I. REMINDER

Consider an arbitrary $SU(N)$-invariant theory in an arbitrary gauge with massive fermions:

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{a;\mu\nu}F_a^{\mu\nu} - \frac{1}{2\xi}\left(\partial_\mu A_a^\mu\right)^2 + \left(\partial^\mu \bar{c}_a\right)\left(\delta_{ac}\partial_\mu + gf_{abc}A_{b;\mu}\right)c_c + \\ &+ \bar{\psi}_i\left[\delta_{ij}\left(i\partial\gamma\right) + g\left(A_a\gamma\right)\left(t_a\right)_{ij} - m\delta_{ij}\right]\psi_j\end{aligned} \tag{1}$$

ξ is an arbitrary number, which does not affect any physical observables. The charge g was chosen to be positive to align with the notation adopted in "Quantum Field Theory and the Standard Model" by M.D. Schwartz. Gauge bosons will be most of the time conventionally called gluons (even though we consider some arbitrary Yang-Mills gauge field, not QCD).

Gluon, ghost and fermion propagators are:

$$D_{ab}^{\mu\nu} = i\frac{-g^{\mu\nu} + (1-\xi)\frac{k^\mu k^\nu}{k^2}}{k^2 + i\varepsilon}\delta_{ab} \tag{2}$$

$$G_{ab} = \frac{i\delta_{ab}}{p^2 + i\varepsilon} \tag{3}$$

$$F_{ij} = i\frac{\hat{p} + m}{p^2 - m^2 + i\varepsilon}\delta_{ij} \tag{4}$$

For the vertex with three incoming gluons (labelled as μ, ν and ρ with momenta k, p, and q, respectively) we have the factor:

$$gf_{abc}\cdot\left[g^{\mu\rho}\left(q-k\right)^\nu + g^{\nu\rho}\left(p-q\right)^\mu + g^{\mu\nu}\left(k-p\right)^\rho\right] \tag{5}$$

Four gluons vertex:

$$-ig^2\left[f_{abe}f_{cde}\left(g^{\mu\rho}g^{\nu\sigma} - g^{\nu\rho}g^{\mu\sigma}\right) + f_{bde}f_{ace}\left(g^{\sigma\rho}g^{\nu\mu} - g^{\mu\sigma}g^{\rho\nu}\right) + f_{bce}f_{ade}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}\right)\right] \tag{6}$$

Ghost-gluon and fermion-gluon vertices are trivial (p^μ is the antighost momentum - see the Lagrangian above):

$$-gf_{abc}p^\mu \tag{7}$$

$$ig\gamma^\mu\left(t_a\right)_{ij} \tag{8}$$

We will need the Feynman parameters:

$$\frac{1}{AB} = \int_0^1 \frac{1}{\left[A + (B-A)x\right]^2}dx \tag{9}$$

$$\frac{1}{ABC} = 2\int_0^1 dx\int_0^x dy\frac{1}{\left[A + (B-A)x + (C-B)y\right]^3} \tag{10}$$

The dimensional regularization:

$$\int\frac{p^\mu p^\nu}{\left(p^2-\Delta\right)^b}\frac{d^Dp}{(2\pi)^D} = \frac{g^{\mu\nu}}{D}\int\frac{p^2}{\left(p^2-\Delta\right)^b}\frac{d^Dp}{(2\pi)^D} \tag{11}$$

$$\int\frac{p^{2a}}{\left(p^2-\Delta\right)^b}\frac{d^Dp}{(2\pi)^D} = \frac{i\left(-1\right)^{a-b}}{(4\pi)^{D/2}}\frac{1}{\Delta^{b-a-D/2}}\frac{\Gamma\left(a+\frac{D}{2}\right)\Gamma\left(b-a-\frac{D}{2}\right)}{\Gamma\left(b\right)\Gamma\left(\frac{D}{2}\right)} \tag{12}$$

And some theory group identities:

$$\mathrm{Tr}\{t_a t_b\} = T_F \delta_{ab} \tag{13}$$
$$f_{abc} f_{dbc} = C_A \delta_{ad} \tag{14}$$

C_A and T_F are normalizing coefficients. We will also use the notation:

$$\tilde{\mu}^2 = 4\pi e^{-\gamma_E} \mu^2 \tag{15}$$

Where γ_E is Euler's constant, μ is usually introduced into dimensional regularization to preserve the correct dimensionality of the answer.

Homework 1 (50 points)

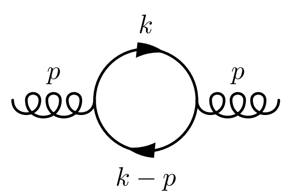
Derive the full expression for the gluon self-energy with the Lagrangian written above.

A large part of this calculation can be found in the file below - reproduce them and complete the evaluation of the ξ-dependent part.

Deadline: June 20th.

II. FERMION LOOP

The first contribution is obvious - a fermion loop similar to QED:



The corresponding expression is given by:

$$iM^{\mu\nu}_{1,ab} = -\,\mathrm{Tr}\{t_a t_b\}\,(ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{(k-p)^2 - m^2 + i\varepsilon} \frac{i}{k^2 - m^2 + i\varepsilon}\, \mathrm{Tr}\Big\{\gamma^\mu \left(\hat{k} - \hat{p} + m\right) \gamma^\nu \left(\hat{k} + m\right)\Big\} \tag{16}$$

Hat denotes the contraction of a 4-vector with gamma matrices. Up to a color factor, this expression is identical to the QED case. The calculations are quite tedious but straightforward:

$$\begin{aligned} iM^{\mu\nu}_{1,ab} &= (-ig)^2 (-1)\, i^2 \mu^{4-D} \cdot T_F \delta_{ab} \int \frac{\mathrm{Tr}\Big\{\left(\hat{k} - \hat{p} + m\right)\gamma^\nu \left(\hat{k} + m\right)\gamma^\mu\Big\}}{\left(k^2 - m^2 + i\varepsilon\right)\left((p-k)^2 - m^2 + i\varepsilon\right)} \frac{d^D k}{(2\pi)^n} = \\ &= -g^2 f(D) \cdot T_F \delta_{ab} \int \frac{2k^\nu k^\mu - p^\nu k^\mu - k^\nu p^\mu - g^{\nu\mu}\left(-k^2 + (pk) + m^2\right)}{\left(k^2 - m^2 + i\varepsilon\right)\left((p-k)^2 - m^2 + i\varepsilon\right)} \frac{d^D k}{(2\pi)^n} \end{aligned} \tag{17}$$

μ^{4-D} factor guarantees that the coupling preserves correct dimension. $f(D)$ is the trace normalization in D dimensions.

Now we use the Feynman parametrization with $A = (p-k)^2 - m^2 + i\varepsilon$ and $B = k^2 - m^2 + i\varepsilon$:

$$\begin{aligned} &A + (B - A)\,x = (p-k)^2 - m^2 + i\varepsilon + \left[k^2 - (k-p)^2\right] x = \\ &= [k - p\,(1-x)]^2 + p^2 x\,(1-x) - m^2 + i\varepsilon \end{aligned} \tag{18}$$

Let's shift the integration variable $k = k' + p\,(1-x)$:

$$k'^2 + p^2 x\,(1-x) - m^2 + i\varepsilon \tag{19}$$

The numerator is (only even powers of k' make contribution):

$$\begin{aligned} &g^{\mu\nu}\left(-k'^2 + p^2 x\,(1-x) + m^2\right) - 2\,(1-x)\,p^\mu p^\nu + 2\,(1-x)^2\,p^\mu p^\nu + 2k'^\mu k'^\nu = \\ &= g^{\mu\nu}\left(-k'^2 + p^2 x\,(1-x) + m^2\right) + 2k'^\mu k'^\nu + 2x^2 p^\mu p^\nu - 2x p^\mu p^\nu \end{aligned} \tag{20}$$

And in total:

$$\begin{aligned} &iM^{\mu\nu}_{1,ab} = -g^2 f(D)\mu^{4-D}\cdot T_F\delta_{ab}\times \\ &\times\int \frac{g^{\mu\nu}\left(-k'^2 + p^2 x\,(1-x) + m^2\right) + 2k'^\mu k'^\nu + 2x^2 p^\mu p^\nu - 2x p^\mu p^\nu}{\left[k'^2 + p^2 x\,(1-x) - m^2 + i\varepsilon\right]^2}\,\frac{d^D k'}{(2\pi)^D}dx \end{aligned} \tag{21}$$

Using standard integrals we get:

$$\begin{aligned} &iM^{\mu\nu}_{1,ab} = -g^2 f(D)\frac{i\mu^{4-D}}{(4\pi)^{D/2}}\cdot T_F\delta_{ab}\int_0^1\left[\frac{1}{\left(m^2 - p^2 x\,(1-x) - i\varepsilon\right)^{2-D/2}}\Gamma\left(2-\frac{D}{2}\right)\times\right. \\ &\times\left[g^{\mu\nu}\left(p^2 x\,(1-x) + m^2\right) + 2x^2 p^\mu p^\nu - 2x p^\mu p^\nu\right] - \\ &\left. -\frac{D}{2}\frac{i}{(4\pi)^{D/2}}\frac{g^{\mu\nu}\left(\frac{2}{D}-1\right)}{\left(m^2 - p^2 x\,(1-x) - i\varepsilon\right)^{1-D/2}}\Gamma\left(1-\frac{D}{2}\right)\right]dx \end{aligned} \tag{22}$$

Note that:

$$\left(1-\frac{D}{2}\right)\Gamma\left(1-\frac{D}{2}\right) = \Gamma\left(2-\frac{D}{2}\right) \tag{23}$$

Then:

$$\begin{aligned} &iM^{\mu\nu}_{1,ab} = -g^2 f(D)\frac{i\mu^{4-D}}{(4\pi)^{D/2}}\cdot T_F\delta_{ab}\int_0^1 \frac{1}{\left(m^2 - p^2 x\,(1-x) - i\varepsilon\right)^{2-D/2}}\Gamma\left(2-\frac{D}{2}\right)\times \\ &\times\left[g^{\mu\nu}\left(p^2 x\,(1-x) + m^2 - m^2 + p^2 x\,(1-x)\right) + 2x^2 p^\mu p^\nu - 2x p^\mu p^\nu\right]\,dx \end{aligned} \tag{24}$$

Simplifying:

$$\begin{aligned} &iM^{\mu\nu}_{1,ab} = -ig^2 f(D)\mu^{4-D}\left[g^{\mu\nu}p^2 - p^\mu p^\nu\right]\Gamma\left(2-\frac{D}{2}\right)\times \\ &\times T_F\delta_{ab}\int_0^1 \frac{2x\,(1-x)}{\left(m^2 - p^2 x\,(1-x) - i\varepsilon\right)^{2-D/2}}\,\frac{dx}{(4\pi)^{D/2}} \end{aligned} \tag{25}$$

The Taylor expansion at $D = 4-\epsilon$ implies $f(D) = 4$ and we obtain:

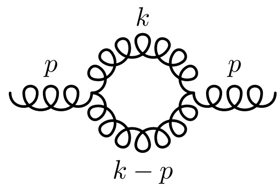
$$iM^{\mu\nu}_{1,ab} = -\frac{ig^2}{2\pi^2}\left[g^{\mu\nu}p^2 - p^\mu p^\nu\right]T_F\delta_{ab}\int_0^1 x\,(1-x)\left[\frac{2}{\epsilon} + \ln\left(\frac{\tilde{\mu}^2}{m^2 - p^2 x\,(1-x) - i\varepsilon}\right) + \mathcal{O}(\epsilon)\right]dx \tag{26}$$

This integral can be evaluated analytically, but we will restrict ourselves to the massless fermion case, which produces a quite nice formula:

$$iM^{\mu\nu}_{1,ab} = T_F\delta_{ab}\frac{g^2}{16\pi^2}\left(g^{\mu\nu}p^2 - p^\mu p^\nu\right)\cdot\left(-\frac{8}{3\epsilon} - \frac{20}{9} - \frac{4}{3}\ln\left(-\frac{\tilde{\mu}^2}{p^2}\right)\right) \tag{27}$$

III. GLUON LOOP

The next contribution is new - it arises from the gluon-gluon interaction:



The corresponding integral is:

$$iM^{\mu\nu}_{2,ab} = i^2 \frac{g^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{-g_{\alpha\beta} + (1-\xi)\frac{k_\alpha k_\beta}{k^2}}{k^2 + i\varepsilon} \frac{-g_{\rho\sigma} + (1-\xi)\frac{(k-p)_\rho (k-p)_\sigma}{(k-p)^2}}{(k-p)^2 + i\varepsilon} \delta_{cf}\delta_{ed} f_{ace} f_{bdf} N^{\beta\nu\rho} N^{\sigma\mu\alpha} \tag{28}$$

With the numerator of the form:

$$N^{\mu\alpha\rho} = \left[g^{\mu\alpha}(p+k)^\rho + g^{\alpha\rho}(p-2k)^\mu + g^{\rho\mu}(k-2p)^\alpha\right] \tag{29}$$

$$N^{\nu\beta\sigma} = \left[g^{\nu\beta}(k+p)^\sigma + g^{\beta\sigma}(p-2k)^\nu + g^{\sigma\nu}(k-2p)^\beta\right] \tag{30}$$

It looks like a scary beast, but the calculations are actually not that bad. First of all we use the group identities:

$$\delta_{cf}\delta_{ed} f_{ace} f_{bdf} = -C_A \delta_{ab} \tag{31}$$

Next we denote:

$$N_1^{\mu\nu} = g_{\alpha\beta} g_{\rho\sigma} \cdot N^{\mu\alpha\rho} N^{\nu\beta\sigma} \tag{32}$$

$$N_2^{\mu\nu} = g_{\alpha\beta} (k-p)_\rho (k-p)_\sigma \cdot N^{\mu\alpha\rho} N^{\nu\beta\sigma} \tag{33}$$

$$N_3^{\mu\nu} = k_\alpha k_\beta g_{\rho\sigma} \cdot N^{\mu\alpha\rho} N^{\nu\beta\sigma} \tag{34}$$

$$N_4^{\mu\nu} = k_\alpha k_\beta (k-p)_\rho (k-p)_\sigma \cdot N^{\mu\alpha\rho} N^{\nu\beta\sigma} \tag{35}$$

So the amplitude becomes:

$$iM^{\mu\nu}_{2,ab} = \frac{g^2}{2} C_A \delta_{ab} \int \left[\frac{N_1^{\mu\nu}}{ab} - (1-\xi)\frac{N_2^{\mu\nu}}{a^2 b} - (1-\xi)\frac{N_3^{\mu\nu}}{ab^2} + (1-\xi)^2 \frac{N_4^{\mu\nu}}{a^2 b^2}\right] \frac{d^4k}{(2\pi)^4} \tag{36}$$

With $a = (k-p)^2 + i\varepsilon$ and $b = k^2 + i\varepsilon$. We can make use of the formula:

$$\frac{1}{A^n B^m} = \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} \int_0^1 \frac{(1-x)^{n-1} x^{m-1}}{[A + (B-A)x]^{m+n}} dx \tag{37}$$

Applying also the results from previous sections, we write:

$$iM_{2,ab}^{\mu\nu} = \frac{g^2}{2} C_A \delta_{ab} \int_0^1 \int \left[\frac{N_1^{\mu\nu}}{\left[\left[k - p\left(1-x\right) \right]^2 + p^2 x \left(1-x\right) + i\varepsilon \right]^2} - \right.$$
$$- 2\left(1-\xi\right) \frac{\left(1-x\right) N_2^{\mu\nu}}{\left[\left[k - p\left(1-x\right) \right]^2 + p^2 x \left(1-x\right) + i\varepsilon \right]^3} -$$
$$- 2\left(1-\xi\right) \frac{x N_3^{\mu\nu}}{\left[\left[k - p\left(1-x\right) \right]^2 + p^2 x \left(1-x\right) + i\varepsilon \right]^3} + \tag{38}$$
$$\left. + \left(1-\xi\right)^2 \frac{x\left(1-x\right) N_4^{\mu\nu}}{\left[\left[k - p\left(1-x\right) \right]^2 + p^2 x \left(1-x\right) + i\varepsilon \right]^4} \right] \frac{d^4 k}{\left(2\pi\right)^4} dx$$

First let's replace $x \to 1 - x$:

$$iM_{2,ab}^{\mu\nu} = \frac{g^2}{2} C_A \delta_{ab} \int_0^1 \int \left[\frac{N_1^{\mu\nu}}{\left[\left(k - px\right)^2 + p^2 x \left(1-x\right) + i\varepsilon \right]^2} - 2\left(1-\xi\right) \frac{x N_2^{\mu\nu}}{\left[\left(k - px\right)^2 + p^2 x \left(1-x\right) + i\varepsilon \right]^3} - \right.$$
$$\left. -2\left(1-\xi\right) \frac{\left(1-x\right) N_3^{\mu\nu}}{\left[\left(k - px\right)^2 + p^2 x \left(1-x\right) + i\varepsilon \right]^3} + \left(1-\xi\right)^2 \frac{x\left(1-x\right) N_4^{\mu\nu}}{\left[\left(k - px\right)^2 + p^2 x \left(1-x\right) + i\varepsilon \right]^4} \right] \frac{d^4 k}{\left(2\pi\right)^4} dx \tag{39}$$

The next step is to shift the variable $k = k' + px$ and introduce $\Delta = -p^2 x\left(1-x\right)$:

$$iM_{2,ab}^{\mu\nu} = \frac{g^2}{2} C_A \delta_{ab} \int_0^1 \int \left[\frac{N_1^{\mu\nu}}{\left[k^2 - \Delta + i\varepsilon\right]^2} - 2\left(1-\xi\right) \frac{x N_2^{\mu\nu}}{\left[k^2 - \Delta + i\varepsilon\right]^3} - \right.$$
$$\left. -2\left(1-\xi\right) \frac{\left(1-x\right) N_3^{\mu\nu}}{\left[k^2 - \Delta + i\varepsilon\right]^3} + \left(1-\xi\right)^2 \frac{x\left(1-x\right) N_4^{\mu\nu}}{\left[k^2 - \Delta + i\varepsilon\right]^4} \right] \frac{d^4 k}{\left(2\pi\right)^4} dx \tag{40}$$

After some algebra we obtain for numerators:

$$N_1^{\mu\nu} \to g^{\mu\nu} k^2 \left(2 - \frac{6 - 4D}{D} \right) - \left[6\left(x^2 - x + 1\right) - D\left(1 - 2x\right)^2 \right] p^\mu p^\nu + \left(2x^2 - 2x + 5\right) p^2 g^{\mu\nu} \tag{41}$$
$$N_4^{\mu\nu} \to \frac{k^2 p^2}{D} \left(p^2 g^{\mu\nu} - p^\mu p^\nu \right) \tag{42}$$

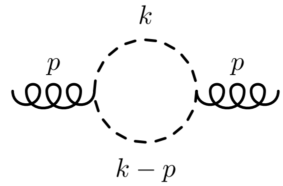
Remarkably, $N_4^{\mu\nu}$ has quite a compact form and is explicitly gauge-invariant. The remaining two are quite cumbersome and will be discussed separately. In this section we will finish the calculation only for the first, ξ-independent term. It gives:

$$iM_{2,ab}^{\mu\nu} = -i \frac{g^2}{2} \frac{\mu^{4-D}}{\left(4\pi\right)^{D/2}} C_A \delta_{ab} \int_0^1 \frac{1}{\left(\Delta - i\varepsilon\right)^{2-D/2}} \times \left[g^{\mu\nu} 3\left(D-1\right) \Gamma\left(1 - \frac{D}{2}\right) \cdot \Delta + \right.$$
$$+ p^\mu p^\nu \left(6\left(x^2 - x + 1\right) - D\left(1 - 2x\right)^2 \right) \Gamma\left(2 - \frac{D}{2}\right) + \tag{43}$$
$$\left. + g^{\mu\nu} p^2 \left(2x^2 - 2x + 5\right) \Gamma\left(2 - \frac{D}{2}\right) \right]$$

Before proceeding further, let's analyze other diagrams - we will see that a lot of terms will be cancelled.

IV. GHOST LOOP

Now we come to the calculation of ghost-gluon term:



The diagram corresponds to the following expression:

$$iM^{\mu\nu}_{3,ab} = -\left(-g\right)^2 \int \frac{d^4k}{\left(2\pi\right)^4} \frac{i}{\left(k-p\right)^2 + i\varepsilon} \frac{i}{k^2 + i\varepsilon} \times f_{cad} f_{dbc} k^\mu \left(k-p\right)^\nu \tag{44}$$

prefactor -1 comes from the fact that ghost fields are Grassmann variables. The Feynman parameters are introduced in a similar way as before:

$$\begin{aligned} &\frac{1}{\left(k-p\right)^2} \frac{1}{k^2} = \int_0^1 \frac{dx}{\left[\left[k - p\left(1-x\right)\right]^2 + p^2 x\left(1-x\right) + i\varepsilon\right]^2} = \\ &= \left\langle k - p\left(1-x\right) = k' \right\rangle = \int_0^1 \frac{dx}{\left[k'^2 + p^2 x\left(1-x\right) + i\varepsilon\right]^2} \end{aligned} \tag{45}$$

So we obtain:

$$iM^{\mu\nu}_{3,ab} = -g^2 C_A \delta_{ab} \int \frac{d^4k'}{\left(2\pi\right)^4} \frac{\left(k'^\mu + p^\mu\left(1-x\right)\right) \cdot \left(k'^\nu - p^\nu x\right)}{\left[k'^2 + p^2 x\left(1-x\right) + i\varepsilon\right]^2} \tag{46}$$

The numerator can be easily transformed:

$$\begin{aligned} &\left(k'^\mu + p^\mu\left(1-x\right)\right) \cdot \left(k'^\nu - p^\nu x\right) = k'^\mu k'^\nu - p^\mu p^\nu x\left(1-x\right) - k'^\mu p^\nu x + p^\mu\left(1-x\right) k'^\nu \to \\ &\to \frac{g^{\mu\nu}}{D} k^2 - p^\mu p^\nu x\left(1-x\right) \end{aligned} \tag{47}$$

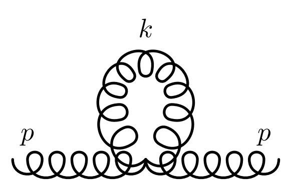
After that the integral can be taken with the general formula shown above:

$$\begin{aligned} &iM^{\mu\nu}_{3,ab} = ig^2 \frac{\mu^{4-d}}{\left(4\pi\right)^{D/2}} C_A \delta_{ab} \times \\ &\times \int_0^1 \frac{1}{\left(\Delta - i\varepsilon\right)^{2-D/2}} x\left(1-x\right) \left[-\frac{g^{\mu\nu}}{2} p^2 \Gamma\left(1 - \frac{D}{2}\right) + p^\mu p^\nu \Gamma\left(2 - \frac{D}{2}\right)\right] \end{aligned} \tag{48}$$

Where Δ was introduced.

V. GLUON TADPOLE

The last term is:



Using the standard expressions, we obtain:

$$\begin{aligned} iM^{\mu\nu}_{4,ab} = -\frac{i^2 g^2}{2}\mu^{4-D}\int\frac{d^D k}{(2\pi)^D}\frac{-g_{\rho\sigma}+(1-\xi)\frac{k_\rho k_\sigma}{k^2}}{k^2+i\varepsilon}\delta_{cd}\times \\ \times\left[f_{abe}f_{cde}\left(g^{\mu\rho}g^{\nu\sigma}-g^{\nu\rho}g^{\mu\sigma}\right)+f_{bde}f_{ace}\left(g^{\sigma\rho}g^{\nu\mu}-g^{\mu\sigma}g^{\rho\nu}\right)+f_{bce}f_{ade}\left(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma}\right)\right] \end{aligned} \tag{49}$$

The symmetry coefficient 1/2 was also added. Formally this integral vanishes in dimensional regularization (this can be verified with the general formula from the first section). We will, however, keep it for now - it will help us to simplify the final answer.

First we use the group identities to write:

$$\begin{aligned} &\delta_{cd}\times\left[f_{abe}f_{cde}\left(g^{\mu\rho}g^{\nu\sigma}-g^{\nu\rho}g^{\mu\sigma}\right)+f_{bde}f_{ace}\left(g^{\sigma\rho}g^{\nu\mu}-g^{\mu\sigma}g^{\rho\nu}\right)+f_{bce}f_{ade}\left(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma}\right)\right]= \\ &=\left[f_{ceb}f_{cea}\left(g^{\sigma\rho}g^{\nu\mu}-g^{\mu\sigma}g^{\rho\nu}\right)+f_{ceb}f_{cea}\left(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma}\right)\right]= \\ &=C_A\delta_{ab}\left(g^{\sigma\rho}g^{\nu\mu}-g^{\mu\sigma}g^{\rho\nu}+g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma}\right) \end{aligned} \tag{50}$$

The contraction with the metric tensor is trivial:

$$g_{\rho\sigma}\left(g^{\sigma\rho}g^{\nu\mu}-g^{\mu\sigma}g^{\rho\nu}+g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma}\right)=2\left(D-1\right)g^{\mu\nu} \tag{51}$$

With the momentum 4-vectors:

$$k_\rho k_\sigma\left(g^{\sigma\rho}g^{\nu\mu}-g^{\mu\sigma}g^{\rho\nu}+g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma}\right)=2\left(k^2g^{\mu\nu}-k^\mu k^\nu\right) \tag{52}$$

So the total integral becomes quite trivial:

$$\begin{aligned} iM^{\mu\nu}_{4,ab} = -g^2\delta_{ab}g^{\mu\nu}C_A\left(D-1\right)\mu^{4-D}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2+i\varepsilon}+ \\ +g^2\delta_{ab}C_A\left(1-\xi\right)\mu^{4-D}\int\frac{d^Dk}{(2\pi)^D}\frac{k^2g^{\mu\nu}-k^\mu k^\nu}{k^2+i\varepsilon}\frac{1}{k^2} \end{aligned} \tag{53}$$

After the simplification:

$$\begin{aligned} iM^{\mu\nu}_{4,ab} &= -g^2\delta_{ab}g^{\mu\nu}C_A\left(D-1\right)\mu^{4-D}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2+i\varepsilon}+ \\ &+g^2\delta_{ab}C_Ag^{\mu\nu}\left(1-\frac{1}{D}\right)\left(1-\xi\right)\mu^{4-D}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2+i\varepsilon}= \\ &=-g^2\delta_{ab}C_A\left(D-1\right)\mu^{4-D}\times g^{\mu\nu}\left[1-\frac{1-\xi}{D}\right]\times\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2+i\varepsilon} \end{aligned} \tag{54}$$

Now we perform the trick - let's multiply the integrand with the factor:

$$1=\frac{(p-k)^2}{(p-k)^2} \tag{55}$$

The numerator will be transformed as:

$$(p-k)^2=(k'-px)^2\to k'^2+p^2x^2 \tag{56}$$

In this case we obtain:

$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 + i\varepsilon} = \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 + i\varepsilon} \frac{(p-k)^2}{(p-k)^2 + i\varepsilon} = \int_0^1 \int \frac{dx}{\left[k'^2 + p^2 x (1-x) + i\varepsilon\right]^2} \frac{d^D k}{(2\pi)^D} =$$
$$= \frac{i}{(4\pi)^{D/2}} \int_0^1 \left[-\frac{D}{2} \frac{1}{(-p^2 x (1-x) - i\varepsilon)^{1-D/2}} \Gamma\left(1 - \frac{D}{2}\right) + \frac{p^2 x^2}{(-p^2 x (1-x) - i\varepsilon)^{2-D/2}} \Gamma\left(2 - \frac{D}{2}\right) \right] dx = \quad (57)$$
$$= \frac{i}{(4\pi)^{D/2}} \int_0^1 \frac{1}{(-p^2 x (1-x) - i\varepsilon)^{2-D/2}} \times \left[\frac{D}{2} \cdot p^2 x (1-x) \cdot \Gamma\left(1 - \frac{D}{2}\right) + p^2 x^2 \Gamma\left(2 - \frac{D}{2}\right) \right] dx$$

It is recommended to also perform $x \to 1 - x$ change of variable. The final answer is:

$$iM_{4,ab}^{\mu\nu} = -i \frac{g^2}{(4\pi)^{D/2}} \delta_{ab} C_A (D-1) \mu^{4-D} \times g^{\mu\nu} \left[1 - \frac{1-\xi}{D} \right] \times$$
$$\times \int_0^1 \frac{1}{(\Delta - i\varepsilon)^{2-D/2}} \times \left[-\frac{D}{2} \Delta \cdot \Gamma\left(1 - \frac{D}{2}\right) + (1-x)^2 p^2 \Gamma\left(2 - \frac{D}{2}\right) \right] dx \quad (58)$$

VI. BRINGING ALL TOGETHER

We obtain for the ξ-independent part:

$$iM_{2,ab}^{\mu\nu} + iM_{3,ab}^{\mu\nu} + iM_{4,ab}^{\mu\nu} =$$
$$= \frac{ig^2}{(4\pi)^{D/2}} \mu^{4-D} \delta_{ab} C_A \int_0^1 \frac{1}{(\Delta - i\varepsilon)^{2-D/2}} \left[\Delta \cdot \frac{g^{\mu\nu} D}{2} \left(\frac{3}{D} (1-D) + (D-1) + \frac{1}{D} \right) \Gamma\left(1 - \frac{D}{2}\right) + \right.$$
$$+ p^\mu p^\nu \left(-3 \left(x^2 - x + 1\right) + \frac{D}{2} (1-2x)^2 + x(1-x) \right) \Gamma\left(2 - \frac{D}{2}\right) + \quad (59)$$
$$\left. + g^{\mu\nu} p^2 \left(x^2 - x + \frac{5}{2} - (1-x)^2 (d-1) \right) \Gamma\left(2 - \frac{D}{2}\right) \right]$$

For this type of integrals $i\varepsilon$ can be safely dropped.
Using the identity:

$$(D-2)\,\Gamma\left(1 - \frac{D}{2}\right) = -2\Gamma\left(2 - \frac{D}{2}\right) \quad (60)$$

We simplify the integral to the form:

$$iM_{2,ab}^{\mu\nu} + iM_{3,ab}^{\mu\nu} + iM_{3,ab}^{\mu\nu} =$$
$$= \frac{ig^2}{(4\pi)^{D/2}} \mu^{4-D} \delta_{ab} C_A \int_0^1 \frac{1}{\Delta^{2-D/2}} \Gamma\left(2 - \frac{D}{2}\right) \times \quad (61)$$
$$\times \left[g^{\mu\nu} p^2 \left(\left(-2x^2 + 3x - 1\right) D + x(4x-5) + \frac{7}{2} \right) + p^\mu p^\nu \left(\frac{D}{2} (1-2x)^2 - 4x^2 + 4x - 3 \right) \right]$$

In $D = 4 - \epsilon$ the final result reads:

$$iM_{ab}^{\mu\nu} = iM_{1,ab}^{\mu\nu} + iM_{2,ab}^{\mu\nu} + iM_{3,ab}^{\mu\nu} + iM_{4,ab}^{\mu\nu} =$$
$$= i\delta_{ab} \frac{g^2}{16\pi} \left(g^{\mu\nu} p^2 - p^\mu p^\nu \right) \left[C_A \left(\frac{10}{3\epsilon} + \frac{5}{3} \ln\left(-\frac{\tilde{\mu}^2}{p^2} \right) \right) - T_F \left(\frac{8}{3\epsilon} + \frac{4}{3} \ln\left(-\frac{\tilde{\mu}^2}{p^2} \right) \right) \right] \quad (62)$$

Homework 2 (50 points)

Consider an arbitrary $SU(N)$ gauge field interacting with n_f fermions and n_s scalar. Prove that the lowest-order expression for the beta-function reads:

$$\beta_0 = \frac{11}{3}N - \frac{4}{3}n_f T_f - \frac{1}{3}n_s T_s$$

Hint: it is advised (but not necessary) to obtain this formula from the renormalization of ghost-gluon interaction.

Homework 3* (50 bonus points)

The running mass is defined in a similar way as the running coupling:

$$\frac{\mu}{m_R}\frac{dm_R}{d\mu} = \gamma_m \tag{63}$$

m_R is the renormalized mass, γ_m is commonly known as the anomalous dimension. With the same Lagrangian as above prove that at the lowest order of the perturbation theory the running masses for both scalars and fermions γ_m have the form:

$$\gamma_m = -\frac{6g_R}{16\pi^2}C_F \tag{64}$$

With g_R being the renormalized charge and C_F is the following group coefficient:

$$C_F = \frac{N^2-1}{N} \tag{65}$$

Hint: Schwartz M.D.: Quantum Field Theory and the Standard Model - section 23.2.

VII. GAUGE-DEPENDENT PART

First we complete we evaluation of:

$$\begin{aligned} &-(1-\xi)\, g^2 C_A \delta_{ab} \times \mu^{4-D} \int_0^1 \int \frac{xN_2^{\mu\nu} + (1-x)\, N_3^{\mu\nu}}{\left[k^2 - \Delta + i\varepsilon\right]^3} \frac{d^D k}{(2\pi)^D} dx = \\ &= -(1-\xi)\, g^2 C_A \delta_{ab} \times \mu^{4-D} \int_0^1 \int \frac{\alpha^{\mu\nu} + \beta^{\mu\nu} k^2 + \gamma^{\mu\nu} k^4}{\left[k^2 - \Delta + i\varepsilon\right]^3} \frac{d^D k}{(2\pi)^D} dx \end{aligned} \tag{66}$$

Explicitly the numerator is given by:

$$\begin{aligned} &xN_2^{\mu\nu} + (1-x)\, N_3^{\mu\nu} \to k^4 g^{\mu\nu} \cdot \frac{D-1}{D} + \left(p^2 g^{\mu\nu} - p^\mu p^\nu\right) \cdot p^2 x\,(x-1)\left(5x^2 - 5x - 1\right) + \\ &+ \frac{k^2 p^2}{D}\left[5 + (9+6D)\,(x-1)\,x\right] g^{\mu\nu} - \frac{k^2}{D} p^\mu p^\nu \left[10 + (D+4)\left(3x^2 - 3x - 1\right)\right] \end{aligned} \tag{67}$$

So the coefficients are:

$$\alpha^{\mu\nu} = \left(p^2 g^{\mu\nu} - p^\mu p^\nu\right) \cdot p^2 x \left(x-1\right) \left(5x^2 - 5x - 1\right) \tag{68}$$

$$\beta^{\mu\nu} = \frac{p^2}{D} \left[5 + \left(9 + 6D\right)\left(x-1\right)x\right] g^{\mu\nu} - \frac{1}{D} p^\mu p^\nu \left[10 + \left(D+4\right)\left(3x^2 - 3x - 1\right)\right] \tag{69}$$

$$\gamma^{\mu\nu} = g^{\mu\nu} \cdot \frac{D-1}{D} \tag{70}$$

Finally, the effective mass is:

$$\Delta = -p^2 x \left(1 - x\right) \tag{71}$$

The following integrals are needed then:

$$\int \frac{1}{\left(p^2 - \Delta\right)^3} \frac{d^D p}{\left(2\pi\right)^D} = \frac{-i}{2\left(4\pi\right)^{D/2}} \frac{1}{\Delta^{3-D/2}} \Gamma\left(3 - \frac{D}{2}\right) \tag{72}$$

$$\int \frac{p^2}{\left(p^2 - \Delta\right)^3} \frac{d^D p}{\left(2\pi\right)^D} = \frac{D}{4} \frac{i}{\left(4\pi\right)^{D/2}} \frac{1}{\Delta^{2-D/2}} \Gamma\left(2 - \frac{D}{2}\right) \tag{73}$$

$$\int \frac{p^4}{\left(p^2 - \Delta\right)^3} \frac{d^D p}{\left(2\pi\right)^D} = \frac{D\left(2+D\right)}{8} \frac{-i}{\left(4\pi\right)^{D/2}} \frac{1}{\Delta^{1-D/2}} \Gamma\left(1 - \frac{D}{2}\right) \tag{74}$$

The first contribution is (the integral is finite, so we can set $D = 4$ right away):

$$\begin{aligned} &\mu^{4-D} \int_0^1 \int \frac{\alpha^{\mu\nu}}{\left[k^2 - \Delta + i\varepsilon\right]^3} \frac{d^D k}{\left(2\pi\right)^D} dx = -\frac{i}{2 \cdot 16\pi^2} \int_0^1 \frac{\alpha^{\mu\nu}}{\Delta}\, dx = \\ &= \frac{i}{2 \cdot 16\pi^2} \cdot \left(g^{\mu\nu} p^2 - p^\mu p^\nu\right) \cdot \int_0^1 \left(5x^2 - 5x - 1\right)\, dx = \\ &= -\frac{i}{2 \cdot 16\pi^2} \cdot \left(g^{\mu\nu} p^2 - p^\mu p^\nu\right) \cdot \frac{11}{6} = -\frac{i}{64\pi^2} \cdot \left(g^{\mu\nu} p^2 - p^\mu p^\nu\right) \cdot \frac{11}{3} \end{aligned} \tag{75}$$

Obviously, it is proportional to the tensor $g^{\mu\nu} p^2 - p^\mu p^\nu$, i.e. possesses a correct gauge-invariant structure. Next we consider k^4:

$$\mu^{4-D} \int_0^1 \int \frac{\gamma^{\mu\nu} k^4}{\left[k^2 - \Delta + i\varepsilon\right]^3} \frac{d^D k}{\left(2\pi\right)^D} dx = \mu^{4-D} \frac{D\left(2+D\right)}{8} \frac{-i g^{\mu\nu}}{\left(4\pi\right)^{D/2}} \left(\frac{D-1}{D}\right) \cdot \Gamma\left(1 - \frac{D}{2}\right) \cdot \int_0^1 \frac{1}{\Delta^{1-D/2}}\, dx \tag{76}$$

After the integration over x, the divergent part becomes:

$$\begin{aligned} &\text{Divergent}\left[\mu^{4-D} \int_0^1 \int \frac{\gamma^{\mu\nu} k^4}{\left[k^2 - \Delta + i\varepsilon\right]^3} \frac{d^D k}{\left(2\pi\right)^D} dx\right] = \\ &= -\frac{i}{16\pi^2 \epsilon} \left(g^{\mu\nu} p^2 - p^\mu p^\nu\right) = -\frac{i}{16\pi^2 \epsilon} \left(g^{\mu\nu} p^2 - p^\mu p^\nu\right) \end{aligned} \tag{77}$$

The constant prefactor in front of this is $-\left(1 - \xi\right) g^2 C_A \delta_{ab}$. The corresponding counterterm has the form:

$$\mathcal{L}_{\text{C.-T.}} = -\frac{\delta_3}{2} \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2 \tag{78}$$

Indeed, in case of one incoming and one outcoming photon this is proportional to:

$$\mathcal{L}_{\text{C.-T.}} \to -\delta_3 \left(\partial^{a,\,\mu} A^\nu \cdot \partial_\mu A^a_\nu - \partial^\mu A^a_\nu \cdot \partial^\nu A^a_\mu\right) \tag{79}$$

So the vertex becomes:

$$-i\delta_3 \cdot \left[p^2 g^{\mu\nu} - p^\mu p^\nu\right] \cdot \delta_{ab} \tag{80}$$

So we obtain:

$$\delta_3 = (1-\xi)\, \frac{g^2}{16\pi^2 \epsilon} C_A \delta_{ab} \tag{81}$$

And finally:

$$\mu^{4-D} \int_0^1 \int \frac{\beta^{\mu\nu} k^2}{\left[k^2 - \Delta + i\varepsilon\right]^3} \frac{d^D k}{(2\pi)^D} dx = \mu^{4-D} \frac{D}{4} \frac{i}{(4\pi)^{D/2}} \cdot \Gamma\left(2 - \frac{D}{2}\right) \cdot \int_0^1 \frac{\beta^{\mu\nu}}{\Delta^{2-D/2}}\, dx \tag{82}$$

After some algebra the finite part obtains the form:

$$\begin{aligned}
&\text{Finite}\left[\mu^{4-D} \int_0^1 \int \frac{\beta^{\mu\nu} k^2 + \gamma^{\mu\nu} k^4}{\left[k^2 - \Delta + i\varepsilon\right]^3} \frac{d^D k}{(2\pi)^D} dx\right] = \\
&= \frac{i}{64\pi^2} \int_0^1 \Bigg\{ \left[g^{\mu\nu} p^2 \left(5 + 45x\,(x-1)\right) + 2p^\mu p^\nu \left(12x\,(1-x) - 1\right) \times \left(\ln\left(-\frac{\tilde{\mu}^2}{p^2}\right) - \ln\left(x\,(1-x)\right)\right)\right] - \\
&- \left[g^{\mu\nu} p^2 12x\,(x-1)\right) + \left(2 + 6x\,(1-x)\, p^\mu p^\nu\right] \Bigg\} dx = \frac{i}{64\pi^2} \left(g^{\mu\nu} p^2 - p^\mu p^\nu\right) \times \left[-2\ln\left(-\frac{\tilde{\mu}^2}{p^2}\right) + \frac{1}{3}\right]
\end{aligned} \tag{83}$$

Finally, there was a $(1-\xi)^2$ contribution:

$$(1-\xi)^2\, \frac{g^2}{2} C_A \delta_{ab} \times \mu^{4-D} \int_0^1 \int \frac{x\,(1-x)\, N_4^{\mu\nu}}{\left[k^2 - \Delta + i\varepsilon\right]^4} \frac{d^4 k}{(2\pi)^4} dx \tag{84}$$

Where we denoted:

$$N_4^{\mu\nu} \to \frac{k^2 p^2}{D} \left(p^2 g^{\mu\nu} - p^\mu p^\nu\right) \to \frac{k^2 p^2}{4} \left(p^2 g^{\mu\nu} - p^\mu p^\nu\right) \tag{85}$$

And we need the formula:

$$\int \frac{p^2}{\left(p^2 - \Delta\right)^4} \frac{d^D p}{(2\pi)^D} = -\frac{D}{12} \frac{i}{(4\pi)^{D/2}} \frac{1}{\Delta^{3-D/2}} \Gamma\left(3 - \frac{d}{2}\right) \to -\frac{i}{3\Delta} \frac{1}{16\pi^2} \tag{86}$$

Leading to:

$$\begin{aligned}
&\int_0^1 \int \frac{x\,(1-x)\, N_4^{\mu\nu}}{\left[k^2 - \Delta + i\varepsilon\right]^4} \frac{d^4 k}{(2\pi)^4} dx = \frac{1}{4} \left(p^2 g^{\mu\nu} - p^\mu p^\nu\right) \times \left(\frac{i}{3} \frac{1}{16\pi^2}\right) = \\
&= \frac{i}{64\pi^2} \cdot \left(p^2 g^{\mu\nu} - p^\mu p^\nu\right) \cdot \frac{1}{3}
\end{aligned} \tag{87}$$

The total gauge-dependent part of the vacuum polarization finally becomes:

$$\frac{ig^2}{16\pi^2} C_A \delta_{ab} \left(p^2 g^{\mu\nu} - p^\mu p^\nu\right) \left[(1-\xi) \cdot \left(\frac{1}{\epsilon} + \frac{1}{2} \ln\left(-\frac{\tilde{\mu}^2}{p^2}\right) - \frac{1}{12} + \frac{11}{12}\right) + \frac{(1-\xi)^2}{24}\right] \tag{88}$$

The final result in case of massless fermions is:

$$iM_{ab}^{\mu\nu} = i\delta_{ab}\frac{g^2}{16\pi}\left(g^{\mu\nu}p^2 - p^\mu p^\nu\right)\times$$
$$\times\left[C_A\left(\frac{10}{3\epsilon}+\frac{5}{3}\ln\left(-\frac{\tilde{\mu}^2}{p^2}\right)+(1-\xi)\cdot\left(\frac{1}{\epsilon}+\frac{1}{2}\ln\left(-\frac{\tilde{\mu}^2}{p^2}\right)+\frac{10}{12}+\frac{1-\xi}{24}\right)\right)-T_F\left(\frac{8}{3\epsilon}+\frac{4}{3}\ln\left(-\frac{\tilde{\mu}^2}{p^2}\right)\right)\right] \quad (89)$$

The full counterterm for the divergent part is:

$$\delta_3 = \frac{1}{\epsilon}\frac{g^2}{16\pi^2}\left[\frac{10}{3}C_A - \frac{8}{3}T_F + (1-\xi)\,C_A\right] \quad (90)$$