I. REMINDER

During previous classes the path integral approach to QFT was studied, as well as non-ableian gauge theories and perturbation theory. In particular, the correlation function:

$$\langle \Omega \left| T \left[A_a^\mu \left(q_1 \right) A_b^\nu \left(q_2 \right) \right] \right| \Omega \rangle = \int A_a^\mu \left(q_1 \right) A_b^\nu \left(q_2 \right) e^{iS} \mathcal{D} A \tag{1}$$

Was investigated. Fields $A_a^\mu \left(q_1 \right)$ and $A_b^\nu \left(q_2 \right)$ represent the external legs, the Taylor expansion of the non-gaussian part of the action gives rise to loops.

Even though there were a lot of details and the explicit calculations were quite tedious, the underlying philosophy is clear:

$$\text{Lagrangian} \Rightarrow \text{Feynman rules} \Rightarrow \text{Diagrams} \Rightarrow \text{Observables} \tag{2}$$

In addition to that, it appeared that the coupling constant isn't constant at all, it is a running quantity:

$$\mu \frac{d g_R}{d \mu} = \beta \left(g_R \right) \tag{3}$$

With g_R being the renormalized coupling.

An important property of quantum field theory is the existence of a Landau pole, where $g_R \to \infty$. This means that the perturbation theory at some point loses its applicability and the non-perturbative part of the path integral must be studied.

Since we mentioned non-perturbative physics, let's discuss another fundamental property of field theory that is closely related to it, namely, the shift of the derivative in the Lagrangian. This is a common operation, but there is a catch that needs to be kept in mind:

$$\int f \cdot (\partial g) \; d^D q = - \int (\partial f) \cdot g \, d^D q \tag{4}$$

This implies that functions f and g vanish at infinity, which is always true in perturbation theory, but may be incorrect in general. In particular, we know that QCD potential grows linearly with distance - the effect, which is known as confinement. It is mistakenly believed that this effect is an exclusive property of QCD, but a similar situation is observed, for example, in the two-dimensional electrodynamics, which is a well-known toy model for the studies of confinement (indeed, the two-dimensional Coulomb potential is a linear function, while the three-dimensional one is log-function - this can be easily seen from the Gauss law).

Homework 1 (50 points)

Prove that two-dimensional photon self-energy in massless spinor QED is not divergent and has the following form:

$$\Pi^{\mu\nu}(k) = \frac{e^2}{\pi k^2} \left(k^2 g^{\mu\nu} - k^\mu k^\nu \right)$$

And thus the exact propagator has non-trivial poles:

$$\mathcal{D}^{\mu\nu}(k) = - \frac{g^{\mu\nu}}{k^2 - \frac{e^2}{\pi} + i\varepsilon}$$

Which means loop corrections produced a photon mass equal to $\frac{e}{\sqrt{\pi}}$ (show that prove that this mass can't be removed by any gauge-invariant counterterms). Taking this into account, how many polarization does photon have in $D = 2$?

II. THETA

Let's examine this using a specific example. Consider an arbitrary Yang-Mills field interacting with some matter fields:

$$\mathcal{L} = -\frac{1}{4}F^2 + \mathcal{L}_{\text{Matter}} + \mathcal{L}_{\text{Interaction}} \tag{5}$$

Corresponding terms can be uniquely defined solely from the Lorentz and gauge invariance requirements.

However, we know that in the Standard Model Lorentz-invariance is weakened to proper transformations (P and T are explicitly violated). This allows us to add one more term:

$$\mathcal{L}_\theta = \theta \, \text{Tr}\Big\{F\tilde{F}\Big\}; \quad \tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} \tag{6}$$

θ is a proportionality coefficient. This contribution is also known as the topological topological.

$\text{Tr}\Big\{F\tilde{F}\Big\}$ is a total derivative - let's prove it. The field tensor is defined as:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu - ig\,[A^\mu, A^\nu] \tag{7}$$

So we get:

$$\begin{aligned} F\tilde{F} &= 2\cdot\varepsilon_{\mu\nu\alpha\beta}\cdot(\partial^\mu A^\nu - igA^\mu A^\nu)\cdot\left(\partial^\alpha A^\beta - igA^\alpha A^\beta\right) = \\ &= 2\cdot\varepsilon_{\mu\nu\alpha\beta}\cdot\left[(\partial^\mu A^\nu)\cdot\left(\partial^\alpha A^\beta\right) - ig\,(\partial^\mu A^\nu)\cdot A^\alpha A^\beta - igA^\mu A^\nu\cdot\left(\partial^\alpha A^\beta\right) - g^2A^\mu A^\nu\cdot A^\alpha A^\beta\right] \end{aligned} \tag{8}$$

The last term:

$$\varepsilon_{\mu\nu\alpha\beta}\cdot\text{Tr}\Big\{A^\mu A^\nu\cdot A^\alpha A^\beta\Big\} = \varepsilon_{\mu\nu\alpha\beta}\cdot A^\mu_a A^\nu_b\cdot A^\alpha_c A^\beta_d\cdot\text{Tr}\{t_a t_b\cdot t_c t_d\} \tag{9}$$

Vanishes. Indeed, we can use the group identities to write:

$$\text{Tr}\{t_a t_b t_c t_d\} = \frac{1}{4N}\delta_{ab}\delta_{cd} + \frac{1}{8}\left(d_{abe} + if_{abe}\right)\left(d_{cde} + if_{cde}\right) \tag{10}$$

$$f_{abe}f_{cde} = \frac{2}{N}\left(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}\right) + d_{ace}d_{bde} - d_{bce}d_{ade} \tag{11}$$

Where d_{abc} and δ_{ab} are symmetric objects. Thus the contraction with the totally antisymmetric Levi-Civita gives zero. Indeed:

$$\begin{aligned} &\varepsilon^{\mu\nu\alpha\beta}\cdot d_{abc}A_{a;\nu}A_{b;\alpha} = \langle a\leftrightarrow b; \alpha\leftrightarrow\nu\rangle = \\ &= \varepsilon^{\mu\alpha\nu\beta}\cdot d_{bac}A_{b;\alpha}A_{a;\nu} = -\varepsilon^{\mu\nu\alpha\beta}\cdot d_{abc}A_{a;\nu}A_{b;\alpha} = 0 \end{aligned} \tag{12}$$

Next we note that the second term is equal to the third one:

$$\begin{aligned} &\varepsilon_{\mu\nu\alpha\beta}\cdot\text{Tr}\Big\{(\partial^\mu A^\nu)\cdot A^\alpha A^\beta\Big\} = \varepsilon_{\mu\nu\alpha\beta}\cdot\text{Tr}\Big\{A^\alpha A^\beta\cdot(\partial^\mu A^\nu)\Big\} = \langle\alpha\leftrightarrow\mu; \beta\leftrightarrow\nu\rangle = \\ &= \varepsilon_{\alpha\beta\mu\nu}\cdot\text{Tr}\Big\{A^\mu A^\nu\cdot\left(\partial^\alpha A^\beta\right)\Big\} = \varepsilon_{\mu\nu\alpha\beta}\cdot\text{Tr}\Big\{A^\mu A^\nu\cdot\left(\partial^\alpha A^\beta\right)\Big\} \end{aligned} \tag{13}$$

Leading to:

$$\text{Tr}\Big\{F\tilde{F}\Big\} = 2\cdot\varepsilon_{\mu\nu\alpha\beta}\cdot\text{Tr}\Big\{(\partial^\mu A^\nu)\cdot\left(\partial^\alpha A^\beta\right) - 2ig\,(\partial^\mu A^\nu)\cdot A^\alpha A^\beta\Big\} \tag{14}$$

Now we introduce the new object (also known as the topological current):

$$K^{\mu} = \varepsilon^{\mu\nu\alpha\beta} \cdot \left[A_{a;\nu} \left(\partial_{\alpha} A_{a;\beta} \right) + \frac{g}{3} f_{abc} A_{a;\nu} A_{b;\alpha} A_{c;\beta} \right] = 4 \cdot \varepsilon^{\mu\nu\alpha\beta} \cdot \mathrm{Tr} \left\{ \frac{1}{2} A_{\nu} \left(\partial_{\alpha} A_{\beta} \right) - \frac{ig}{3} A_{\nu} A_{\alpha} A_{\beta} \right\} \tag{15}$$

We used the fact that:

$$\mathrm{Tr}\{t_a t_b t_c\} = \frac{1}{4} \left(d_{abc} + i f_{abc} \right) \tag{16}$$

The derivative becomes:

$$\partial_{\mu} K^{\mu} = 4 \cdot \varepsilon_{\mu\nu\alpha\beta} \cdot \mathrm{Tr} \left\{ \frac{1}{2} \left(\partial^{\mu} A_{\nu} \right) \left(\partial_{\alpha} A_{\beta} \right) - ig \left(\partial_{\mu} A_{\nu} \right) \cdot A_{\alpha} A_{\beta} \right\} \tag{17}$$

Where we used symmetry properties again:

$$\varepsilon^{\mu\nu\alpha\beta} \partial_{\mu} \partial_{\alpha} A_{\beta} = 0 \tag{18}$$

$$\varepsilon^{\mu\nu\alpha\beta} \partial_{\mu} \left(A_{\nu} A_{\alpha} A_{\beta} \right) = 3! \left(\partial_{\mu} A_{\nu} \right) \cdot A_{\alpha} A_{\beta} \tag{19}$$

So finally:

$$\partial_{\mu} K^{\mu} = \mathrm{Tr} \left\{ F \tilde{F} \right\} \tag{20}$$

Total derivative does not contribute to the equations of motion, but still leads to observable effect - for example, it contributes to electric dipole moment. We will not develop this idea any further right now, but we'll come back to it a little later.

III. QUANTUM FIELD ANOMALIES

Today we will get acquainted with a new concept in quantum field theory - anomalies. This is a highly-involved topic, but we will try to learn it's basics.

Generally speaking, anomaly is property of a classical theory violated by quantization. Usually this property is some symmetry, but not necessarily (not to be confused with the symmetry breaking, these two effects are completely different).

The typical explanation is as follows:

1) Anomaly arises due to the non-commutativity of operators in quantum mechanics. For instance, Noether's theorem, which holds for classical variables, may not apply directly to operators.

2) In the path integral formulation, the classical Lagrangian serves as just one ingredient in the entire path integral. In particular, the symmetry of the classical Lagrangian does not necessarily translate to symmetry in the path integral.

Or we could just say that the introduction of the UV-scale cut-off, which is always present in QFT (at least implicitly), may spoil some properties of the classical theory.

We will start from the so-called scale anomaly and then proceed to the more involved example of an axial anomaly.

IV. ENERGY-MOMENTUM TENSOR

Noether theorem implies that if there exists a continuous transformation of fields and coordinates:

$$q^{\mu} \to q'^{\mu} \tag{21}$$

$$\phi(q) \to \phi'(q') \tag{22}$$

Such that the action remains unchanged, then there is a conserved current corresponding to it. Mathematically it can be expressed as:

$$0 = \delta S = \int \frac{d}{dq^{\mu}} \left[\left(\frac{\partial \mathcal{L}}{\partial \left(\partial_{\mu} \phi \right)} \partial_{\nu} \phi - \delta^{\mu}_{\nu} \mathcal{L} \right) \delta q^{\nu} - \frac{\partial \mathcal{L}}{\partial \left(\partial_{\mu} \phi \right)} \Delta \phi \right] d^D q \tag{23}$$

Where denoted:

$$\delta q^{\mu} = q'^{\mu} - q^{\mu} \tag{24}$$
$$\Delta \phi = \phi' \left(q' \right) - \phi \left(q \right) \tag{25}$$

ϕ is the field of any spin, Lorentz indices are ommited.

In particular, the energy-momentum tensor is the conserved Noether current:

$$\partial_{\mu} \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_{\mu} \phi \right)} \partial_{\nu} \phi - \delta^{\mu}_{\nu} \mathcal{L} \right) = \partial_{\mu} T^{\mu}_{\nu} = 0 \tag{26}$$

Associated with spacetime translations:

$$q'^{\mu} - q^{\mu} = [\text{const}]^{\mu} \tag{27}$$
$$\phi' \left(q' \right) = \phi \left(q \right) \tag{28}$$

Conserved quantities are defined up to some transformation. For example, we can redefine:

$$T^{\mu\nu} \to T^{\mu\nu} + \partial_{\lambda} f^{\lambda\mu\nu} \tag{29}$$

Where $f^{\lambda\mu\nu}$ is an arbitrary rank-3 tensor obeying:

$$f^{\lambda\mu\nu} = -f^{\mu\lambda\nu} \tag{30}$$

Though Noether theorem by itself does not impose any additional constraints on $T^{\mu\nu}$, we can use this ambiguity to make the energy-momentum tensor symmetric. The motivation for that is, for example, the fact that in the Einstein equations:

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{31}$$

We have $R_{\mu\nu} = R_{\nu\mu}$ and $g_{\mu\nu} = g_{\nu\mu}$.

Noether theorem provides an explicit formula for a conserved current which corresponds to an arbitrary symmetry transformation. A nice explanation and derivation of this material can be found in the book "Quantum Field Theory" by L.H. Ryder. However, for the energy-momentum tensor there exists a compact and beautiful formula:

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} \tag{32}$$

Which is quite trivial - since $T^{\mu\nu}$ is related to the shift invariance of the action, it can be constructed from only S and $g_{\mu\nu}$. The only reasonable way to combine them in a meaningful tensor is to take a functional derivative and the factor $2/\sqrt{-g}$ is just a conventional proportionality coefficient. Notably, it provides an answer which is automatically symmetric in μ and ν. It is sometimes referred to as the Belinfante formula.

V. SCALE INVARIANCE

Theories, which do not contain dimensional parameters (masses, dimensional couplings), appear to have much broader symmetry than just the Poincare invariance. This extremely powerful symmetry is called conformal and it plays a crucial role in many aspects of theoretical physics from condensed matter to string theory.

Remarkably, Yang-Mills theory is conformal if and only if $D = 4$.

We will only consider a special case of conformal transformation, namely - the scale symmetry:

$$q \to \lambda^{-1} q \tag{33}$$

$$[\text{field}] \to \lambda^{\text{dim}} [\text{field}] \tag{34}$$

Where "dim" stands for the dimension of the field:

$$[\varphi] = \frac{D-2}{2}, \qquad [\psi] = \frac{D-1}{2}, \qquad [A^\mu] = \frac{D-2}{2} \tag{35}$$

This symmetry gives rise to the current of the form:

$$j^\mu = T^{\mu\nu} x_\nu + V^\mu \quad \Rightarrow \quad \partial_\mu j^\mu = T^\mu_\mu + \partial_\mu V^\mu = 0 \tag{36}$$

V^μ is called the virial current and originates from the transformation of fields $\delta\phi$.

Using the ambiguity mentioned in the previous section, we can ensure that in case of the conformal theory the energy-momentum tensor is symmetric and traceless. Such tensor if often called "improved" and denoted $\theta^{\mu\nu}$.

Homework 2 (25 points)

Prove that the energy-momentum tensor of the gauge field is:

$$T_{\mu\nu} = -\,\text{tr}\left\{ F_{\mu\alpha} F_\nu{}^\alpha - \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \right\}$$

And check that at $D = 4$ it is indeed traceless.

Hint: if you decide to use the Belinfante formula, you may need Jacobi's matrix identity:

$$\delta g = g g^{\mu\nu} \delta g_{\mu\nu} = -g g_{\mu\nu} \delta g^{\mu\nu}$$

VI. SCALE ANOMALY

Despite all its power, the fate of conformal symmetry at the quantum level is predetermined - it cannot be preserved simultaneously with regularization. Indeed, renormalization implies that we introduce some scale Λ of the UV cut-off. This contradicts the statement that scale invariance requires the absence of dimensional parameters in a given theory.

To prove this, let's consider the Yang-Mills theory in the dimensional regularization. For convenience, we make the replacement $gA \to A$, so there is a factor $1/g^2$ is front of the F^2 term (also remember that the topological term makes no contribution to the perturbation theory):

$$S = -\frac{1}{4g_0^2} \int F^2 \, d^{4-\epsilon} q \tag{37}$$

g_0 is the bare charge, which includes the counterterms. Under the scale transformation:

$$F^2 \, d^{4-\epsilon} q \to \lambda^\epsilon \cdot F^2 \, d^{4-\epsilon} q = \underbrace{(\lambda^\epsilon - 1)}_{\text{shift}} F^2 \, d^{4-\epsilon} q + F^2 \, d^{4-\epsilon} q \tag{38}$$

Thus we obtain:

$$\delta S = -\frac{\lambda^\epsilon - 1}{4g_0^2} \int F^2 \, d^{4-\epsilon}q \tag{39}$$

But this is not the end of the story, since the bare charge contains the pole $1/\epsilon$:

$$g_0^2 = \mu^\epsilon g_R^2 \left[\frac{Z_1}{Z_2\sqrt{Z_3}} \right]^2 \tag{40}$$

After expanding in series with respect to g_R^2, we obtain:

$$g_0^2 = \mu^\epsilon g_R^2 \left(1 - \frac{\alpha}{2\pi} \frac{\beta_0}{\epsilon} \right) \tag{41}$$

In the absence of any fermions or scalars interacting with the gauge field we obtain $\beta_0 = \frac{11}{3} C_A$. Finally:

$$\delta S = -\frac{\lambda^\epsilon - 1}{4\mu^\epsilon g_R^2} \left(1 + \frac{\alpha}{2\pi} \frac{\beta_0}{\epsilon} \right) \int F^2 \, d^{4-\epsilon}q \to -\ln\lambda \frac{\beta_0}{32\pi^2} \int F^2 \, d^4q \tag{42}$$

And we conclude that the trace anomaly is:

$$T_\mu^\mu = -\beta_0 \frac{F^2}{32\pi^2} \to -\alpha\beta_0 \frac{F^2}{8\pi} \tag{43}$$

This formula remains valid even in the presence of massless matter fields, β_0 is changed accordingly (the last step restored the original normalization of the gauge field).

VII. CHANGE OF VARIABLE

Now we proceed to the next example, namely - axial anomaly, but before relevaling it we need to perform some preparatory work.

Normally if we make an arbitrary change of variable in a path integral:

$$\phi(q) = \Delta(q) \, \phi'(q) \tag{44}$$

We get:

$$d\phi(q) = \frac{d\phi(q)}{d\phi'(q)} d\phi'(q) \equiv |\mathrm{Det}\,\Delta(q)| \, d\phi'(q) \tag{45}$$

Where $|\mathrm{Det}\,\Delta(q)|$ is just a Jacobian - it works in the same way as with usual function.

But if we deal with Grassmann variables one should remember that for them integration works as a derivative:

$$\int d\theta X = \frac{d}{d\theta} X \tag{46}$$

Meanwhile, the derivative transforms in an opposite way. Let's assume $\theta = \Delta\theta'$:

$$\frac{d}{d\theta} = \frac{d\theta'}{d\theta} \frac{d}{d\theta'} \equiv \frac{1}{|\mathrm{Det}\,\Delta(q)|} \frac{d}{d\theta'} \tag{47}$$

Thus if we make a change of variables in a fermionic path integral:

$$d\psi\left(q\right)=\Delta\left(q\right)d\psi'\left(q\right) \tag{48}$$
$$d\bar{\psi}\left(q\right)=\Delta^{+}\left(q\right)d\bar{\psi}'\left(q\right) \tag{49}$$

We get:

$$d\bar{\psi}\left(q\right)d\psi\left(q\right)=\frac{1}{\left|\mathrm{Det}\,\Delta\left(q\right)\right|^{2}}d\bar{\psi}'\left(q\right)d\psi'\left(q\right) \tag{50}$$

Note that determinant can be written as a trace of the logarithm:

$$\mathrm{Det}\,\Delta=\exp\{\mathrm{Tr}\{\ln\Delta\left(q\right)\}\} \tag{51}$$

Field variables like ψ are objects which belong to Hilbert space and Δ is basically an operator acting in this space (it transforms a function). The trace of an operator can be written as:

$$\mathrm{Tr}\{\ln\Delta\left(q\right)\}\equiv\int d^{D}q\langle q\left|\ln\Delta\left(q\right)\right|q\rangle \tag{52}$$

But note that if Δ has a matrix structure, we additionally have to take a trace in the usual matrix indices - i.e. we have to trace it in both Hilbert space and in the usual matrix sense! It gives:

$$\mathrm{Tr}\{\ln\Delta_{ik}\left(q\right)\}\equiv\int d^{D}q\langle q\left|\mathrm{tr}\{\ln\Delta_{ik}\left(q\right)\}\right|q\rangle \tag{53}$$

Big Trace stands for the overall trace and small trace stands for just matrix indices.

Homework 3 (25 points)

Prove that at the classical $D=2$ massless QED the numbers of both left and right particles are fixed and can't be changed.

Tip: $D=2$ gamma matrices in the standard form are:

$$\gamma^{0}=\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad\gamma^{1}=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\qquad\gamma^{5}=\gamma^{0}\gamma^{1}=\begin{pmatrix}-1&0\\0&1\end{pmatrix}$$

Hint: use the fact that there is an additional (axial) symmetry.

VIII. AXIAL ANOMALY

In the massless case the Lagrangian is:

$$\mathcal{L}=-\frac{1}{4}F^{2}+\bar{\psi}\left(i\gamma D\right)\psi+\theta\,\mathrm{Tr}\Big\{F\tilde{F}\Big\} \tag{54}$$

Let's perform a change of variable:

$$\psi\rightarrow e^{i\gamma^{5}\beta}\psi \tag{55}$$

γ^{5} is Hermitian and anticommutes with all gamma-matrices. Thus we have:

$$\bar{\psi} \to \bar{\psi} e^{i\gamma^5 \beta} \tag{56}$$

The Lagrangian goes to:

$$\mathcal{L} \to \mathcal{L} + \bar{\psi}\left[i\gamma^\mu \cdot \partial_\mu \left(i\gamma^5 \beta\right)\right]\psi + PI = \mathcal{L} + \beta \cdot \partial_\mu \left(\bar{\psi}\gamma^\mu \gamma^5 \psi\right) + PI \tag{57}$$

The quantity $\left(\bar{\psi}\gamma^\mu\gamma^5\psi\right)$ is a classical current j^5 associated with the axial symmetry transformation.

The path integral measure must be transformed as well and PI denotes the corresponding contribution. We have (let D be equal to $1+3$):

$$\operatorname{Det}\Delta_{ik} = \exp\{\operatorname{Tr}\{\ln \Delta_{ik}(q)\}\} = \exp\left\{i\beta \int d^4 q \langle q \left|\operatorname{tr}\{\gamma^5_{ik}\}\right| q\rangle\right\} \tag{58}$$

Naively one would expect the trace to be zero. But we also know that QFT requires a renormalization and thus we cannot be sure about this! Indeed, remember that in the dimensional regularization γ^5 is not a good object, because it was initially defined in $D = 1+3$.

The expression above must be regularized with a factor $\frac{1}{\Lambda}$, where Λ is a cut-off scale, i.e. very large quantity, which has a dimension of mass.

$\operatorname{Det}\Delta_{ik}$ must be kept dimensionless in order not to spoil the initial path integral. So we have to introduce something dimensionless like:

$$\text{cut-off} \propto \frac{M}{\Lambda} \tag{59}$$

But electrons are massless and thus we have only one quantity which has a dimension of mass - derivative, which is a four-vector. Indices must be contracted and the only dimensionless vector quantity which we can use is gamma. We write:

$$\gamma\partial = -i^2\gamma\partial = -i\gamma\left(i\partial\right) \Rightarrow -i\gamma\left(i\partial - gA\right) \tag{60}$$

I.e. we introduced the covariant derivative to preserve the gauge invariance:

$$\text{cut-off} \propto \frac{\gamma\left(i\partial - gA\right)}{\Lambda} \tag{61}$$

Now we take this expression squared - it doesn't change anything, because this function anyway goes zero:

$$\left(\gamma\left(i\partial - gA\right)\right)^2 = \gamma^\mu\gamma^\nu\left(i\partial - gA\right)_\mu\left(i\partial - gA\right)_\nu \tag{62}$$

Next we note that:

$$\gamma^\mu\gamma^\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} + \frac{1}{2}\left[\gamma^\mu, \gamma^\nu\right] = g^{\mu\nu} - i\sigma^{\mu\nu} \tag{63}$$

Where we denoted the commutator of gamma matrices divided by two as $-i\sigma^{\mu\nu}$. Thus we get:

$$\left(\gamma\left(i\partial - gA\right)\right)^2 = \left(i\partial - gA\right)^2 - i\sigma^{\mu\nu}\left(i\partial - gA\right)_\mu\left(i\partial - gA\right)_\nu \tag{64}$$

Now note that we can write:

$$\left(i\partial - gA\right)_\mu\left(i\partial - gA\right)_\nu = \frac{1}{2}\left[\left(i\partial - gA\right)_\mu, \left(i\partial - gA\right)_\nu\right] + \frac{1}{2}\left\{\left(i\partial - gA\right)_\mu, \left(i\partial - gA\right)_\nu\right\} \tag{65}$$

I.e. we divided it into commutator and anticommutator again, just like with the gamma above.

The symmetric anticommutator must be contracted with antisymmetric commutator $\sigma^{\mu\nu}$, which gives zero. Meanwhile, the first term gives:

$$\left[(i\partial - gA)_\mu (i\partial - gA)_\nu - \mu \leftrightarrow \nu\right] \phi = -igF_{\mu\nu}\phi \tag{66}$$

In total we have:

$$(\gamma (i\partial - gA))^2 = (i\partial - gA)^2 - \frac{g\sigma^{\mu\nu}F_{\mu\nu}}{2} \tag{67}$$

Finally, we write:

$$\text{cut-off} \propto f\left[\frac{(i\partial - gA)^2 - \frac{g\sigma^{\mu\nu}F_{\mu\nu}}{2}}{\Lambda^2}\right] \tag{68}$$

For simplicity we omit the matrix indices in the expression below:

$$\langle q \left| \mathrm{tr}\{\gamma^5\} \right| q\rangle \Rightarrow \lim_{\Lambda\to\infty} \langle q \left| \mathrm{tr}\left\{\gamma^5 \cdot f\left[\frac{(i\partial - gA)^2 - \frac{g\sigma^{\mu\nu}F_{\mu\nu}}{2}}{\Lambda^2}\right]\right\} \right| q\rangle \tag{69}$$

Note that now the trace is taken with the assumption $\Lambda \to \infty$, i.e. in the regularization-free limit when $D = 4$, which means we can use the normal rules of trace calculation.

The question is - which function f to choose? Actually, the most natural way is to take the exponent, since it decreases most rapidly at infinity:

$$\lim_{\Lambda\to\infty} \langle q \left| \mathrm{tr}\left\{\gamma^5 \cdot \exp\left\{\frac{(i\partial - gA)^2 - \frac{g\sigma^{\mu\nu}F_{\mu\nu}}{2}}{\Lambda^2}\right\}\right\} \right| q\rangle \tag{70}$$

We can now use the Zassenhaus formula:

$$e^{A+B} \approx e^A e^B \times O(1) \tag{71}$$

Small corrections are proportional to:

$$e^{[A,B]} \propto \exp\left\{\frac{1}{\Lambda^4}\right\} \to 1 \tag{72}$$

Thus we can neglect them and all further factors - we are interested only in the leading order of $\frac{1}{\Lambda}$. It gives:

$$\lim_{\Lambda\to\infty} \langle q \left| \exp\left\{\frac{(i\partial - gA)^2}{\Lambda^2}\right\} \mathrm{tr}\left\{\gamma^5 \cdot \exp\left\{\frac{-g\sigma^{\mu\nu}F_{\mu\nu}}{2\Lambda^2}\right\}\right\} \right| q\rangle \tag{73}$$

$\exp\left\{\frac{-g\sigma^{\mu\nu}F_{\mu\nu}}{2\Lambda^2}\right\}$ can be expanded in series, but constant and linear terms vanish due to the trace properties of γ^5. The first non-vanishing term is quadratic.

Note that F also has a matrix structure, but it belongs to the color space, i.e. γ and F can be traced independently:

$$\mathrm{tr}\{\gamma^5\sigma^{\mu\nu}\sigma^{\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\} = \mathrm{tr}\{\gamma^5\sigma^{\mu\nu}\sigma^{\alpha\beta}\} \cdot \mathrm{tr}\{F_{\mu\nu}F_{\alpha\beta}\} \tag{74}$$

We have to calculate the following:

$$\operatorname{tr}\{\gamma^5\sigma^{\mu\nu}\sigma^{\alpha\beta}\} = -\frac{1}{4}\left(\operatorname{tr}\{\gamma^5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\} - \operatorname{tr}\{\gamma^5\gamma^\nu\gamma^\mu\gamma^\alpha\gamma^\beta\} - \operatorname{tr}\{\gamma^5\gamma^\mu\gamma^\nu\gamma^\beta\gamma^\alpha\} + \operatorname{tr}\{\gamma^5\gamma^\nu\gamma^\mu\gamma^\beta\gamma^\alpha\}\right) \tag{75}$$

We use the well-known identity:

$$\operatorname{tr}\{\gamma^5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\} = -4i\epsilon^{\mu\nu\alpha\beta}; \quad \epsilon^{0123} = -1 \tag{76}$$

And Levi-Civita antisymmetry property. It gives:

$$\operatorname{tr}\{\gamma^5\sigma^{\mu\nu}\sigma^{\alpha\beta}\} = 4i\epsilon^{\mu\nu\alpha\beta} \tag{77}$$

We obtain:

$$\frac{ig^2}{2}\cdot\epsilon^{\mu\nu\alpha\beta}\operatorname{tr}\{F_{\mu\nu}F_{\alpha\beta}\}\cdot\lim_{\Lambda\to\infty}\frac{1}{\Lambda^4}\langle q\left|\exp\left\{\frac{(i\partial - gA)^2}{\Lambda^2}\right\}\right|q\rangle \tag{78}$$

1/2 comes from Taylor expansion.

Let's assume we are interested in the leading order with respect to the small parameter g (i.e. the perturbtation theory). Then we can set A to be zero in the exponent - it makes a small correction:

$$\frac{ig^2}{2}\cdot\epsilon^{\mu\nu\alpha\beta}\operatorname{tr}\{F_{\mu\nu}F_{\alpha\beta}\}\cdot\lim_{\Lambda\to\infty}\frac{1}{\Lambda^4}\langle q\left|\exp\left\{-\frac{\partial^2}{\Lambda^2}\right\}\right|q\rangle \tag{79}$$

Let's make a Fourier transformation:

$$|q\rangle = \int\frac{d^4p}{(2\pi)^4}e^{-ipq}|p\rangle \tag{80}$$

$$\langle q| = \int\frac{d^4p'}{(2\pi)^4}e^{ip'q}\langle p'| \tag{81}$$

It gives:

$$\langle q\left|\exp\left\{-\frac{\partial^2}{\Lambda^2}\right\}\right|q\rangle = \langle q\left|\exp\left\{-\frac{\partial^2}{\Lambda^2}\right\}\right|q\rangle = \int\frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}e^{ip'q}\langle p'|\exp\left\{-\frac{\partial^2}{\Lambda^2}\right\}e^{-ipq}|p\rangle \tag{82}$$

Simplifying, we obtain:

$$\int\frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}e^{ip'q}e^{-ipq}\exp\left\{-\frac{p^2}{\Lambda^2}\right\}\underbrace{\langle p'|p\rangle}_{(2\pi)^4\delta(p-p')} = \int\frac{d^4p}{(2\pi)^4}e^{ipq}e^{-ipq}\exp\left\{\frac{p^2}{\Lambda^2}\right\} \tag{83}$$

The remaining integral is gaussian and can be taken after contour rotation $p^0 \to ip_E^0$:

$$\int\frac{d^4p}{(2\pi)^4}\exp\left\{\frac{p^2}{\Lambda^2}\right\} = i\int\frac{d^4p_E}{(2\pi)^4}\exp\left\{-\frac{p_E^2}{\Lambda^2}\right\} = i\frac{\Lambda^4}{16\pi^2} \tag{84}$$

Where we also used:

$$p^2 = p_0^2 - \vec{\mathbf{p}}^2 \to -p_4^2 - \vec{\mathbf{p}}^2 = p_E^2 \tag{85}$$

And finally:

$$\mathrm{Det}\,\Delta=\exp\left\{-\frac{ig^2}{32\pi^2}\beta\cdot\epsilon^{\mu\nu\alpha\beta}\int d^4q\,\mathrm{tr}\left\{F\tilde{F}\right\}\right\} \tag{86}$$

I.e. we got the non-vanishing result even though Λ was taken to be infinitely large. This means that we have:

$$\frac{1}{\left|\mathrm{Det}\,\Delta\left(q\right)\right|^2}=\exp\left\{\frac{ig^2}{16\pi^2}\beta\cdot\int d^4q\,\mathrm{tr}\left\{F\tilde{F}\right\}\right\} \tag{87}$$

And one finds the new Lagrangian:

$$\mathcal{L}\to\mathcal{L}+\beta\cdot\partial_\mu\left(\bar{\psi}\gamma^\mu\gamma^5\psi\right)+\frac{g^2}{16\pi^2}\beta\cdot\mathrm{tr}\left\{F\tilde{F}\right\} \tag{88}$$

The change of variables must not affect the path integration result, thus we can write:

$$\partial_\mu\left(\bar{\psi}\gamma^\mu\gamma^5\psi\right)=-\frac{g^2}{16\pi^2}\,\mathrm{tr}\left\{F\tilde{F}\right\} \tag{89}$$

I.e. the axial symmetry of Lagrangian leads to the non-conserving current. In case if we have n_f interacting massless fermions:

$$\partial_\mu\left(\bar{\psi}\gamma^\mu\gamma^5\psi\right)=-\frac{n_f g^2}{16\pi^2}\,\mathrm{tr}\left\{F\tilde{F}\right\} \tag{90}$$

It is usually assumed that gauge currents must be anomaly-free. Otherwise the theory is considered to be unphysical.

IX. STRONG CP PROBLEM

Alternatively, one could shift the derivative back to the β, which is assumed to be a constant. In this case the term with fermions vanish and we get the following Lagrangian:

$$\mathcal{L}=-\frac{1}{4}F^2+\bar{\psi}\left(i\gamma D\right)\psi+\theta\,\mathrm{Tr}\left\{F\tilde{F}\right\}+\frac{\beta g^2}{16\pi^2}\,\mathrm{tr}\left\{F\tilde{F}\right\} \tag{91}$$

Which says that the axial transformation shifts the θ parameter.

In principle, it means that in theories with axial symmetry the θ term can always be removed with this change of variable and thus unphysical.

This works in $SU\left(2\right)\times U\left(1\right)$ sector of Standard Model, but not in QCD, which has axially broken interaction terms with other particles in the Standard Model. But for some reason θ_{QCD} appears to be very small (unobservable with the current accuracy of the experiment), currently it was verified that $\theta_{QCD}\lesssim 10^{-11}$.

It could be explained if one of the quarks is massless, but this possibility seems to be ruled out by various experiments.

Another possible solution to this strong CP problem is axion, which has an effective coupling to the gauge field of the following form:

$$\mathcal{L}_f=-a\,\mathrm{tr}\left\{F\tilde{F}\right\} \tag{92}$$

We can assume that the field a has a spontaneously broken symmetry (Peccei-Quinn) at the scale θ, so it cancels the QCD topological term. The exact mechanism of this symmetry breaking is model-dependent. Axions have never been observed.

X. AXIAL ANOMALY MEETS THETA

Let's now think about QCD where the topological term is non-trivial and look at the determinant once again:

$$\mathrm{Det}\,\Delta = \exp\left\{ i\beta \int d^4q \langle q \left| \mathrm{tr}\{\gamma^5\} \right| q \rangle \right\} \tag{93}$$

It says that β is periodic parameter, i.e. $\beta \in [0; 2\pi)$ (because it appears in the exponent with purely imaginary argument). This means that the term:

$$\theta\, \mathrm{Tr}\left\{ F\tilde{F} \right\} + \frac{\beta g^2}{16\pi^2}\, \mathrm{tr}\left\{ F\tilde{F} \right\} = \theta'\, \mathrm{tr}\left\{ F\tilde{F} \right\} \tag{94}$$

Can be considered as a periodic potential of some sort with respect to θ. In other words, we have a lot of minima $|n\rangle$ with the same potential energy. Transitions between different minima are called instantons and they are essentially non-perturbative.

The true state of the theory is a mixing of all minima (actually this is a Bloch theorem):

$$|\theta\rangle = \sum e^{-in\theta} |n\rangle \tag{95}$$

It is invariant to shifts between different vacuums, i.e.:

$$|\theta\rangle = \sum e^{-in\theta} |n+m\rangle = \langle n+m = n' \rangle = \sum e^{-i\left(n'-m\right)\theta} |n'\rangle = e^{im\theta} \sum e^{-in'\theta} |n'\rangle \tag{96}$$

I.e. the global phase of the state $|\theta\rangle$ was changed. But the global phase of quantum state is unobservable.

XI. NOTE ON TUNNELING

Fields cannot tunnel in the entire space at once. This can be easily read off the Euclidean path integral:

$$\int e^{-S_E}\, \mathcal{D}\phi; \quad S_E = \int d^D q\, (K + V) \tag{97}$$

Here ϕ is any field or any combination of fields, K is the kinetic part and V is the potential part. If the tunneling happens at the entire space, the evolution is independent of $\vec{\mathbf{x}}$, i.e. the Lagrangian along this evolution does not contain $\vec{\mathbf{x}}$ at all:

$$S_E = \mathrm{Volume} \cdot \int d\tau\, (K + V) \tag{98}$$

The volume of space is infinitely large and S_E is positive by it's definition, thus e^{-S_E} vanishes. In other words, transition may only happen locally and then spread to the entire Universe. This is a general feature of tunneling processes for fields. Typically, such processes are essentially non-perturbative.

Finally, note that it is important to separate instantons (transitions between two vacuums with the same energy) and false vacuum decay (transitions to the lower energy state).

XII. QCD IN THE CHIRAL LIMIT

Consider the pure QCD with the three lightest quarks (up, down and strange) without any other interactions. Their masses are small and in some cases can be neglected (the topological term also becomes removable):

$$\mathcal{L} = -\frac{1}{4}F^2 + \bar{\psi}_u\, (i\gamma D)\, \psi_u + \bar{\psi}_d\, (i\gamma D)\, \psi_d + \bar{\psi}_s\, (i\gamma D)\, \psi_s \tag{99}$$

The theory above has a local $SU(3)$ invariance (strong interaction) - quarks can be rotated in the color space.
We can introduce a new variable:

$$\Psi = \begin{pmatrix} \psi_u \\ \psi_d \\ \psi_s \end{pmatrix} \tag{100}$$

And write the Lagrangian in a short form:

$$\mathcal{L} = -\frac{1}{4}F^2 + \bar{\Psi}\left(i\gamma D\right)\Psi \tag{101}$$

It is invariant under the mixing of quarks between each other:

$$\Psi \to \exp\{it_a\alpha_a\}\Psi \tag{102}$$

We have 3 quarks, so this is an $SU(3)$ global symmetry.
It also leads to the global axial symmetry of quark mixing:

$$\exp\left\{i\gamma^5 t_a\beta_a\right\} \tag{103}$$

There is also a global $U(1)$ invarince, because we can rotate all quarks together:

$$\exp\{i\varphi\}; \quad \exp\left\{i\gamma^5\omega\right\} \tag{104}$$

Global transformation means that α_a, β_a, φ and ω do not dependent on x. The full global symmetry is usually written as:

$$SU_v(3) \times SU_a(3) \times U_v(1) \times U_a(1) \tag{105}$$

The last symmetry is anomalous. Curiously, $SU_a(3)$ is not anomalous, because generators are traceless matrices and do not suffer from renormalization. Indeed, all the divergences belong to the Lorentz space-time and the definition of γ^5 implies that $D = 1 + 3$, which is broken by the dimensional regularization. Internal symmetries matrices do not suffer from this.

Because they belong to different spaces, they commute and can be traced separately, thus:

$$\operatorname{tr}\left\{\gamma^5 t_a\right\} = \operatorname{tr}\left\{\gamma^5\right\}\operatorname{tr}\{t_a\} \tag{106}$$

$\operatorname{tr}\left\{\gamma^5\right\}$ is a poorly defined quantity, but $\operatorname{tr}\{t_a\}$ is equal to zero without any doubt.

However, it was discovered by Nambu that $SU_a(3)$ is spontaneously broken. It produces 8 (by the amount of generators) Goldstone bosons, which are supposed (by definition) to be massless, but they must acquire mass because in reality quarks are massive. Thus they are usually called pseudo-Goldstone bosons.

Using the chiral theory one can predict masses and other parameters of those particles. Experimentally we observe 8 pseudoscalar mesons which have exactly the predicted parameters, so they are usually associated with each other.

If $U_a(1)$ had not been anomalous, there would have been 9 pseudo-Goldstone bosons. That is why in the nonet of pseudoscalar mesons one meson (η') looks "odd" - it has very different parameters compared to the rest. This effect is usually called $U(1)$ problem.

Homework 4* (25 bonus points)

Prove that the energy-momentum tensor of the Chern-Simons theory:

$$\mathcal{L} = \frac{\kappa}{4\pi}\int \varepsilon^{\mu\nu\rho}\operatorname{Tr}\left\{A_\mu\partial_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho\right\}d^3q$$

Is zero and explain it's meaning.

Homework 5* (50 bonus points)

Prove that Chern-Simons theory is gauge invariant if κ is an integer number.