I. WHAT HAPPENED BEFORE

Let's briefly remind ourselves what happened in the previous course and recall what we now know about the reality.

1) Reality is spanned by the Lorentz group, which describes rotations in 4-dimensional Minkowski space-time. As any other group it has different representations by some sets of matrices. They are historically denoted with a letter s, which takes integer (including zero) and half-integer values

$$s = 0,\, \frac{1}{2},\, 1,\, ... \tag{1}$$

First 3 representations have specific names

$$s = 0 \to \text{scalar representation} \tag{2}$$

$$s = \frac{1}{2} \to \text{bispinor representation} \tag{3}$$

$$s = 1 \to \text{vector representation} \tag{4}$$

Objects which transform under those representations are called scalars

$$\phi' = \phi \tag{5}$$

Bispinors

$$\psi'_i = \Lambda_{ik}\psi_k \tag{6}$$

And 4-vectors

$$A'^{\mu} = a^{\mu}{}_{\nu}A^{\nu} \tag{7}$$

I.e. the information about the Lorentz structure is encoded in the index, which is thus called the Lorentz index.

2) The n-dimensional Lorentz invariant action

$$S = \int \mathcal{L}\left[\text{field}, \partial\left(\text{field}\right)\right]\, d^n q \tag{8}$$

Is uniquely defined by s and some general reasoning.

3) Klein-Gordon, Dirac and Maxwell equations were studied and it was shown that they contain various inconsistencies (Klein paradox, negative energies, etc.). To overcome them, second quantization was proposed, i.e.

$$\text{field} \to \hat{\text{field}} \quad + \quad \text{(anti)-commutation relations} \tag{9}$$

After that the perturbation theory was developed, diagrams invented, matrix elements, cross sections, all that kind of things.

Is that all?

Of course, not.

II. WHAT IS GOING TO HAPPEN NOW

Despite being a powerful technique that has led to significant advancements in physics, second quantization has some limitations:

1) It encounters difficulties when dealing with gauge theories, particularly non-abelian gauge fields.

2) Its capability to investigate non-perturbative effects remains unclear.

3) It is rooted in the Hamiltonian formulation rather than the Lagrangian one, which raises concerns about Lorentz invariance, among others.

However, there exists an approach that can comprehensively cover physics from classical mechanics to string theory, through a single concept known as the

Path integral

But before we proceed, let's solve a simple exercise to understand, what is going on.

III. TIME EVOLUTION OPERATOR

Let's first define the rules of the game.

Rule 1: an arbitrary state at any moment of time can be described with a function $|\psi\rangle$, which in turn can be treated as a vector in a functional Hilbert space.

Rule 2: vectors are typically projected on some orthogonal basis. As an example, let's consider the coordinate space as such basis

$$|q\rangle \tag{10}$$

Be definition

$$\langle q|\psi\rangle = \psi(q) \tag{11}$$

This is just a straightforward analogue to the normal linear algebra language

$$(\mathbf{e}_x, \mathbf{v}) = v_x \tag{12}$$

With $\mathbf{e}_x$ being a basis vector along x-axis, $\mathbf{v}$ is an arbitrary vector. Orthogonality also implies that

$$\langle q'|q\rangle = \delta\left(q' - q\right) \tag{13}$$

Where the delta-function was introduced. This is the analogue of the relation

$$(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij} \tag{14}$$

Where the Kronecker delta is used.

The normalization

$$\int |q\rangle\langle q|\, dq = 1 \tag{15}$$

Then is fulfilled automatically. Indeed, let's multiply it with $|q'\rangle$ and $\langle q''|$. On the right-hand side we immediately obtain $\delta\left(q'' - q'\right)$. On the left-hand side

$$\int \langle q''|q\rangle\langle q|q'\rangle\, dq = \int \delta\left(q'' - q\right)\delta\left(q - q'\right)\, dq = \delta\left(q'' - q'\right) \tag{16}$$

That is, the condition is well-defined. Another way to write the expansion above is then

$$|\psi\rangle = \int |q\rangle\langle q|\psi\rangle\, dq = \int \psi(q)|q\rangle\, dq \tag{17}$$

This can also be understood as

$$\mathbf{v} = \sum_i \mathbf{e}_i(\mathbf{e}_i, \mathbf{v}) = \sum_i v_i \mathbf{e}_i \tag{18}$$

Rule 3: physical variables are represented in terms of operators. Observables are simply the eigenvalues of these operators. The state

$$|q'\rangle \tag{19}$$

Satisfies the equation

$$\hat{q}|q'\rangle = q'|q'\rangle \tag{20}$$

I.e. in the $|q\rangle$ basis the operator $\hat{q}$ is diagonal.

Rule 4: states typically want to evolve in time. If a state does not carry the time-dependence explicitly, it is assumed to be simply taken at some given moment of time. For definiteness, let's say that further it will be $t = 0$.

This can be formulated in terms of the time evolution operator

$$|q, t\rangle = \hat{U}\left(t, 0\right)|q\rangle \tag{21}$$

Now, let's consider a state $|q_i, t_i\rangle$, describing a particle being at the point q_i at the moment t_i. The quantity

$$\langle q_f, t_f | q_i, t_i\rangle \tag{22}$$

Is known as an overlap of states (or an amplitude), sometimes we will also denote it by $K(q_f, t_f; q_i, t_i)$. The modulus square of this quantity

$$|\langle q_f, t_f | q_i, t_i\rangle|^2 \tag{23}$$

Gives the transition probability from one state to another in a given time $t_f - t_i$. Note that the global (constant) phase does not of the states does not affect the transition amplitude.

Time evolution operator is normalized as $U\left(t, t\right) \equiv 1$ by obvious reasons. It also satisfies the unitarity

$$\hat{U}^+\left(t, t_0\right) = \hat{U}\left(t_0, t\right) \tag{24}$$

And the group property

$$\hat{U}\left(t_1, t_c\right)\hat{U}\left(t_c, t_2\right) = \hat{U}\left(t_1, t_2\right) \tag{25}$$

So we can write

$$K(q_f, t_f; q_i, t_i) = \langle q_f, t_f | q_i, t_i\rangle = \langle q_f|\hat{U}\left(t_f, t_i\right)|q_i\rangle \tag{26}$$

Using the group property of the time evolution operator and the completeness relation of $|q\rangle$ states, we can show that

$$\begin{aligned}\int K(q_f, t_f; q_c, t_c)K(q_c, t_c; q_i, t_i)\, dq_c = \int \langle q_f|\hat{U}\left(t_f, t_c\right)|q_c\rangle\langle q_c|\hat{U}\left(t_c, t_i\right)|q_i\rangle\, dq_c = \\ = \langle q_f|\hat{U}\left(t_f, t_c\right)\hat{U}\left(t_c, t_i\right)|q_i\rangle = \langle q_f|\hat{U}\left(t_f, t_i\right)|q_i\rangle = \langle q_f, t_f|q_i, t_i\rangle = K(q_f, t_f; q_i, t_i)\end{aligned} \tag{27}$$

The physical meaning of this formula is clear - we added the intermediate point q_c (which, for the definiteness, the particle occupies at the moment t_c) to the evolution from the initial state $|q_i, t_i\rangle$ to the final one $|q_f, t_f\rangle$ and integrated over all possible q_c.

Homework 1 (10 points)

As a self-test to see if you understand this concept and notation, prove the formula

$$\psi\left(q_f, t_f\right) = \int K(q_f, t_f; q_i, t_i)\psi\left(q_i, t_i\right) dq_i \tag{28}$$

I.e. $K(q_f, t_f; q_i, t_i)$ can be treated as the kernel of the integral transformation which moves the system from the initial state to the final one.

Let's now try to understand how this wonderful operator $\hat{U}$ looks like explicitly.

IV. TIME EVOLUTION OPERATOR 2

The state of free particles is described with a position q, but that is not the only parameter that can be associated with it. There is also a momentum p, which is a conserved quantity whose conservation is associated with the homogeneity of space.

Let's find some operator which could correspond to $\hat{p}$ in the $|q\rangle$ basis. In principle, the only possible operator which one could deduce for such a quantity (associated with the homogeneity of space) is a derivative operator

$$\hat{p} \propto \partial_x \tag{29}$$

Space homogeneity requires the proportionality constant to be purely imaginary - in this case the shift $x \to x + a$ results in a constant phase, which is unobservable

$$\hat{p} = -i\partial_x, \qquad \hat{p}e^{ipq} = pe^{ipq} \tag{30}$$

By definition, the eigenfunction of the momentum operator $\hat{p}$ projected on the $|q\rangle$ space must be treated as $\langle q|p\rangle$, i.e. we deduce

$$\langle q|p\rangle = e^{ipq} \tag{31}$$

The normalization of the momentum states is (by convention) slightly different

$$\int |p\rangle\langle p| \frac{dp}{2\pi} = 1 \tag{32}$$

Sanity check can be easily performed. Multiply if first with $|q\rangle$

$$\int |p\rangle e^{-ipq} \frac{dp}{2\pi} = |q\rangle \tag{33}$$

And then - with $\langle q'|$

$$\int e^{ip(q-q')} \frac{dp}{2\pi} = \delta\left(q' - q\right) \tag{34}$$

I.e. the notation system is self-consistent. Note also that by definition

$$\langle p|q\rangle = \langle q|p\rangle^* = e^{-ipq} \tag{35}$$

If time is homogeneous as well, there is one more conserved quantity, called energy (or Hamiltonian). Using the same reasoning as before, we get

$$\hat{E} = i\partial_t \tag{36}$$

The fact that $\hat{p}$ and $\hat{E}$ come with an opposite sign, as we will see later, actually corresponds to the Lorentz signature.

The relation between E and p can be deduced from the general reasoning. In fact, since the Galilean invariance ($c \to \infty$ limit of the Lorentz invariance) implies that p^2 does not change under the corresponding group symmetry transformation, the only possible form of such dependence (in non-relativistic case) is

$$E = \frac{p^2}{2m} + V(q) \tag{37}$$

With m being a constant, conventionally called "mass", $V(q)$ is called a potential energy. This identity is based solely on Galilean invariance and thus must hold for the operators as well

$$\hat{E} = \frac{\hat{p}^2}{2m} + V(\hat{q}) \tag{38}$$

Operators themselves do not live and must act on some states, therefore

$$\hat{E}|\psi(t)\rangle = \underbrace{\left(\frac{\hat{p}^2}{2m} + V(\hat{q})\right)}_{\hat{H}}|\psi(t)\rangle \tag{39}$$

This equation describes the evolution of a state $|\psi(t)\rangle$ in time.

Further details can be found in the following textbooks:

1) L. D. Landau, E. M. Lifshitz Course of Theoretical Physics, Volume 1, Mechanics - sections 4 and 6.

2) L. D. Landau, E. M. Lifshitz Course of Theoretical Physics, Volume 3, Quantum mechanics - section 15.

3) L. D. Landau, E. M. Lifshitz Course of Theoretical Physics, Volume 4, Quantum Electrodynamics - section 10.

Let's now factorize

$$|\psi(t)\rangle = \hat{U}(t,0)|\psi\rangle \tag{40}$$

So the equation above is transformed to

$$i\frac{\partial \hat{U}(t,0)}{\partial t} = \hat{H}\hat{U}(t,0) \tag{41}$$

Which can be easily solved. It gives

$$\hat{U}(t,0) = \exp\Big\{-i\hat{H}t\Big\} \tag{42}$$

It was assumed that the potential energy is time-independent. In case of the time-dependent potential energy we can slice the time in intervals Δt so small that at each interval the potential energy can be treated as a constant. Then the derivation of the $\hat{U}(t,0)$ operator above can be repeated, but one has to keep the required time-ordering, i.e. we get

$$\hat{U}(t,0) = T\exp\left\{-i\int_0^t \hat{H}(\tau)d\tau\right\} \tag{43}$$

Where T denotes the time-ordered product. We see that this operator satisfies the properties derived in the previous section.

Homework 2 (10 points)

Consider the transition amplitude $K(q_f, t_f; p, t)$, which gives the overlap of the initial state with the definite momentum p and the final state with the definite coordinate q'. Express it in the form

$$K(q_f, t_f; p_i, t_i) = \int f(p_i, q_i)\, K(q_f, t_f; q_i, t_i)\, dq_i \tag{44}$$

$f(p, q)$ is a function to be defined.

V. GAUSSIAN INTEGRALS

In order to proceed further we will need to deal with some integrals and it is always better to be prepared in advance. The first integral of interest will be

$$I = \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}sp^2 + Jp\right\} dp \tag{45}$$

Note that $s > 0$ is required for this integral to exist. In this case it can be calculated by completing the square

$$\begin{aligned} I = \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}s\left(p - \frac{J}{s}\right)^2 + \frac{J^2}{2s}\right\} dp = \langle p - \frac{J}{s} = p' \rangle \\ = \exp\left\{\frac{J^2}{2s}\right\} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}sp'^2\right\} dp' = \langle \sqrt{s}p' = p'' \rangle = \\ = \frac{1}{\sqrt{s}} \exp\left\{\frac{J^2}{2s}\right\} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}p''^2\right\} dp'' = \sqrt{\frac{2\pi}{s}} \exp\left\{\frac{J^2}{2s}\right\} \end{aligned} \tag{46}$$

Now the integral with a positively defined matrix S

$$\mathcal{I} = \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\mathbf{p}^T S \mathbf{p} + \mathbf{J}^T \mathbf{p}\right\} dp \tag{47}$$

Without the loss of generality we can consider the matrix S to be symmetric. Indeed, any matrix can be splitted into symmetric and antisymmetric parts

$$M_{ik} = \underbrace{\frac{M_{ik} + M_{ki}}{2}}_{S_{ik}} + \underbrace{\frac{M_{ik} - M_{ki}}{2}}_{A_{ik}} \tag{48}$$

And the antisymmetric part will anyway cancel

$$\mathbf{p}^T A \mathbf{p} = p_i A_{ij} p_j = -p_i A_{ji} p_j = 0 \tag{49}$$

Then we write

$$\mathcal{I} = \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\left(\mathbf{p} - S^{-1}\mathbf{J}\right)^T S \left(\mathbf{p} - S^{-1}\mathbf{J}\right) + \frac{\mathbf{J}^T S^{-1} \mathbf{J}}{2}\right\} dp \tag{50}$$

Indeed

$$\left(\mathbf{p}-S^{-1}\mathbf{J}\right)^T S\left(\mathbf{p}-S^{-1}\mathbf{J}\right)=\mathbf{p}^T S\mathbf{p}+\mathbf{J}^T S^{-1}\mathbf{p}-\mathbf{p}^T\mathbf{J}-\mathbf{J}^T\mathbf{p} \tag{51}$$

The fact that $S^T=S$ was used. By the definition of a scalar product $\mathbf{p}^T\mathbf{J}=\mathbf{J}^T\mathbf{p}$. Thus we write

$$\mathbf{p}-S^{-1}\mathbf{J}=\mathbf{p}' \tag{52}$$

And obtain

$$\mathcal{I}=\exp\left\{\frac{\mathbf{J}^T S^{-1}\mathbf{J}}{2}\right\}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2}\mathbf{p}'^T S\mathbf{p}'\right\}d\mathbf{p}' \tag{53}$$

Let's perform the orthogonal transformation

$$\mathbf{p}'=U\mathbf{p}'' \tag{54}$$

The modulus of the determinant U is equal to 1, so there is no corresponding Jacobian. We can safely require this matrix U to diagonalize S

$$U^T SU=S'=\begin{pmatrix} S'_{11} & 0 & ... & 0 \\ 0 & S'_{22} & ... & ... \\ ... & ... & ... & ... \\ 0 & ... & ... & S'_{nn} \end{pmatrix} \tag{55}$$

In this case the integral over $\mathbf{p}''$ is splitted into the product of one-dimensional integrals over the components of $\mathbf{p}''$. The product

$$\frac{1}{\sqrt{S'_{11}...S'_{nn}}}=\frac{1}{\sqrt{\det\{S\}}} \tag{56}$$

Arises in the denominator and we finally get

$$\mathcal{I}=\sqrt{\frac{(2\pi)^n}{\det\{S\}}}\exp\left\{\frac{\mathbf{J}^T S^{-1}\mathbf{J}}{2}\right\} \tag{57}$$

VI. PATH INTEGRAL IN QUANTUM MECHANICS

Let's now get back to the transition amplitude

$$K(q_f,t_f;q_i,t_i)=\langle q_f,t_f|q_i,t_i\rangle=\langle q_f|\hat{U}\left(t_f,t_i\right)|q_i\rangle \tag{58}$$

And recall the property

$$K(q_f,t_f;q_i,t_i)=\int K(q_f,t_f;q_c,t_c)K(q_c,t_c;q_i,t_i)dq_c \tag{59}$$

Let's enhance this idea and insert more than one intermediate points, denoted as q_j

$$K(q_f,t_f;q_i,t_i)=\int K(q_f,t_f;q_n,t_n)\,K(q_n,t_n;q_{n-1},t_{n-1})\,...\,K(q_2,t_2;q_1,t_1)\,K(q_1,t_1;q_i,t_i)\,dq_n...dq_1 \tag{60}$$

The time intervals between each two points will be denoted as Δt and are assumed to be equal. There are $n+1$ intervals in total.

In the general case of the time-dependent potential the evolution operator has the form

$$\hat{U}(t,0) = T\exp\left\{-i\int_0^t \hat{H}(\tau)d\tau\right\} \tag{61}$$

But since $n \to \infty$ the path between each two points is so small that the potential can be treated as a constant. Additionally, we assume $t_f > t_n > ... > t_1 > t_i$, i.e. the time-ordering is preinstalled and we get

$$\int \langle q_f|e^{-i\hat{H}\Delta t}|q_n\rangle \langle q_n|e^{-i\hat{H}\Delta t}|q_{n-1}\rangle ... \langle q_2|e^{-i\hat{H}\Delta t}|q_1\rangle \langle q_1|e^{-i\hat{H}\Delta t}|q_i\rangle \, dq_n...dq_1 \tag{62}$$

Next we use the orthogonality of the momentum space

$$\langle q_{j+1}|e^{-i\hat{H}\Delta t}|q_j\rangle = \int \langle q_{j+1}|p\rangle\langle p|e^{-i\hat{H}\Delta t}|q_j\rangle \, \frac{dp}{2\pi} \tag{63}$$

The explicit form of the Hamiltonian is

$$\hat{H} \equiv \hat{H}(\hat{q},\hat{p},t) = \frac{\hat{p}^2}{2m} + V(\hat{q},\hat{p},t) \tag{64}$$

Without any loss of generality we can assume $V(\hat{q},\hat{p},t)$ to be a Weyl-ordered operator, which means that all $\hat{p}$ are dragged to the left position using the commutation relation of $\hat{q}$ and $\hat{p}$. Unless otherwise stated, we assume that this condition is always true.

Let's expand the exponent

$$\exp\left\{-i\left(\frac{\hat{p}^2}{2m} + V(\hat{q},\hat{p},t)\right)\Delta t\right\} \approx 1 - i\left(\frac{\hat{p}^2}{2m} + V(\hat{q},\hat{p},t)\right)\Delta t \tag{65}$$

The remaining terms are of order $(\Delta t)^2$ or higher and are neglected. Note that the higher-order terms would be proportional to the commutator $\left[\hat{p}^2, V(\hat{q},\hat{p},t)\right]$.

Since $\hat{p}^+ = \hat{p}$ we can write

$$\langle p|\hat{p}|q_j\rangle = \langle q_j|\hat{p}|p\rangle^* \tag{66}$$

Thus the operator $\hat{p}$ can be used to act on the state $\langle p|$, the operator $V(\hat{q},t)$ can act on the state $|q_j\rangle$. The exponent can be combined back again and we get

$$\langle p|e^{-i\hat{H}\Delta t}|q_j\rangle \approx \exp\left\{-i\left(\frac{p^2}{2m} + V(q_j,p,t)\right)\Delta t\right\}\langle p|q_j\rangle \tag{67}$$

We remind

$$\langle q|p\rangle = e^{ipq} \tag{68}$$

Therefore

$$\langle q_{j+1}|e^{-i\hat{H}\Delta t}|q_j\rangle = e^{-iV(q_j,t)\Delta t}\int e^{-i\frac{p^2}{2m}\Delta t}e^{ip(q_{j+1}-q_j)}\,\frac{dp}{2\pi} \tag{69}$$

This is the Gaussian integral of the form

$$I = \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}sp^2 + Jp\right\} dp = \sqrt{\frac{2\pi}{s}} \exp\left\{\frac{J^2}{2s}\right\} \tag{70}$$

Where denoted

$$s = \frac{i\Delta t}{m} \tag{71}$$

$$J = i\left(q_{j+1} - q_j\right) \tag{72}$$

And we obtain

$$\begin{aligned} \langle q_{j+1}|e^{-i\hat{H}\Delta t}|q_j\rangle &= \left[\frac{m}{2\pi i\Delta t}\right]^{1/2} \exp\left\{i\frac{m}{2}\frac{\left(q_{j+1}-q_j\right)^2}{\left(\Delta t\right)^2}\Delta t - iV(q_j,p_j,t)\Delta t\right\} = \\ &= \left[\frac{m}{2\pi i\Delta t}\right]^{1/2} \exp\{iL\left(q_j,\dot{q}_j\right)\Delta t\} \end{aligned} \tag{73}$$

Where $L\left(q_j,\dot{q}_j\right)$ is the Lagrangian at the point q_j and we finally get

$$K(q_f,t_f;q_i,t_i) = \left[\frac{m}{2\pi i\Delta t}\right]^{\frac{n+1}{2}} \int \exp\left\{i\sum_{j=0}^{n} L\left(q_j,\dot{q}_j\right)\Delta t\right\} dq_n...dq_1 \tag{74}$$

The following notation was adopted

$$q_0 = q_i \tag{75}$$

$$q_{n+1} = q_f \tag{76}$$

Let's now consider the limit $n \to \infty$ and $\Delta t \to 0$. Denote

$$\lim_{n\to\infty} \left[\frac{m}{2\pi i\Delta t}\right]^{\frac{n+1}{2}} dq_n...dq_1 = \mathcal{D}q \tag{77}$$

The prefactor formally becomes infinite, but we want it to be consumed into the integration measure $\mathcal{D}q$. The exponent is also trivially modified

$$\sum_{j=0}^{n} L\left(q_j,\dot{q}_j\right)\Delta t = \int_{t_i}^{t_f} L\left(q,\dot{q}\right)\, dt \tag{78}$$

The integration limit will be omitted most of the time to save ink. The total answer reads

$$K\left(q_f,t_f;q_i,t_i\right) = \int_{q[t_i]=q_i}^{q[t_f]=q_f} e^{iS[q]}\,\mathcal{D}q; \qquad S[q] = \int_{t_i}^{t_f} L\left(q,\dot{q}\right)\, dt \tag{79}$$

And is called the path integral.

VII. THE PHYSICAL MEANING OF PATH INTEGRAL

Let's summarize what we just did

1) Starting from the operator formalism, we developed the new approach to the quantum mechanics, based on the study of the function $K(q_f,t_f;q_i,t_i)$. It allows to express the transition probability (as well as the energy spectrum and other observable, as we will see later) in terms of only **classical Lagrangian**.

2) The integration measure $\mathcal{D}q$ was achieved by the integration over all possible intermediate points. The symbol $\mathcal{D}q$ can be treated as the integration over all paths, some of which may require faster-than-light motion, etc.

The representation of the $K(q_f,t_f;q_i,t_i)$ in this terms is thus called the path integral.

Homework 3 (10 points)

The representation above is called the path integral in the coordinate space. Prove that it is possible to as well formulate it in terms of the phase space

$$K(q_f,t_f;q_i,t_i)=\int \exp\left\{i\sum_{j=0}^{n} p_j \frac{q_{j+1}-q_j}{\Delta t} - i\sum_{j=0}^{n}\left(\frac{p_j^2}{2m}+V\left(q_j,\dot{q}_j\right)\right)\Delta t\right\} dq_n...dq_1 \frac{dp_n...dp_0}{(2\pi)^{n+1}} \tag{80}$$

Where $p_0 \equiv p$. In the continuous limit it can be written in the form

$$K(q_f,t_f;q_i,t_i)=\int \exp\left\{i\int\left(p\dot{q}-H\left(p,q\right)\right)dt\right\}\mathcal{D}q\mathcal{D}p \tag{81}$$

Which can be very useful for some applications.

It this new approach one does not need to deal with any operators. The only necessary object is the classical Lagrangian - the quantization is automatically applied.

Clearly, we could repeat this derivation for the many-dimensional system

$$K(\mathbf{q}_f,t_f;\mathbf{q}_f,t_f)=\int_{\mathbf{q}[t_i]=\mathbf{q}_i}^{\mathbf{q}[t_f]=\mathbf{q}_f} e^{iS[\mathbf{q}]}\,\mathcal{D}\mathbf{q}; \qquad S[\mathbf{q}]=\int_{t_i}^{t_f} L\left(\mathbf{q},\dot{\mathbf{q}}\right)dt \tag{82}$$

In the real scalar field theory the role of coordinate is played by the field $\phi(x)$ itself. Assume that at the moment t the field ϕ was spread around the Universe, symbolically this configuration will be denoted as ϕ_i. Then it started evolving and at the given moment t' found itself in a state ϕ_f. We could then easily generalize the derivation above and write the path integral for the real scalar field in the form

$$K(\phi_f,t_f;\phi_i,t_i)=\int_{\phi[t_i]=\phi_i}^{\phi[t_f]=\phi_f} e^{iS[\phi]}\,\mathcal{D}\phi; \qquad S[\phi]=\int_{t_i}^{t_f}\left(\frac{1}{2}\left(\partial\phi\right)^2-V\left(\phi\right)\right)d^nq \tag{83}$$

Since it takes a lot of time and space to write the limits of integration explicitly, we will usually omit them because they are straightforward to be understood.

Next, complex scalar field can be treated as the sum two real scalar field - number of variables is simply doubled. In this case we just need to replace

$$\mathcal{D}\phi \to \mathcal{D}\phi\mathcal{D}\phi^* \tag{84}$$

$$\frac{1}{2}\left(\partial\phi\right)^2-\frac{1}{2}m^2\phi^2 \to \left|\partial\phi\right|^2-m^2\left|\phi\right|^2 \tag{85}$$

And so on. In fact, path integral

1) Provides a straightforward treatment for the gauge theories.

2) Contains a theory in a complete way, with all possible perturbative and non-perturbative effects.

3) Provides manifestly Lorentz invariant description of any physical system.

And much more, and all of it - using only classical Lagrangian as an input. For instance, the quantum gravity path integral in 4 dimensions is given by

$$\int \mathcal{D}g_{\mu\nu}g^{-5/2}\exp\{iS_g\}; \qquad S_g=\int R\sqrt{g}\,d^4q \tag{86}$$

$g^{-5/2} = \det\{g_{\mu\nu}\}^{-5/2}$ is added because $\mathcal{D}g_{\mu\nu}$ integration measure is not Lorentz invariant by itself (in other dimensions the power of this factor is different).

Homework 4 (10 points)

Deduce the Lagrangian of the Schrodinger equation, write the corresponding path integral, and explain its meaning.

VIII. AXIOMATIC APPROACH TO QUANTUM MECHANICS

Let's now think more hard on how the physics works. Assume that the system evolves from the given state A to another state B. Obviously, there are many ways to reach A from B, so let's compare each trajectory with a certain number S, called the action along the path.

It is clear that S must be an additive quantity. Indeed, if the path AB is broken into two parts with some given C, then the total action becomes

$$S_{AB} = S_{AC} + S_{CB} \tag{87}$$

Since there are so many possibilities, there is no prior reasons for any of them to be anyhow specific. Let's introduce the so-called transition amplitude

$$K(A;B) = \int F\left(S_{AB}\right) \mathcal{D}q \tag{88}$$

With $F\left(S\right)$ being the "weight" of each trajectory, so far some unknown function of S. It is clear that nothings changes if we break the path with an arbitrary C and express $K(A;B)$ in terms of $K(A;C)$ and $K(C;B)$ - just to remember to integrate over dq_C in order to get rid of the q_C dependence

$$K(A;B) = \int K(A;C)K(C;B)dq_C = \int\int\int F\left(S_{AC}\right) F\left(S_{CB}\right) \mathcal{D}q_{AC}\mathcal{D}q_{CB}dq_C = \int F\left(S_{AB}\right) \mathcal{D}q \tag{89}$$

The integration

$$\int\int\int ... \mathcal{D}q_{AC}\mathcal{D}q_{CB}dq_C \tag{90}$$

Is obviously equivalent to

$$\int ... \mathcal{D}q_{AB} \equiv \int ... \mathcal{D}q \tag{91}$$

Thus we have

$$F\left(S_{AB}\right) = F\left(S_{AC}\right) F\left(S_{CB}\right) \tag{92}$$

The only function F which satisfies this equation combined with the $S_{AB} = S_{AC} + S_{CB}$ condition is exponential

$$F = e^{\mathrm{const}\times S} \tag{93}$$

The constant is, in principle, arbitrary, but as we will see later a self-consistent approach requires it to be imaginary. In other words

$$K(A;B) = \int e^{iS_{AB}} \mathcal{D}q \tag{94}$$

And this path integral, in principle, stands as the sole quantity capable of describing the evolution of the system (and necessary for this). All the observables can be extracted from it.

The integral above can be approximated with the saddle-point method, which says that the dominant contribution to the transition is given by the extremization of S_{AB}, i.e.

$$\delta S_{AB} = 0 \tag{95}$$

Which is also known as the least action principle.

So the recipe to win physics is the following deduce the classical Lagrangian from the general reasoning (Lorentz/-Galilean invariance). The classical theory is given just by the condition $\delta S = 0$, the quantum effects arise from the sum over paths.

The operator method can be deduced from it, just as previously the path integral was derived from the operator formalism. These are just two different approaches to solving the same problems, but the path integral is way more fundamental and allows to cover physics from the classical mechanics to quantum gravity and even beyond.

As a concluding remark, we would like to emphasize a detail that was previously left behind. In general, initial and final states are considered arbitrary; they serve as boundary conditions for the path integral. However, in some specific cases (such as when the problem exists in a holomorphic space, etc.), there may also be boundary effects, which explicitly depend on A and B

$$K(A;B) = \int e^{iS_{AB}} \times [\text{boundary terms}]_{A,B}\, \mathcal{D}q \tag{96}$$

We will not deal with such examples in this course.

IX. EXPECTATION VALUES

It is in principle clear from the definition of the path integral that the path integral of the form

$$\int O_1 ... O_n\, e^{iS}\, \mathcal{D}q \tag{97}$$

Expresses nothing but the time-ordered matrix element of some operator $\hat{O}_1 ... \hat{O}_n$ between the initial and final states (indeed, it has a form an average being weighted by the exponential of the action, integrated over a functional measure)

$$\langle q_f, t_f | T\left(\hat{O}_1 ... \hat{O}_n\right) | q_i, t_i\rangle \tag{98}$$

If $q_f = q_i$ holds, this expression is also known as the expectation value in the state $|q_i\rangle$ over a time period $t_f - t_i$.

Let's prove this explicitly, at first - for the single coordinate operator $\hat{q}(t_1)$, taken at some given moment t_1 which satisfies $t' > t_1 > t$.

Recall that we derived the path integral by inserting an infinite amount of coordinate states, i.e. using the definition of how $\hat{q}$ acts on the coordinate space we could write

$$\langle q_f, t_f|\hat{q}(t_1)|q_i, t_i\rangle = \int \langle q_f, t_f|\hat{q}(t_1)|q_1, t_1\rangle\langle q_1, t_1|q_i, t_i\rangle dq_1 = \int q(t_1)\langle q_f, t_f|q_1, t_1\rangle\langle q_1, t_1|q_i, t_i\rangle dq_1 \tag{99}$$

And then proceed with the standard derivation to obtain

$$\langle q_f, t_f|\hat{q}(t_1)|q_i, t_i\rangle = \int q(t_1)\, e^{iS}\, \mathcal{D}q \tag{100}$$

Note that since the inserted states were time-ordered, the expectation value must be also time-ordered, otherwise the argument above would simply not work

$$\langle q_f, t_f|T\left[\hat{q}(t_1)\hat{q}(t_2)\right]|q_i, t_i\rangle = \int q(t_1)q(t_2)\, e^{iS}\, \mathcal{D}q \tag{101}$$

X. THE ENERGY SPECTRUM

In the context of the path integral there is one more technique that we would like to demonstrate, namely - how to conveniently extract the discrete energy spectrum.

First of all, note that in such a case the set of $|E_n\rangle$ orthogonal states is produced. Denote

$$\langle q|E_n\rangle = \psi_n(q) \tag{102}$$

Since this is a discrete spectrum (not a continuous one), we have

$$\langle E_n|E_m\rangle = \delta_{nm} \tag{103}$$

$$\sum_n |E_n\rangle\langle E_n| = 1 \tag{104}$$

Let's use it to write

$$\begin{aligned} K(q_f,t_f;q_i,t_i) &= \langle q_f|\exp\left\{-i\hat{H}(t_f-t_i)\right\}|q_i\rangle = \sum_n \langle q_f|\exp\left\{-i\hat{H}(t_f-t_i)\right\}|E_n\rangle\langle E_n|q_i\rangle = \\ &= \sum_n e^{-iE_n(t_f-t_i)}\langle q_f|E_n\rangle\langle E_n|q_i\rangle = \sum_n e^{-iE_n(t_f-t_i)}\psi_n(q_f)\,\psi_n^*(q_i) \end{aligned} \tag{105}$$

If one is able to factorize that path integral in such a way it is possible to extract both $\psi_n(q)$ and the energy spectrum.

If one just needs to explore the spectrum, this formula can be simplified. Consider the case when $q_i = q_f$, i.e. the particle goes back to the initial state. Let's integrate over this point q and introduce the new letter Z (the path integral over all closed paths)

$$Z = \int K(q_i,t_f;q_i,t_i)\,dq = \sum_n e^{-iE_n(t_f-t_i)}\int \langle q|E_n\rangle\langle E_n|q_i\rangle\,dq \tag{106}$$

Obviously

$$\langle q|E_n\rangle\langle E_n|q\rangle = \left|\langle E_n|q\rangle\right|^2 = \left|\langle q|E_n\rangle\right|^2 \tag{107}$$

And thus we can use the orthogonality of the $|q_i\rangle$ states

$$Z = \sum_n e^{-iE_n(t_f-t_i)}\underbrace{\langle E_n|E_n\rangle}_{=1} = \sum_n e^{-iE_n(t_f-t_i)} \tag{108}$$

These formulas work best in the imaginary time.

XI. FREE PARTICLE

The only remaining step now to exploit physics is to understand how to evaluate the path integral. Let's start from the case of a free particle - the simplest way to solve the problem is to explicitly write the sliced time transition amplitude

$$K(q_f,t_f;q_i,t_i) = \left[\frac{m}{2\pi i\Delta t}\right]^{\frac{n+1}{2}}\int \exp\left\{\frac{im}{2\Delta t}\sum_{j=0}^{n}(x_{j+1}-x_j)^2\Delta t\right\}dq_n...dq_1 \tag{109}$$

It can be easily calculated if the make use of the formula

$$\int e^{ia(q-q_1)^2+ib(q_2-q)^2}\,dq = \sqrt{\frac{i\pi}{a+b}}\exp\left\{\frac{ab}{a+b}\left(q_1-q_2\right)^2\right\} \tag{110}$$

The integral over dq_1 reads

$$\left[\frac{m}{2\pi i\Delta t}\right]^{\frac{2}{2}}\int\exp\left\{\frac{im}{2\Delta t}\left(q_1-q_2\right)^2+\frac{im}{2\Delta t}\left(q_i-q_1\right)^2\right\}dq_1 = \sqrt{\frac{m}{2\cdot 2\pi i\Delta t}}\exp\left\{\frac{im}{2\cdot 2\Delta t}\left(q_2-q_i\right)^2\right\} \tag{111}$$

The next is the integral over dq_2

$$\left[\frac{m}{2\pi i\Delta t}\right]^{\frac{1}{2}}\sqrt{\frac{m}{2\cdot 2\pi i\Delta t}}\int\exp\left\{\frac{im}{2\Delta t}\left(q_2-q_3\right)^2+\frac{im}{2\cdot 2\Delta t}\left(q_2-q_i\right)^2\right\}dq_2 = \sqrt{\frac{m}{2\cdot 3\pi i\Delta t}}\exp\left\{\frac{im}{2\cdot 3\Delta t}\left(q_3-q_i\right)^2\right\} \tag{112}$$

After repeating this procedure n times we get

$$K(q_f,t_f;q_i,t_i) = \sqrt{\frac{m}{2\cdot(n+1)\,\pi i\Delta t}}\exp\left\{\frac{im}{2\cdot(n+1)\,\Delta t}\left(q_f-q_i\right)^2\right\} \tag{113}$$

Recall

$$(n+1)\,\Delta t \equiv t_f - t_i \tag{114}$$

So the function becomes

$$K(q_f,t_f;q_i,t_i) = \sqrt{\frac{m}{2\cdot\pi i\,(t_f-t_i)}}\exp\left\{\frac{im}{2}\frac{\left(q_f-q_i\right)^2}{t_f-t_i}\right\} \tag{115}$$

Homework 5 (15 points)

Calculate $K(p_f,t_f;p_i,t_i)$.
Hint: Hagen Kleinert: Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets - chapter 2.

XII. IMAGINARY TIME

Let's have another look at the path integral

$$K(q_f,t_f;q_i,t_i) = \int e^{i\int L(q,\dot q)\,dt}\mathcal{D}q, \qquad L\left(q,\dot q\right) = \frac{m\dot q^2}{2} - V\left(q\right) \tag{116}$$

The time t is assumed to be a real parameter describing the evolution. However, we note that the path integral can be analytically continued to the complex values of t

$$t \to t - i\tau, \quad \tau > 0 \tag{117}$$

Consider now the case when $t = 0$ and the path integral lives in purely imaginary time

$$idt \to d\tau \tag{118}$$

$$\dot q^2 \to -\dot q^2 \tag{119}$$

We obtain the so-called Euclidean path integral

$$K(q_f,\tau_f;q_i,\tau_i)=\int e^{-\int L_E(q,\dot{q})\,d\tau}\mathcal{D}q, \qquad L_E\left(q,\dot{q}\right)=\frac{m\dot{q}^2}{2}+V\left(q\right) \tag{120}$$

With the sign of the potential is now reversed. $L_E > 0$, so the path integral is now purely Gaussian and not oscillating. Or, in other words, integrated function is finite and decreasing.

Additionally, if we face a discontinuous path, where the derivative goes infinity, it does not make contribution. Indeed, $\dot{q}^2 \to \infty$ and the contribution of this path to the integral vanishes.

And actually, this is a theorem - path integral with only first order derivatives is undefined (this can be avoided if the field is represented in terms of Grassmann variables, as we will see later).

The reason why this approach is called Euclidean is straightforward if we consider the (real scalar, just as an example) field theory

$$K(\phi_f,t_f;\phi_i,t_i)=\int \exp\left\{i\int L\left(\phi,\partial\phi\right)\,d^nq\right\}\mathcal{D}\phi, \qquad \mathcal{L}=\frac{1}{2}\left(\partial\phi\right)^2-V\left(\phi\right) \tag{121}$$

Which transforms to

$$K(\phi_f,\tau_f;\phi_i,\tau_i)=\int \exp\left\{-\int \mathcal{L}_E\left(\phi,\partial\phi\right)\,d^nq_E\right\}\mathcal{D}\phi, \qquad \mathcal{L}_E=\frac{1}{2}\left(\partial\phi\right)_E^2+V\left(\phi\right) \tag{122}$$

Where denoted

$$\left(\partial\phi\right)_E^2=\left(\frac{\partial\phi}{\partial t}\right)^2+\left(\frac{\partial\phi}{\partial\vec{\mathbf{q}}}\right)^2 \tag{123}$$

I.e. the metric now can be treated as $(1,1,1,1)$.

XIII. PARTITION FUNCTION

Let's have at the path integral from a different angle (with the time-independent Hamiltonian for now)

$$K(q_f,\tau_f;q_i,\tau_i)=\langle q_f|e^{-\hat{H}(\tau_f-\tau_i)}|q_i\rangle=\int e^{-\int L_E(q,\dot{q})\,d\tau}\mathcal{D}q \tag{124}$$

The Euclidean partition function becomes

$$Z\equiv\int K(q,\tau_f;q,\tau_i)\,dq=\int\langle q|e^{-\hat{H}(\tau_f-\tau_i)}|q\rangle\,dq\equiv\mathrm{Tr}\left\{e^{-\hat{H}(\tau_f-\tau_i)}\right\} \tag{125}$$

The last transition is just the definition of a trace as the sum of diagonal elements.

Note that the integration measure $\mathcal{D}q$ can be safely redefined if we add this additional point - there is anyway an infinite amount of them

$$Z=\oint e^{-\int L_E(q,\dot{q})\,d\tau}\mathcal{D}q \tag{126}$$

Meanwhile, in the statistical physics the partition function is known to be the key object which can be used to extract all the thermodynamic potentials, etc. It is defined as

$$Z\equiv\mathrm{Tr}\left\{e^{-\beta\hat{H}}\right\} \tag{127}$$

With β being the inverse temperature. In other words, imaginary time τ can be exactly treated as the inverse temperature β and the transformation above maps the quantum mechanics onto the statistical physics. Now recall

$$K(q_f, \tau_f; q_i, \tau_i) = \sum_n e^{-E_n(\tau_f - \tau_i)} \psi_n(q') \, \psi_n^*(q) \tag{128}$$

$$Z = \sum_n e^{-E_n(\tau_f - \tau_i)} \tag{129}$$

In practice this relation turns out to be very useful to extract the energy spectrum from the partition function Z. In the large Euclidean time limit (corresponding to the zero temperature limit in the statistical physics) one obtains

$$\lim_{\tau_f - \tau_i \to \infty} Z \to e^{-E_0(\tau_f - \tau_i)} \tag{130}$$

I.e. by setting $\tau_f - \tau_i \to \infty$ one can obtain the ground state E_0. This can be useful is case if, for example, the potential is very complicated and it is not possible to extract the entire spectrum.

XIV. ABOUT SCALAR FIELDS AND FEYNMAN PRESCRIPTION

The only super-renormalizable field theory in 4 dimensions is ϕ^3:

$$\mathcal{L}_3 = \frac{1}{2} (\partial\phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{a}{3!} \phi^3 \tag{131}$$

The path integral for this theory does not exist, which is especially clear if we rotate ourselves to the imaginary time. Indeed, the integral

$$\int_{-\infty}^{\infty} e^{-x^3} \, dx \tag{132}$$

Is divergent. The physical meaning of this is that the potential x^3 does not produce a stable minimum and the system tends to infinitely fall. It is noteworthy that this divergence does not appear in the perturbative expansion, but the perturbation theory series itself does not converge.

The theory can be regularized/stabilized if we add the quartic interaction

$$\mathcal{L}_4 = \frac{1}{2} (\partial\phi)^2 - \frac{m^2}{2} \phi^2 - \frac{a}{3!} \phi^3 - \frac{b}{4!} \phi^4 \tag{133}$$

In this case the path integral will be obviously finite, but means that there are no self-consistent super-renormalizable theories in 4-dimensional space-time.

Let's now have a look at the free scalar field

$$\mathcal{L}_2 = \frac{1}{2} (\partial\phi)^2 - \frac{m^2}{2} \phi^2 \tag{134}$$

The Feynman prescription $m^2 \to m^2 - i\varepsilon$ is also consistent with the definition of the path integral

$$\exp\left\{ i \int \left(i\varepsilon\phi^2 \right) \, d^n q \right\} = \exp\left\{ -\varepsilon \int \phi^2 \, d^n q \right\} \tag{135}$$

This exponent produces a well-defined Gaussian exponent and, in some sence, controls the path integral convergence.

XV. SEMICLASSICAL APPROXIMATION

One of the most straightforward approaches for the path integral calculation in more complicated example is the steepest descent method. In many important cases, such as the harmonic oscillator, it gives an exact answer.

Let's start from afar and factorize an arbitrary path in the form

$$q = q_{cl} + \delta q \tag{136}$$

And assume the boundary condition

$$\delta q\left(t_i\right) = \delta q\left(t_f\right) = 0 \tag{137}$$

To avoid confusion in notation, here t_i and t_f denote the initial and final moments of evolution.

Now expand the action around q_{cl}

$$S \approx S_{cl} + \int_{t_i}^{t_f}\left[\frac{\partial L}{\partial \dot{q}}\delta\dot{q} + \frac{\partial L}{\partial q}\delta q\right] dt + \frac{1}{2}\int_{t_i}^{t_f}\left[\frac{\partial^2 L}{\partial \dot{q}^2}\left(\delta\dot{q}\right)^2 + 2\frac{\partial L}{\partial \dot{q}}\frac{\partial L}{\partial q}\delta\dot{q}\delta q + \frac{\partial^2 L}{\partial q^2}\left(\delta q\right)^2\right] dt \tag{138}$$

All the derivatives in this expression are assumed to be evaluated under the condition $q = q_{cl}$, i.e.

$$\frac{\partial L}{\partial q} \equiv \left.\frac{\partial L}{\partial q}\right|_{q=q_{cl}} \tag{139}$$

This is nothing but the Taylor series.

The first term is just the action on the path q_{cl}. The second term can be transformed if we note

$$\frac{\partial L}{\partial \dot{q}}\delta\dot{q} + \frac{\partial L}{\partial q}\delta q = \left[-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{\partial L}{\partial q}\right]\delta q \tag{140}$$

The equation in the square brackets is just the Euler-Lagrange equation.

Meanwhile, the saddle-point approximation states that the dominant contribution to the integral of the form

$$\int \exp\{iS\}\,\mathcal{D}q \tag{141}$$

Comes from the path where the exponent is extremized, i.e. from the path which satisfies the Euler-Lagrange equation. Assume q_{cl} to be exactly this path (previously it was just an unspecified expansion point), so we have

$$\left[-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{\partial L}{\partial q}\right]_{q=q_{cl}} = 0 \tag{142}$$

δq can be treated as quantum fluctuations around the classical path.

The third term we transform using the identities

$$\delta\dot{q}\frac{\partial^2 L}{\partial \dot{q}^2}\delta\dot{q} = \frac{d}{dt}\left(\delta q\frac{\partial^2 L}{\partial \dot{q}^2}\delta\dot{q}\right) - \delta q\frac{d}{dt}\left(\frac{\partial^2 L}{\partial \dot{q}^2}\delta\dot{q}\right) \tag{143}$$

$$2\delta q\frac{\partial L}{\partial \dot{q}}\frac{\partial L}{\partial q}\delta\dot{q} = \frac{d}{dt}\left(\delta q\frac{\partial L}{\partial \dot{q}}\frac{\partial L}{\partial q}\delta q\right) - \delta q\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\frac{\partial L}{\partial q}\right)\delta q \tag{144}$$

The terms

$$\frac{d}{dt}\left(\delta q\frac{\partial^2 L}{\partial \dot{q}^2}\delta\dot{q}\right) \quad (145)$$

$$\frac{d}{dt}\left(\delta q\frac{\partial L}{\partial \dot{q}}\frac{\partial L}{\partial q}\delta q\right) \quad (146)$$

Are full derivatives - they vanish after the integration if we use the boundary conditions on the function δq written above. All in all

$$S \approx S_{cl} + \frac{1}{2}\int_{t_i}^{t_f} \delta q\left[\frac{d}{dt}\left(-\frac{\partial^2 L}{\partial \dot{q}^2}\delta\dot{q}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\frac{\partial L}{\partial q}\right)\delta q + \frac{\partial^2 L}{\partial q^2}\delta q\right] dt \quad (147)$$

Recall now

$$L = \frac{m\dot{q}^2}{2} - V(q) \quad (148)$$

Note that if $V(q)$ contains terms which are only quadratic and linear in q the semiclassical approximation gives an exact answer, since all the subsequent terms in the expansion of S require at least 3 derivatives, which vanish. We obtain

$$S \approx S_{cl} + \frac{1}{2}\int_{t_i}^{t_f} \delta q\hat{D}\delta q\, dt \quad (149)$$

$$\hat{D} = -m\frac{\partial^2}{\partial t^2} - \left.\frac{\partial^2 V(q)}{\partial q^2}\right|_{q=q_{cl}} \quad (150)$$

The path integral is then equal to

$$K\left(q_f, t_f; q_i, t_i\right) = e^{iS_{cl}}\int \exp\left\{\frac{1}{2}\int_{t_i}^{t_f}\delta q\hat{D}\delta q\, dt\right\}\mathcal{D}\delta q = \frac{e^{iS_{cl}}}{\sqrt{\det\left\{\hat{D}\right\}}} \quad (151)$$

I.e. we need to find the determinant of some Sturm-Liouville differential operator. The easiest way to do it is to find its eigenvalues and calculate their product.

Remarkably, with the boundary conditions the path integral can also be expressed in the form

$$K\left(q_f, t_f; q_i, t_i\right) = e^{iS_{cl}}K\left(0, t_f; 0, t_i\right) \quad (152)$$

XVI. HARMONIC OSCILLATOR

The harmonic oscillator is defined by the potential

$$V = \frac{m\omega^2 q^2}{2} \quad (153)$$

So we need to solve the eigenvalue problem

$$\left(-\frac{\partial^2}{\partial t^2} - \omega^2\right)y_n = \lambda_n y_n \quad (154)$$

Note that in principle this defines the classical equations of motion

$$\ddot{q} + \omega^2 q = 0 \tag{155}$$

So everything is bounded to classical-level physics only.
We have to keep the boundary condition

$$y(t_i) = y(t_f) = 0 \tag{156}$$

The corresponding solution is

$$y_n(t) = \sin\left(\frac{\pi n (t - t_i)}{T}\right) \tag{157}$$

Where for simplicity we denoted $t_f - t_i = T$ and n is a non-zero integer (as usual, the trivial solution $y_n = 0$ is discarded). Eigenvalues are

$$\lambda_n = -\omega^2 + \frac{\pi^2 n^2}{T^2} \tag{158}$$

The product of the eigenvalues

$$\det\left\{\hat{D}\right\} = \prod_{i=1}^{\infty}\left(-\omega^2 + \frac{\pi^2 n^2}{T^2}\right) \tag{159}$$

Is obviously divergent (look at the $n \to \infty$ limit). However, we know that for $\omega = 0$ this must reproduce the free particle answer, which can be chosen as the normalization

$$\prod_{i=1}^{\infty}\left(-\omega^2 + \frac{\pi^2 n^2}{T^2}\right) = \frac{\prod_{i=1}^{\infty}\left(-\omega^2 + \frac{\pi^2 n^2}{T^2}\right)}{\prod_{i=1}^{\infty}\frac{\pi^2 n^2}{T^2}} \times \prod_{i=1}^{\infty}\frac{\pi^2 n^2}{T^2} \equiv \prod_{i=1}^{\infty}\left(1 - \frac{\omega^2 T^2}{\pi^2 n^2}\right) \times \frac{2\pi i T}{m} \tag{160}$$

I.e. we assumed that $\frac{2\pi i T}{m}$ is the definition of the divergent product $\prod_{i=1}^{\infty}\frac{\pi^2 n^2}{T^2}$ in order for the free particle limit to be reproduced correctly. The object $\prod_{i=1}^{\infty}\left(1 - \frac{\omega^2 T^2}{\pi^2 n^2}\right)$ is convergent and gives

$$\det\left\{\hat{D}\right\} = \frac{\sin(\omega T)}{\omega T} \times \frac{2\pi i T}{m} \tag{161}$$

So the final answer reads

$$K(q_f, t_f; q_i, t_i) = \sqrt{\frac{m\omega}{2\pi i \sin(\omega T)}} e^{iS_{cl}} \tag{162}$$

Homework 6 (15 points)

Prove that in the case of harmonic oscillator the classical path is defined by

$$q_{cl}(t) = \frac{q_f \sin(\omega(t - t_i)) + q_i \sin(\omega(t_f - t))}{\sin(\omega T)} \tag{163}$$

And the classical action is

$$S_{cl} = \frac{m\omega}{2\sin(\omega T)}\left[\left(q_i^2 + q_f^2\right)\cos(\omega T) - 2q_f q_i\right] \tag{164}$$

Hint: Hagen Kleinert: Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets - chapter 2.

Now set $q_i = q_f$ and integrate over it, leading to

$$Z = \int K\left(q, t_f; q, t_i\right) \, dq = \sqrt{\frac{m\omega}{2\pi i \sin\left(\omega T\right)}} \int \exp\left\{ i \frac{m\omega q^2}{\sin\left(\omega T\right)} \cdot \left[\cos\left(\omega T\right) - 1\right] \right\} dq = \\ = \sqrt{\frac{1}{2\left[\cos\left(\omega T\right) - 1\right]}} = \frac{1}{2i\sin\left(\omega T/2\right)} = \frac{e^{-i\omega T/2}}{1 - e^{-i\omega T}} \tag{165}$$

Sum of a convergent geometric progression is

$$\sum_{i=0}^{\infty} b_i = \frac{b_0}{1-f} \tag{166}$$

With f being the progression scale factor. Use imaginary time to make it well-defined

$$Z = \frac{e^{-\omega\beta/2}}{1 - e^{-\omega\beta}} = \sum_{n=0}^{\infty} e^{-\omega\beta(n+1/2)}; \qquad iT = \beta \tag{167}$$

I.e. $b_0 = e^{-\omega\beta/2}$ and $f = e^{-\omega\beta}$. Obviously, the spectrum is then

$$E_n = \omega\left(n + \frac{1}{2}\right) \tag{168}$$

Finally, we illustrate how to extract the ground state wave function. In the large τ limit only the ground state contributes

$$K(q_f, \tau_f; q_i, \tau_i) = \sum_n e^{-E_n\beta} \psi_n\left(q_f\right) \psi_n^*\left(q_i\right) \to e^{-E_0\beta} \psi_0\left(q_f\right) \psi_0^*\left(q_i\right) \to e^{-iE_0 T} \psi_0\left(q_f\right) \psi_0^*\left(q_i\right) \tag{169}$$

And we have to look for the coefficient in front of the exponent

$$\exp\{-i\omega T/2\} \tag{170}$$

In this limit we have

$$\frac{m\omega\cos\left(\omega T\right)}{2\sin\left(\omega T\right)} \to i\frac{m\omega}{2}\coth\omega\tau \to \frac{im\omega}{2} \tag{171}$$

$$\frac{m\omega}{\sin\left(\omega T\right)} = 2i\frac{m\omega e^{-i\omega T}}{1 - e^{-2i\omega T}} \to 2im\omega e^{-i\omega T} \tag{172}$$

So the path integral is decaying into

$$K\left(q_f, t_f; q_i, t_i\right) = \sqrt{\frac{m\omega}{2\pi i \sin\left(\omega T\right)}} \exp\left\{ \frac{im\omega}{2\sin\left(\omega T\right)} \left[\left(q_i^2 + q_f^2\right)\cos\left(\omega T\right) - 2q_f q_i\right] \right\} \to \\ \to \sqrt{\frac{m\omega}{\pi}} \exp\{-i\omega T/2\} \exp\left\{ -\frac{m\omega}{2}\left(q_i^2 + q_f^2\right) \right\} \tag{173}$$

So one clearly obtains

$$\psi_0\left(q\right) = \left(\frac{m\omega}{\pi}\right)^{1/4} \exp\left\{ -\frac{m\omega}{2} q^2 \right\} \tag{174}$$

Which is a well-known result from the quantum mechanics. By finding the coefficients in front of the higher-order exponents, one can construct all other wave functions.

Homework 7 (15 points)

Calculate the path integral for the particle moving in the linear potential

$$V(q) = \alpha q \tag{175}$$

And derive the corresponding spectrum.

XVII. ANHARMONIC OSCILLATOR

So far we discussed only Gaussian path integrals, but let's now have a look at the more complicated example

$$V(q) = \frac{m\omega^2 q^2}{2} + \frac{gq^4}{4!} \tag{176}$$

But in this case we can split the action into Gaussian and non-Gaussian parts

$$S = S^{(\text{int})} + S^{(0)} \tag{177}$$

If $g \ll 1$ is fulfilled, $S^{(\text{int})}$ expands in Taylor series, the path integral can be calculated perturbatively (there are also non-perturbative techniques to calculate such integrals, but let's not think about them for now).

The calculation of corrections comes down to evaluating some matrix elements. Moreover, we note that

$$\int q(t_1)\, e^{iS[q]}\, \mathcal{D}q \equiv \left[\frac{1}{i}\frac{\delta}{\delta J(t_1)} \int \exp\left\{iS[q] + i\int J(t)\, q(t)\; dt\right\} \mathcal{D}q\right]_{J=0} \tag{178}$$

Where we used the functional derivative

$$\frac{\delta J(t)}{\delta J(t_1)} = \delta(t - t_1) \tag{179}$$

In the general case

$$\int q(t_1) \ldots q(t_n)\, e^{iS[q]}\, \mathcal{D}q \equiv \left[\frac{1}{i^n}\frac{\delta}{\delta J(t_1)} \ldots \frac{\delta}{\delta J(t_n)} \int \exp\left\{iS[q] + i\int J(t)\, q(t)\; dt\right\} \mathcal{D}q\right]_{J=0} \tag{180}$$

The path integral with an added Jq term is typically called the generating functional.

This means that in an arbitrary theory the path integral can be calculated with any desired accuracy and only the knowledge of the Gaussian integral is required.

Let's illustrate this with a simple example - namely, let's calculate the correction to the ground state of the anharmonic oscillator. Recall the partition function

$$Z = \int e^{-S_E}\, \mathcal{D}q = \int e^{-S_E^{\text{int}}} e^{-S_E^0}\, \mathcal{D}q \approx e^{-E_0 \beta} \tag{181}$$

After the Taylor expansion is applied

$$Z \approx Z^{(0)} - \frac{g}{4!} \int \left(\int_{\tau_1}^{\tau_2} \left[q(\tau)\right]^4\, d\tau\right) e^{-S_E^0} \mathcal{D}q \approx e^{-E_0 \beta} \tag{182}$$

I.e. we need to calculate the 4-point function and integrate it over τ

$$\int \left(\int_{\tau_1}^{\tau_2} [q(\tau)]^4 \, d\tau \right) e^{-S_E^0} \mathcal{D}q = \int_{\tau_1}^{\tau_2} \left\langle 0 \left| T [q(\tau)]^4 \right| 0 \right\rangle d\tau \tag{183}$$

In principle, the definition of the partition function Z implies that we need to integrate over all possible closed loops. However, from the equations above, it becomes evident that the ground state provides the dominant contribution. Next

$$\begin{aligned} \int [q(\tau)]^4 e^{-S_E^0} \mathcal{D}q &= \left[\left(\frac{\delta}{\delta J(\tau)} \right)^4 \int \exp\left\{ -S_E[q] + \int_{\tau_1}^{\tau_2} J(\tau) q(\tau) \, d\tau \right\} \mathcal{D}q \right]_{J=0} = \\ = Z^{(0)} &\left[\left(\frac{\delta}{\delta J(\tau)} \right)^4 \exp\left\{ \frac{1}{2} \int_{\tau_1}^{\tau_2} J(\tau_1) S^{-1}(\tau_1, \tau_2) J(\tau_2) \, d\tau_1 d\tau_2 \right\} \right]_{J=0} \end{aligned} \tag{184}$$

The formula for the Gaussian integral with the linear term was applied. The summation over matrix indices

$$\mathbf{J}^T S^{-1} \mathbf{J} = J_i \left(S^{-1} \right)_{ik} J_k \equiv \sum_i \sum_k J_i \left(S^{-1} \right)_{ik} J_k \tag{185}$$

But the summation is replaced with integration, since there are infinitely many components. $S^{-1}(\tau, \tau')$ denotes the inverse of the quadratic form from $S_E^{(0)}$

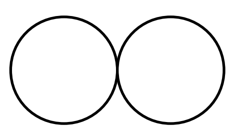
$$S(\tau) = m \left[-\frac{\partial^2}{\partial \tau^2} + \omega^2 \right] \tag{186}$$

In some sense it is degenerate, since it depends only on one parameter (or index).

Let's Taylor the exponent. Fewer than four Js will be annihilated by the derivatives, more than four will vanish when setting $J = 0$

$$\begin{aligned} &Z^{(0)} \left[\left(\frac{\delta}{\delta J(\tau)} \right)^4 \exp\left\{ \frac{1}{2} \int_{\tau_1}^{\tau_2} \int_{\tau_1}^{\tau_2} J(\tau_1) S^{-1}(\tau_1, \tau_2) J(\tau_2) \, d\tau_1 d\tau_2 \right\} \right]_{J=0} = \\ &= \frac{Z^{(0)}}{8} \left[\left(\frac{\delta}{\delta J(\tau)} \right)^4 \int_{\tau_1}^{\tau_2} \int_{\tau_1}^{\tau_2} \int_{\tau_1}^{\tau_2} \int_{\tau_1}^{\tau_2} \left(J(\tau_1) S^{-1}(\tau_1, \tau_2) J(\tau_2) \, d\tau_1 d\tau_2 \right)^2 \right]_{J=0} = \frac{Z^{(0)}}{8} \times 4! \left[S^{-1}(\tau, \tau) \right]^2 \end{aligned} \tag{187}$$

Note that the problem is reduced to the calculation of the enclosed Feynman graph



The last step is to evaluate $S^{-1}(\tau_1, \tau_2) \equiv W(\tau_1, \tau_2)$. It can be found from the equation

$$m \left[-\frac{\partial^2}{\partial \tau_1^2} + \omega^2 \right] W(\tau_1, \tau_2) = \delta(\tau_1 - \tau_2) \tag{188}$$

Which is just an analogue to the ordinary matrix relation

$$M_{ik} \left(M^{-1} \right)_{kj} = \delta_{ij} \tag{189}$$

The answer can be easily obtained with the Fourier transformation

$$W\left(\tau_1,\tau_2\right)=\frac{1}{m}\int_{-\infty}^{\infty}\frac{e^{ik(\tau_1-\tau_2)}}{k^2+\omega^2}\frac{dk}{2\pi}=\frac{e^{-\omega|\tau_1-\tau_2|}}{2m\omega} \tag{190}$$

So we finally get

$$\frac{3g}{4!}\int_{\tau_1}^{\tau_2}\left[W\left(\tau,\tau\right)\right]^2\,d\tau=\frac{g\beta}{32m^2\omega^2} \tag{191}$$

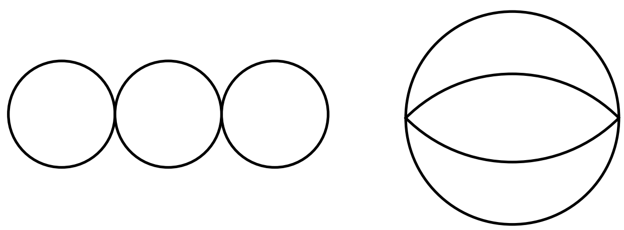
Leading to

$$Z\approx Z^{(0)}\left(1-\frac{g\beta}{32\omega^2}\right)\approx Z^{(0)}\exp\left\{-\frac{g\beta}{32m^2\omega^2}\right\}\approx e^{-E_0\beta} \tag{192}$$

The partition function $Z^{(0)}$ contains the unperturbed result $\omega/2$. Thus the first correction to the energy level is

$$\frac{g}{32m^2\omega^2}\equiv\underbrace{g}_{\text{coupling constant}}\times\underbrace{\frac{1}{8}}_{\text{symmetry coefficient}}\times\underbrace{\frac{g}{4m^2\omega^2}}_{\text{double propagator}} \tag{193}$$

The next order contains the set of diagrams shown below



It can be easily proven that the answer is given by

$$-\frac{21g^2}{128m^4\omega^5}\cdot\frac{1}{36} \tag{194}$$

But note that at the second order the full propagator $W\left(\tau_1,\tau_2\right)$ is contributing, not just $W\left(\tau,\tau\right)$ (because the loop is not enclosed on itself).

XVIII. REAL SCALAR FIELD

The real scalar field path integral is given by

$$K(\phi_f,t';\phi_i,t)=\int\exp\left\{i\int\mathcal{L}\left(\phi,\partial\phi\right)\,d^nq\right\}\mathcal{D}\phi \tag{195}$$

With the Lagrangian

$$\mathcal{L}\left(\phi,\partial\phi\right)=\frac{1}{2}\left(\partial\phi\right)^2-\frac{1}{2}m^2\phi^2 \tag{196}$$

But we know that is describes simply the infinite amount of harmonic oscillators. The Hamiltonian of this system is

$$H=\frac{1}{2}\int\left(\pi^{2}+(\nabla\phi)^{2}+m^{2}\phi^{2}\right)d\mathbf{q} \tag{197}$$

In the momentum space it transforms into

$$H=\frac{1}{2}\int\left[\pi\left(-p\right)\pi\left(p\right)+\omega_{k}^{2}\phi\left(-p\right)\phi\left(p\right)\right]d\mathbf{p},\qquad\omega_{p}^{2}=\mathbf{p}^{2}+m^{2} \tag{198}$$

2π to some power is inside $d\mathbf{p}$ symbol. After repeating the steps above we obtain

$$H=\int\omega_{k}\left(n+\frac{1}{2}\right)d\mathbf{p} \tag{199}$$

The ground-state wave functions, partition function and the rest can be, of course, easily obtained, though they are not of great interest (just a composition of many oscillators) and we will go on to the fermion field, which is less trivial.

XIX. GRASSMANN NUMBERS

Let's now move to the Dirac field

$$\mathcal{L}=\bar{\psi}\left(i\left(\gamma\partial\right)-m\right)\psi \tag{200}$$

It has one specific property which we would like to emphasize. Using the plane wave decomposition

$$\psi=\int\left(a_{p,s}u_{p,s}e^{-ipx}+b_{p,s}^{*}v_{p,s}e^{ipx}\right)d\mathbf{p} \tag{201}$$

Is it possible to write the field energy-momentum as

$$P^{\mu}=\int p^{\mu}\left(a_{p,s}^{*}a_{p,s}-b_{p,s}b_{p,s}^{*}\right)d\mathbf{p} \tag{202}$$

This is a classical expression and it is not well-defined - the energy is not bounded from below. One can fix this by requiring ψ field functions to be anticommutating objects (and so do $a_{p,s}$ and $b_{p,s}$)

$$-b_{p,s}b_{p,s}^{*}=b_{p,s}^{*}b_{p,s} \tag{203}$$

Which fixes the problem. This is a circumstance that is often omitted, but fermions are represented by field variables which anticommute already on the classical level.

Let' first consider just one Grassmann variable θ. From the anticommutativity it immediately follows that

$$\theta\theta=-\theta\theta=0 \tag{204}$$

So an arbitrary function of it can only be linear

$$f(\theta)=a+b\theta \tag{205}$$

In case of two Grassmann variables the following holds

$$\theta\eta=-\eta\theta \tag{206}$$

In the field theory the bispinor ψ is a set of 4 independent Grassmann variables.
First of all, we need to defined the integrals

$$\int d\theta \tag{207}$$

$$\int \theta \, d\theta \tag{208}$$

The integral is definite and goes all over the Grassmann variable space, just like

$$\int_{\infty}^{\infty} f(q) \, dq \tag{209}$$

An important property of such kind of integrals is shift symmetry

$$\int_{\infty}^{\infty} f(q+a) \, dq = \int_{\infty}^{\infty} f(q) \, dq \tag{210}$$

We would like to have it preserved

$$\int f(\theta) \, d\theta = \int f(\theta + \eta) \, d\theta = \int (a + b\theta + b\eta) \, d\theta \tag{211}$$

But the definition of $f(\theta)$ implies that this is equal to

$$\int (a + b\theta) \; d\theta \tag{212}$$

But since both b and η are arbitrary, this means that

$$\int d\theta = 0 \tag{213}$$

The integral of the Grassmann variable over itself can be then treated as

$$\int \theta \, d\theta = 1 \tag{214}$$

It actually appears that for Grassmann variables the integration and differentiation are the same operations

$$\frac{\partial}{\partial \theta} f(\theta) = \int f(\theta) \, d\theta = 1 \tag{215}$$

Let's now consider two Grassmann variables denoted θ and θ^*. The exponent is also trivialized (b is an ordinary number)

$$\exp\{-\theta^* b \theta\} = 1 - \theta^* b \theta \tag{216}$$

And thus the Gaussian integral becomes

$$\int \exp\{-\theta^* b \theta\} \, d\theta^* d\theta = -b \int \theta^* \theta \, d\theta^* d\theta = b \int \theta^* b \theta \, d\theta d\theta^* = b \tag{217}$$

Now say we have a set of θ_i and corresponding θ_i^*. We are interested in the multidimensional integral

$$\int \exp\{-\theta_i^* A_{ij} \theta_j\} \, d\theta_j^* d\theta_i = \int \theta_i^* A_{ij} \theta_j \, d\theta_i d\theta_j^* \tag{218}$$

A_{ij} must be an antisymmetric matrix. Let's diagonalize it with the orthogonal transformation, just like in the case of a normal integral. This leads to

$$\prod_i A_{ii} \int \theta_i^* \theta_i \, d\theta_i d\theta_i^* = \prod_i A_{ii} = \det\{A\} \tag{219}$$

In contrast to the bosonic case, the inverse of A is not required to exist.

The linear term can be treated in the same manner as before (remember to write it in a symmetric way with respect to θ_i and θ_i^*)

$$\int \exp\left\{-\theta_i^* A_{ij} \theta_j + c_i^* \theta_i + c_j \theta_j^*\right\} d\theta_j^* d\theta_i = \det\{A\} \exp\left\{c_i^* \left(A^{-1}\right)_{ij} c_j\right\} \tag{220}$$

c_i are some numbers.

The key difference between bosons and fermions is thus that in the first case we obtained $\frac{1}{\sqrt{\det\{S\}}}$, meanwhile for fermions it is $\det\{A\}$.

XX. DIRAC FIELD

In this section we derive the spectrum of this theory, i.e. the Pauli exclusion principle. This is more tricky than in the case of scalars, since there is no analogy with the classical oscillator. We restrict ourselves to the 4-dimensional space-time. The derivation is based on the section 9.2 of the book "Solitons and Instantons" by R. Rajaraman.

The Lagrangian of the free Dirac field is given by

$$\mathcal{L} = \bar{\psi} \left(i \left(\gamma \partial\right) - m\right) \psi \tag{221}$$

It is of the first order in derivatives, which could be a problem in case of bosons, but Grassmann variables simply cannot produce any divergence due to the way they are defined. We remind that

$$\bar{\psi} = \psi^+ \gamma^0 \tag{222}$$

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu} \tag{223}$$

The path integral above for the Lagrangian above is Gaussian and thus can be easily taken. Let's rewrite the Lagrangian in a form that better suits our purposes

$$\bar{\psi} \left(i \left(\gamma \partial\right) - m\right) \psi = \psi^+ \left(i \partial_t + i \alpha_i \nabla_i - m \beta\right) \psi \tag{224}$$

Where we introduced

$$\gamma^0 = \beta \tag{225}$$

$$\beta \gamma_i = \alpha_i \tag{226}$$

The integration measure $\mathcal{D}\bar{\psi}$ is of course equivalent (up to a constant) to the integration measure $\mathcal{D}\psi^+$. The answer to the integral is then given by the product of the eigenvalues λ of the operator

$$\left(i \partial_t + i \alpha_i \nabla_i - m \beta\right) \varphi(\mathbf{q}, t) = \lambda \varphi(\mathbf{q}, t) \tag{227}$$

But this equation is well-known - this is just a classical Dirac equation with an energy shifted by λ. There are 4 solutions in total, two of them refer to the eigenvalues ($\sqrt{\mathbf{p}^2 + m^2} = \varepsilon_p$)

$$E - \lambda = \varepsilon_p, \quad \Rightarrow \quad \lambda = E - \varepsilon_p \tag{228}$$

And other two give

$$E - \lambda = -\varepsilon_p, \quad \Rightarrow \quad \lambda = E + \varepsilon_p \tag{229}$$

Since we are interested only in the spectrum, let's study the partition function, i.e. the evolution over the closed path (it simply allows you to solve the problem in the most economical way). This implies the boundary conditions

$$\varphi(\mathbf{q}, t_i) = -\varphi(\mathbf{q}, t_f) \tag{230}$$

Which is a well-known fact from the group theory - by moving the spinor along a closed loop, one gets an additional phase factor -1. Thus E is quantized

$$E \to E_n, \qquad E_n T = (2n+1)\pi \tag{231}$$

With n being an arbitrary integer

$$n = \infty, ..., -1, 0, 1, ..., \infty \tag{232}$$

The product of eigenvalues is then given by

$$Z = \prod_p \prod_{n=-\infty}^{\infty} (E_n - \varepsilon_p)^2 (E_n + \varepsilon_p)^2 \tag{233}$$

The tricky part is that one has to include all possible $\mathbf{p}$ and n. The product over $\mathbf{p}$ is continuous. Note that

$$\prod_{n=-\infty}^{\infty} (E_n - \varepsilon_p)^2 = \prod_{n=-\infty}^{\infty} (E_{-n} + \varepsilon_p)^2 = \prod_{n=-\infty}^{\infty} (E_n + \varepsilon_p)^2 \tag{234}$$

So we get

$$Z = \prod_p \prod_{n=-\infty}^{\infty} (E_n + \varepsilon_p)^4 = \prod_p \prod_{n=-\infty}^{\infty} \left(\frac{(2n+1)\,\pi}{T}\right)^4 \left(1 + \frac{\varepsilon_p T}{(2n+1)\,\pi}\right)^4 \tag{235}$$

The first product is a constant which will be consumed into normalization

$$\prod_{n=-\infty}^{\infty} \left(\frac{(2n+1)\,\pi}{T}\right)^4 = [\text{const}] \tag{236}$$

The second one is the definition of the function

$$\prod_{n=-\infty}^{\infty} \left(1 + \frac{\varepsilon_p T}{(2n+1)\,\pi}\right)^4 = \cos\left(\frac{\varepsilon_p T}{2}\right) \tag{237}$$

And finally

$$Z = \prod_p \left(\cos\left(\frac{\varepsilon_p T}{2}\right)\right)^4 \to \prod_p e^{2i\varepsilon_p T}\left(1+e^{-i\varepsilon_p T}\right)^4 \tag{238}$$

The bracket can be expanded if we use the binomial theorem

$$\left(1+e^{-i\varepsilon_p T}\right)^4 = \sum_{n=0}^{4} \frac{4!}{n!\,(4-n)!} e^{-in\varepsilon_p T} \tag{239}$$

After the product is applied, one obtains

$$\prod_p \sum_{n=0}^{4} \frac{4!}{n!\,(4-n)!} e^{-in\varepsilon_p T} = \sum_{\{n_p\}} \prod_p \frac{4!}{n_p!\,(4-n_p)!} e^{-in_p\varepsilon_p T} \tag{240}$$

The notation $\{n_p\}$ means that for each $\mathbf{p}$ there is a set of $n = 0, ..., 4$. To prove this relation, decompose and collect back again

$$\begin{aligned} \sum_{\{n_p\}} \prod_p \frac{4!}{n_p!\,(4-n_p)!} e^{-in_p\varepsilon_p T} &= \sum_{n_{p_1}=0}^{4} ... \sum_{n_{p_l}=0}^{4} \frac{4!}{n_{p_1}!\,(4-n_{p_1})!} e^{-in_{p_1}\varepsilon_p T} ... \frac{4!}{n_{p_l}!\,(4-n_{p_l})!} e^{-in_{p_l}\varepsilon_p T} = \\ = \left(\sum_{n_{p_1}=0}^{4} \frac{4!}{n_{p_1}!\,(4-n_{p_1})!} e^{-in_{p_1}\varepsilon_p T}\right) ... \left(\sum_{n_{p_l}=0}^{4} \frac{4!}{n_{p_l}!\,(4-n_{p_l})!} e^{-in_{p_l}\varepsilon_p T}\right) &= \prod_p \sum_{n=0}^{4} \frac{4!}{n!\,(4-n)!} e^{-in\varepsilon_p T} \end{aligned} \tag{241}$$

Finally

$$\sum_{\{n_p\}} \prod_p \frac{4!}{n!\,(4-n)!} e^{-in_p\varepsilon_p T} = \sum_{\{n_p\}} \frac{4!}{n!\,(4-n)!} \exp\left\{-iT \int n_p \varepsilon_p \frac{d^3p}{(2\pi)^3}\right\} \tag{242}$$

$$\prod_p e^{2i\varepsilon_p T} = \exp\left\{2iT \int \varepsilon_p \frac{d^3p}{(2\pi)^3}\right\} \tag{243}$$

Thus

$$Z = \sum_{\{n_p\}} \frac{4!}{n!\,(4-n)!} \exp\left\{-iT \int_p (n_p - 2)\,\varepsilon_p \frac{d^3p}{(2\pi)^3}\right\} \tag{244}$$

The factor in front of the exponent is the combinatorial degeneracy coefficient. The energy levels then are given by

$$E = \int (n_p - 2)\,\varepsilon_p \frac{d^3p}{(2\pi)^3} \tag{245}$$

Which is precisely the Dirac quantization. Indeed, for each momentum $\mathbf{p}$ one can create up to 4 particles, corresponding to fermion/antifermion with two possible helicities, each with the energy ε_p.

Homework 8 (15 points)

Since Grassmann variables anticommute, does it mean that the term $\mathcal{L}_{int} = \bar{\psi}\psi\bar{\psi}\psi$ always vanish in the path integral? What about higher order interactions, for example $\left(\bar{\psi}\psi\right)^5$? Space-time dimension is set to $D = 1+3$.

Hint: remember that ψ is not a single variable, but a multicomponent object.

Homework 9* (25 bonus points)

Would you get the same tree level amplitude for $e^+e^- \to 4e^+e^-$ process from the $\left(\bar{\psi}\psi\right)^5$ term in the canonical formalism and with the path integral?

Hint: think about the operator ordering and Wick theorem.