$. All other particles are treated as massless and thus:

$$p_1 p_4 = m_1 E_4 \tag{17}$$

$$p_2 p_3 = \frac{\left(p_2 + p_3\right)^2}{2} = \frac{\left(p_1 - p_4\right)^2}{2} = \frac{m_1^2 - 2\left(p_1 p_4\right)}{2} = \frac{m_1^2 - 2 m_1 E_4}{2} \tag{18}$$

The common formula for the decay rate reads:

$$\Gamma = \frac{1}{2}\frac{1}{2m_1}\int d_{LIPS}\left(2\pi\right)^4 \delta^{[4]}\left(p_1 - p_2 - p_3 - p_4\right)|\mathcal{M}|^2 \tag{19}$$

d_{LIPS} stand for the Lorentz-invariant phase space of the three final particles:

$$d_{LIPS} = \frac{d^3 p_2}{2E_2\left(2\pi\right)^3}\frac{d^3 p_3}{2E_3\left(2\pi\right)^3}\frac{d^3 p_4}{2E_4\left(2\pi\right)^3} \tag{20}$$

4-dimensional delta function represents the conservation laws of 3-momentum and energy:

$$\delta\left(p_1 - p_2 - p_3 - p_4\right) = \delta\left(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4\right)\delta\left(m_1 - E_2 - E_3 - E_4\right) \tag{21}$$

Using the first one of them, we can remove the integration over $d^3 p_2$:

$$\delta\left(p_1 - p_2 - p_3 - p_4\right)\frac{d^3 p_2 d^3 p_3 d^3 p_4}{8E_2 E_3 E_4} = \delta\left(m_1 - E_2 - E_3 - E_4\right)\frac{E_3^2 E_4^2}{E_2 E_3 E_4}\frac{dE_3 dE_4 d\Omega_3 d\Omega_4}{8} \tag{22}$$

Where $d\Omega_i = \sin\theta_i d\theta_i d\phi_i$ represents the solid angle element. This also implies that $\mathbf{p}_2$ is now constrained:

$$\mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4 = 0 \quad \Rightarrow \quad |\mathbf{p}_2| = \sqrt{\mathbf{p}_3^2 + \mathbf{p}_4^2 + 2\left|\mathbf{p}_3\right|\left|\mathbf{p}_4\right|\cos\theta_{34}} \tag{23}$$

θ_{23} is the angle between final particles 2 and 3. Since all particles are massless, this can be reduced to:

$$E_2 = \sqrt{E_3^2 + E_4^2 + 2E_3 E_4 \cos\theta_{34}} \tag{24}$$

The remaining delta function removes the integration over dE_3, but we have to be careful with this step, since the considerations above require E_2 to be treated as a function of E_3. In such a case the delta function of a function is expressed as:

$$\delta\left(g\left(x\right)\right) = \frac{\delta\left(x\right)}{\left|g'\left(x_k\right)\right|} \tag{25}$$

x_k is a zero of $g\left(x\right)$. In our example this simply means that the energy conservation $m_1 = E_2 + E_3 + E_4$ can be used.

Explicitly:

$$\begin{aligned} \delta\left(m_1 - E_2 - E_3 - E_4\right) dE_3 &= \frac{\delta\left(E_4\right)}{\left|\frac{\partial}{\partial E_3}\left(m_1 - E_2 - E_3 - E_4\right)\right|} = \frac{\delta\left(E_4\right)}{\left|-\frac{\partial E_2}{\partial E_3} - 1\right|} = \\ &= \delta\left(E_4\right)\frac{E_4}{E_2 + E_3 + E_4\cos\theta_{34}} = \delta\left(E_4\right)\frac{E_4}{m_1 + E_4\left(\cos\theta_{34} - 1\right)} \end{aligned} \tag{26}$$

And thus finally:

$$\delta\left(m_1 - E_2 - E_3 - E_4\right)\frac{E_3^2 E_4^2}{E_2 E_3 E_4}\frac{dE_3 dE_4 d\Omega_3 d\Omega_4}{8} = \delta\left(m_1 - E_2 - E_3 - E_4\right)\frac{E_3 E_4}{m_1 + E_4\left(\cos\theta_{34} - 1\right)}\frac{dE_4 d\Omega_3 d\Omega_4}{8} \tag{27}$$

E_3 in this formula is understood as a root of the equation:

$$m_1 - E_3 - E_4 = \sqrt{E_3^2 + E_4^2 + 2E_3 E_4 \cos\theta_{34}} \quad \Rightarrow \quad E_3 = \frac{m_1\left(\frac{m_1}{2} - E_4\right)}{m_1 + E_4\left(\cos\theta_{34} - 1\right)} \tag{28}$$

As the last step, we need to express $\cos\theta_{34}$. The easiest way to achieve this is to parameterize momentum vectors in spherical coordinates:

$$p_3 = E_3 \begin{pmatrix} 1 \\ \sin\theta_3\cos\phi_3 \\ \sin\theta_3\sin\phi_3 \\ \cos\theta_3 \end{pmatrix}; \quad p_4 = E_4 \begin{pmatrix} 1 \\ \sin\theta_4\cos\phi_4 \\ \sin\theta_4\sin\phi_4 \\ \cos\theta_4 \end{pmatrix} \tag{29}$$

In this case it can be trivially seen that:

$$\cos\theta_{34} = \frac{\mathbf{p}_3\mathbf{p}_4}{E_3 E_4} = \sin\theta_3\sin\theta_4\cos\left(\phi_3 - \phi_4\right) + \cos\theta_3\cos\theta_4 \tag{30}$$

Clearly, the remaining angular integrations are quite cumbersome, but the final answer is actually quite compact:

$$\Gamma = \frac{G_F^2 m_1^5}{192\pi^3} \equiv \frac{\hbar}{\tau} \tag{31}$$

The muon lifetime τ is known to be $\sim 2.2\,\mu\mathrm{s}$, leading to $\Gamma \sim 3 \cdot 10^{-19}\,\mathrm{GeV}^{-1}$ and $G_F \sim 1.16 \cdot 10^{-5}\,\mathrm{GeV}^{-2}$. This allows us to determine the Higgs parameter:

$$v \sim 246\,\mathrm{GeV} \tag{32}$$

IV. DETERMINATION OF MIXING ANGLE PARAMETER

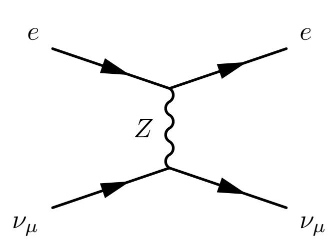
Now, having the value of v we can set the boundary on M_W and M_Z. Indeed, since $\sin\theta_w < 1$, we obtain:

$$M_W \geq 37.4\,\mathrm{GeV} \tag{33}$$
$$M_Z \geq M_W \tag{34}$$

Having an idea where to look helped to motivate the collider experiments at CERN, with which W and Z were eventually discovered. In fact, we can improve this result even further if we are able to measure θ_w directly, which appears to be possible with electron-neutrino scattering:

$$\nu_\mu e \to \nu_\mu e \tag{35}$$

The corresponding diagram is:



Let's denote neutrinos as k_1, k_2 and electrons as p_1, p_2. Both electrons and neutrinos can be treated as massless. Neutrino and electron vertices are, correspondingly:

$$-i\frac{e}{4\cos\theta_w\sin\theta_w}\gamma^\alpha\left(1-\gamma^5\right) \tag{36}$$

$$i\frac{e}{4\cos\theta_w\sin\theta_w}\gamma^\alpha\left(c_v-\gamma^5\right) \tag{37}$$

And then the low-energy amplitude becomes:

$$\begin{aligned}\mathcal{M} &= \frac{ie^2}{16\cos^2\theta_w\sin^2\theta_w}\frac{1}{M_Z^2}\left[\bar{u}\left(k_2\right)\gamma^\alpha\left(1-\gamma^5\right)u\left(k_1\right)\right]\left[\bar{u}\left(p_2\right)\gamma_\alpha\left(c_v-\gamma^5\right)u\left(p_1\right)\right]= \\ &= i\frac{G_F}{2\sqrt{2}}\left[\bar{u}\left(k_2\right)\gamma^\alpha\left(1-\gamma^5\right)u\left(k_1\right)\right]\left[\bar{u}\left(p_2\right)\gamma_\alpha\left(c_v-\gamma^5\right)u\left(p_1\right)\right]\end{aligned} \tag{38}$$

Further calculations are straightforward and, in principle, quite similar to the previous example. The only major difference is that the mixing angle is now explicitly included in the amplitude:

$$c_v = 1-4\sin^2\theta_w \tag{39}$$

Neutrino has only one allowed spin state, thus the average over the initial spins brings 1/2 factor instead of 1/4:

$$|\mathcal{M}|^2 = \frac{G_F^2}{16}\cdot \mathrm{Tr}\Big\{\gamma^\alpha\left(1-\gamma^5\right)\hat{k}_1\gamma^\beta\left(1-\gamma^5\right)\hat{k}_2\Big\}\cdot \mathrm{Tr}\left\{\gamma_\alpha\left(c_v-\gamma^5\right)\hat{p}_1\gamma_\beta\left(c_v-\gamma^5\right)\hat{p}_2\right\} \tag{40}$$

After some trivial algebra:

$$\mathrm{Tr}\Big\{\gamma^\alpha\left(1-\gamma^5\right)\hat{k}_1\gamma^\beta\left(1-\gamma^5\right)\hat{k}_2\Big\} = 8\left[-g^{\alpha\beta}\left(k_1k_2\right)+i\varepsilon^{\alpha\beta\sigma\rho}k_{1,\sigma}k_{2,\rho}+k_1^\alpha k_2^\beta+k_1^\beta k_2^\alpha\right] \tag{41}$$

$$\mathrm{Tr}\left\{\gamma_\alpha\left(c_v-\gamma^5\right)\hat{p}_1\gamma_\beta\left(c_v-\gamma^5\right)\hat{p}_2\right\} = 4\left(c_v^2+1\right)\left(-g_{\alpha\beta}\left(p_1p_2\right)+p_{1,\alpha}p_{2,\beta}+p_{1,\beta}p_{2,\alpha}\right)+8ic_v\cdot\varepsilon_{\alpha\beta\sigma\rho}p_1^\sigma p_2^\rho \tag{42}$$

And in total:

$$|\mathcal{M}|^2 = 4G_F^2\left[\left(c_v-1\right)^2\left(k_1p_2\right)\left(k_2p_1\right)+\left(c_v+1\right)^2\left(k_1p_1\right)\left(k_2p_2\right)\right] \tag{43}$$

The second term can be simplified if we note that it can be expressed in terms of the Mandelstam variable s:

$$s = \left(k_1+p_1\right)^2 = \left(k_2+p_2\right)^2 = 2\left(k_1p_1\right) = 2\left(k_2p_2\right) \tag{44}$$

And thus the matrix element squared is brought to the form:

$$|\mathcal{M}|^2 = G_F^2s^2\left[\left(c_v-1\right)^2\frac{\left(k_1p_2\right)\left(k_2p_1\right)}{\left(k_1p_1\right)\left(k_2p_2\right)}+\left(c_v+1\right)^2\right] \tag{45}$$

Let's calculate the cross section in the center-of-momentum frame. In this case we can parameterize:

$$\frac{\left(k_1p_2\right)\left(k_2p_1\right)}{\left(k_1p_1\right)\left(k_2p_2\right)} = \frac{1}{4}\left(1-\cos\alpha\right)^2 \tag{46}$$

Using the general formula for the cross section in the center-of-momentum frame, we obtain:

$$\sigma = \frac{1}{32\pi s}\int|\mathcal{M}|^2\sin\alpha\,d\alpha = \frac{G_F^2s}{48\pi}\times\left[\left(c_v-1\right)^2+3\left(c_v+1\right)^2\right] = \frac{G_F^2s}{12\pi}\times\left[1+c_v+c_v^2\right] \tag{47}$$

Technically, no particle detectors are capable of measuring the full 4π solid angle, but the corresponding correction is irrelevant.

Measurements of the $\nu_\mu e \to \nu_\mu e$ allowed to access c_v and thus the mixing angle:

$$\sin^2\theta_w \approx 0.22 \tag{48}$$

Which makes it possible to predict the masses of heavy vector bosons:

$$M_W \approx 80\,\mathrm{GeV} \tag{49}$$
$$M_Z \approx 92\,\mathrm{GeV} \tag{50}$$

Which is brilliantly confirmed by experiments.

V. QUARK MIXING

In conclusion, we note that it was historically observed that the Fermi constant G_F measured in pion decays $\pi^- \to \mu^- \bar{\nu}_\mu$ was slightly different from G_F measured in pure leptonic processes:

$$\frac{G_F^{\text{pion decay}}}{G_F} \approx 0.975 \tag{51}$$

Which can be explained if we introduce the mixing between quarks analogous to the electroweak one. In other words, weak interaction couples not to flavour eigenstates, but to their combinations. In particular, it was proposed to form a superposition (only two generations were known these days):

$$\begin{pmatrix} d' \\ s' \end{pmatrix} = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix} \tag{52}$$

$\theta_c \approx 13^\circ$ is called a Cabibbo angle. Such a modification explained many of the hadronic processes anomalies observed in those days. Cabibbo model was later generalized to three quark generations and plays a crucial role in understanding the baryon balance of the Universe.