I. REMINDER

During the previous class the path integral was introduced. The transition amplitude of a single particle from the initial state $|q_i\rangle$ to the final state $|q_f\rangle$ in a time $t_f - t_i$ is given by:

$$K\left(q_f, t_f; q_i, t_i\right) = \int_{q[t_i]=q_i}^{q[t_f]=q_f} e^{iS[q]}\,\mathcal{D}q; \qquad S[q] = \int_{t_i}^{t_f} \left(\frac{m\dot{q}^2}{2} - V\left(q\right)\right) dt \tag{1}$$

We also proved the formula which relates matrix elements and the path integral:

$$\langle q_f, t_f | T\left[\hat{q}(t_1)...\hat{q}(t_n)\right] | q_i, t_i\rangle = \int_{q[t_i]=q_i}^{q[t_f]=q_f} q(t_1)...q(t_n) e^{iS[q]}\,\mathcal{D}q; \tag{2}$$

Field theory is just a trivial generalization of this concept. The real scalar field path integral is:

$$K(\phi_f, t_f; \phi_i, t_i) = \int_{\phi[t_i]=\phi_i}^{\phi[t_f]=\phi_f} e^{iS[\phi]}\,\mathcal{D}\phi; \qquad S[\phi] = \int_{t_i}^{t_f} \left(\frac{1}{2}\left(\partial\phi\right)^2 - V\left(\phi\right)\right) d^n q \tag{3}$$

The expression for the matrix element can be naturally generalized:

$$\left\langle \phi_f, t_f \left| T\left[\hat{\phi}(q_1)...\hat{\phi}(q_n)\right] \right| \phi_i, t_i \right\rangle = \int_{\phi[t_i]=\phi_i}^{\phi[t_f]=\phi_f} \phi\left(q_1\right)...\phi\left(q_n\right) e^{iS[\phi]}\,\mathcal{D}\phi \tag{4}$$

We discussed only Gaussian path integrals, but (as an example) it is very common to have the potential energy of the form:

$$V\left(\phi\right) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{5}$$

But in this case we can split the action into Gaussian and non-Gaussian parts:

$$S = S^{(\text{int})} + S^{(0)} \tag{6}$$

$S^{(\text{int})}$ can be expanded in Taylor series. In this case the calculation of the path integral can be performed perturbatively. The calculation of corrections at each order is reduced to the calculation of some matrix elements.

Finally, we note that:

$$\int_{\phi[t_i]=\phi_i}^{\phi[t_f]=\phi_f} \phi\left(q_1\right) e^{iS[\phi]}\,\mathcal{D}\phi \equiv \left[\frac{1}{i}\frac{\delta}{\delta J\left(q_1\right)} \int_{\phi[t_i]=\phi_i}^{\phi[t_f]=\phi_f} \exp\left\{iS\left[\phi\right] + i\int J\left(q\right)\phi\left(q\right)\, d^n q\right\} \mathcal{D}\phi\right]_{J=0} \tag{7}$$

Where we used the functional derivative:

$$\frac{\delta J\left(q\right)}{\delta J\left(q_1\right)} = \delta\left(q - q_1\right) \tag{8}$$

This is just the continuous limit for the well-known vector calculus identity:

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij} \tag{9}$$

In the general case:

$$\int_{\phi[t_i]=\phi_i}^{\phi[t_f]=\phi_f} \phi(q_1)...\phi(q_n)\, e^{iS[\phi]}\, \mathcal{D}\phi \equiv \left[\frac{1}{i^n}\frac{\delta}{\delta J(q_1)}...\frac{\delta}{\delta J(q_n)} \int_{\phi[t_i]=\phi_i}^{\phi[t_f]=\phi_f} \exp\left\{iS[\phi] + i\int J(q)\,\phi(q)\, d^n q\right\} \mathcal{D}\phi\right]_{J=0} \tag{10}$$

The path integral with an added $J\phi$ term is typically called the generating functional.

This means that in an arbitrary theory the path integral can be calculated with any desired accuracy and only the knowledge of the Gaussian integral is required.

Homework 1 (35 points)

Consider the anharmonic oscillator potential:

$$V(q) = \frac{m\omega^2 q^2}{2} + \frac{gq^4}{4!} \tag{11}$$

Prove that the ground state energy has the form:

$$E_0 = \frac{\omega}{2} + \frac{g}{32m^2\omega^2} - \frac{7g^2}{1536m^4\omega^5} + \mathcal{O}\left(g^3\right) \tag{12}$$

In the context of field theory, analogous calculations can also be carried out, with only a minor increase in complexity (and also - the need to use some kind of regularization)

Hint: a part of this calculation can be found in the script of the first lecture.

II. NON-ABELIAN GAUGE THEORY

Let's consider a situation where instead of one field there is a set of n fields. For definiteness, we will refer to the scalar field, but the same reasoning can be applied for spinors as well.

In this case fields can be conveniently combined in a vector:

$$\phi = \begin{pmatrix}\phi_1\\ ...\\ \phi_n\end{pmatrix}; \qquad \phi^+ = (\phi_1^*, ..., \phi_n^*) \tag{13}$$

And the Lagrangian keeps the form:

$$\mathcal{L} = \partial^\mu \phi_i^+ \partial_\mu \phi_i - V\left(\phi_i^+\phi_i\right); \qquad i = (1, .., n) \tag{14}$$

The index i is Euclidean - there is no need to distinguish covariant and contravariant components. It is also usually not written explicitly to save time and space. The Lagrangian is obviously invariant with respect to the transformations:

$$\phi \to U\phi, \qquad \phi^+ \to \phi^+ U^+ \tag{15}$$

With U being an unitary matrix, $U^{-1} = U^+$ and $|\mathrm{Det}\, U| = 1$. In other words, the Lagrangian possesses the $U(n)$ symmetry, which is called internal.

$U(n)$ group describes rotations and reflections of a complex n-dimensional vector. Reflections, however, are typically not of the great interest (in principle, it is not very clear why this is so, but the reality is organized in such a way). $SU(n)$ subgroup commonly holds physical significance, it implies:

$$\mathrm{Det}\, U = 1 \tag{16}$$

Regardless of whether reflections are taken into account, the total symmetry is a direct product:

$$[\text{space-time symmetry}] \times [\text{internal symmetries}] \tag{17}$$

Coleman–Mandula theorem states that (in the absence of sypersymmetry) there are no other self-consistent symmetry structures besides that. The flat space-time symmetry is a well-known Poincare group (Lorentz transformations combined with shifts), which can be extended to a conformal group if there are no dimensional parameters in a given theory.

Internal symmetries can be global and local. The local symmetry allows U to be a function of coordinate, i.e. $U = U(x)$. In this case the Lagrangian loses its invariance, but we can restore it by introducing the covariant derivative:

$$\mathcal{L} = D^\mu \phi^+ D_\mu \phi - V\left(\phi^+ \phi\right); \qquad D^\mu = \partial^\mu + igA^\mu \tag{18}$$

Where A^μ is a so-called gauge field (or just Yang-Mills field), necessarily a vector quantity with respect to the Lorentz group and $n \times n$ matrix in the internal space. The prefactor ig is just a convention. Schwartz M.D. uses $-ig$ convention in his book.

We require the transformation law of A^μ to be such that:

$$D_\mu \phi \to U D_\mu \phi, \qquad D^\mu \phi^+ \to D^\mu \phi^+ U^{-1} \tag{19}$$

Which can be achieved if:

$$A^\mu \to A'^\mu = U A^\mu U^{-1} - \frac{i}{g} U \partial^\mu U^{-1} \tag{20}$$

Homework 2 (10 points)

Prove the statement above.

Symmetry groups are compact and thus matrix A^μ can be expanded in a basis of so-called generators t_a:

$$A^\mu = A^\mu_a t_a \tag{21}$$

$U(n)$ matrices have n^2 degrees of freedom and n^2 generators, but in case of $SU(n)$ there are $n^2 - 1$, because an additional condition $\operatorname{Det} U$ is implied. This defines the number of gauge bosons (8 gluons in QCD, etc.). Generators are normalized by the condition:

$$\operatorname{Tr}\{t_a t_b\} = \frac{1}{2}\delta_{ab} \tag{22}$$

The last remaining step is to understand the Lagrangian of the Yang-Mills field, which must obey both space-time and internal symmetry. To achieve this, we can follow an approach analogous to quantum electrodynamics (QED). Specifically, we introduce the field strength tensor, but with an additional non-commutative term:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + \alpha \left[A^\mu, A^\nu\right] \tag{23}$$

By demanding that the Lagrangian is gauge invariant, we can uniquely determine the value of α.

Homework 3 (10 points)

Prove that the field strength tensor constructed in the following way:

$$F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} + ig\left[A^{\mu}, A^{\nu}\right] \tag{24}$$

Transforms as follows:

$$F^{\mu\nu} \to U F^{\mu\nu} U^{+} \tag{25}$$

Homework 4 (10 points)

Let's also define the structure constants f_{bca}:

$$\left[t_b, t_c\right] = i f_{bca} t_a$$

These constant are representation-invariant (accept this fact). Prove that for the field strength tensor $F^{\mu\nu} = F_a^{\mu\nu} t_a$ we have:

$$F_a^{\mu\nu} = \partial^{\mu} A_a^{\nu} - \partial^{\nu} A_a^{\mu} - g f_{abc} A_b^{\mu} A_c^{\nu}$$

Note that in can also be defined as a commutator of derivatives:

$$\left[D^{\mu}, D^{\nu}\right] = ig F^{\mu\nu} \tag{26}$$

The expression of interest then reads:

$$\mathcal{L} = -\frac{1}{2} \operatorname{Tr}\left(F_{\mu\nu} F^{\mu\nu}\right) = -\frac{1}{4} F_{\mu\nu;a} F_a^{\mu\nu} \tag{27}$$

It is known from experiment that the local symmetry of the Standard Model is a product $U(1) \times SU(2) \times SU(3)$. Generally speaking, it is unclear where this comes from and why this is so. Maybe this will be once clarified in some Grand Unification Theory or whatever.

In cases like this, when many internal symmetries are involved, the field lives in many spaces, which are separated from each other:

$$\phi \equiv \phi_{ijk...} \tag{28}$$

And the covariant derivative becomes:

$$D^{\mu} = \partial^{\mu} + ig_1 A_1^{\mu} + ig_2 A_2^{\mu} + ... \tag{29}$$

Generators belonging to different groups commute - they simply live in different spaces and do not feel each other. Similarly, they do not see gamma-matrices as matrices. The Lagrangian:

$$\mathcal{L} = \bar{\psi}\left[i\left(\gamma D\right) - m\right]\psi \tag{30}$$

Is obviously invariant.

Finally, we note that the unitary transformation can be written in the form:

$$U(x) = e^{i\theta(x)} \tag{31}$$

Where $\theta(x)$ is a Hermitian matrix. Of course it can also be decomposed in the basis:

$$\theta(x) = \theta_a(x)\, t_a \tag{32}$$

Such reparameterization means the transition from a group to an algebra.

Lastly, it is worth noting that we initially chose to work with the complex scalar field. However, this choice is not essential. Alternatively, we could consider a real-valued field. In this case the symmetry group would be $O(n)$ instead of $U(n)$, but the line of reasoning remains the same.

III. NOTE ON THE REPRESENTATION THEORY

To be completely precise, $U(n)$ group acting on n-dimensional vectors gives rise to the so-called fundamental representation of the rotation group. There are, in principle, infinitely many representations and different choices of them produce different physics.

In the Standard Model physical fields live in the fundamental representation - that is a God-given (experimental) fact.

IV. WHY GAUGE FIXING IS NECESSARY

Yang-Mills field does not have a free non-interacting limit, the Lagrangian requires third and fourth powers of A^μ to be gauge invariant and cannot be self-consistently described with the second quantization procedure.

However, in perturbative case the path integral can be easily evaluated using the procedure which was described in the first section. We only need to know the determinant of the quadratic part, which is (up to a constant) given by:

$$S^{\mu\nu} = g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu \tag{33}$$

But this operator does not have an inverse. Indeed, the eigenvalues equation reads:

$$S^{\mu\nu}V_\mu = \lambda V^\nu \tag{34}$$

And has a solution:

$$V^\nu = \partial^\nu v \tag{35}$$

Which implies $\lambda = 0$, the Gaussian integral diverges. The underlying reason for this divergence is quite evident: the operator remains undefined unless we apply gauge fixing. Consequently, we need to redefine the space $\mathcal{D}A_\mu$ in a manner that ensures the integration respects the selected gauge condition. This ingenious method was initially introduced by Faddeev and Popov.

V. GAUGE FIXING IN THE PATH INTEGRAL

Consider the path integral over a $SU(N)$-symmetric gauge field:

$$\int e^{iS[A]}\,\mathcal{D}A_\mu; \qquad S = -\frac{1}{4}\int F_a^{\mu\nu}F_{\mu\nu;a}\,d^n q \tag{36}$$

The path integral goes over all possible field configurations, i.e. the integration measure is to be understood as:

$$\mathcal{D}A_\mu = \prod_\mu \prod_a dA_{a,\mu} \tag{37}$$

Just like the notation $d\mathbf{q}$ in fact means $\prod_i dq_i$.

We want to fix the gauge and to narrow down the integration space. The easiest way to do it is to introduce the Faddeev-Popov factor, which is basically an integral over all possible fields and gauges:

$$\int\int \delta\left(A_\mu - A'^U_\mu\right)\delta\left(\partial^\mu A'_\mu\right)\mathcal{D}A'_\mu \mathcal{D}U \equiv \frac{1}{J\left[A_\mu\right]} \tag{38}$$

Where U is the gauge transformation:

$$A'^U_\mu = UA^\mu U^{-1} - \frac{i}{g}U\partial^\mu U^{-1} \tag{39}$$

The first delta-function fixes the Lorenz condition, the second delta function fixes the corresponding gauge transformation U. Note that this integral is gauge invariant. Indeed, let's make a gauge transformation v of A_μ:

$$\frac{1}{J\left[A^V_\mu\right]} = \int\int \delta\left(A^v_\mu - A'^U_\mu\right)\delta\left(\partial^\mu A'_\mu\right)\mathcal{D}A'_\mu \mathcal{D}U = \int\int \delta\left(A_\mu - A'^{V^{-1}U}_\mu\right)\delta\left(\partial^\mu A'_\mu\right)\mathcal{D}A'_\mu \mathcal{D}U \tag{40}$$

Now we can change the variable:

$$U = VU' \tag{41}$$

$$\mathcal{D}U = \mathcal{D}U' \tag{42}$$

The last line is true because $\mathcal{D}U$ integral goes over all possible gauges. $V^{-1}U$ is a product of two gauge transformations, thus it is still a gauge transformations and the integration measure hasn't changed. We get:

$$\frac{1}{J\left[A^V_\mu\right]} = \frac{1}{J\left[A_\mu\right]} \tag{43}$$

Let's insert this unit factor in the definition of the path integral:

$$\int J\left[A\right] \times \int\int \delta\left(\partial A'\right)\delta\left(A - A'^U\right)e^{iS[A]}\,\mathcal{D}A'\mathcal{D}A\mathcal{D}U \tag{44}$$

Evaluate the integral over A using the delta-function:

$$\int\int J\left[A'^U\right] \times \delta\left(\partial A'\right)e^{iS\left[A'^U\right]}\,\mathcal{D}A'\mathcal{D}U \tag{45}$$

But the function J and the action S are both gauge invariant, thus we can get rid of U:

$$\int \mathcal{D}U \int J\left[A'\right] \times \delta\left(\partial A'\right)e^{iS\left[A'\right]}\,\mathcal{D}A' \tag{46}$$

The first factor is a constant (gauge group volume), which can be consumed by the normalization:

$$\int J\left[A\right] \times \delta\left(\partial A\right)e^{iS[A]}\,\mathcal{D}A \tag{47}$$

Where we have changed the variable. Now we just have to calculate the J explicitly.

VI. ADJOINT COVARIANT DERIVATIVE

Let's make some preparatory work which will be very helpful in the upcoming sections. Consider an infinitely small gauge transformation:

$$A_\mu^U = UA_\mu\left(x\right)U^+ - \frac{i}{g}U\partial_\mu U^+ \approx \left(1 + it^b\theta^b\right)\left(t^aA_\mu^a\right)\left(1 - it^b\theta^b\right) + \frac{1}{g}t^a\partial_\mu\theta^a \tag{48}$$

It gives:

$$A_\mu^U \approx t^aA_\mu^a + iA_\mu^a\theta^b \cdot \left(t^bt^a - t^at^b\right) + \frac{1}{g}t^a\partial_\mu\theta^a \tag{49}$$

By the definition of structure constants we have:

$$\left[t^b, t^a\right] = if^{bac}t^c \tag{50}$$

Leading to:

$$A_\mu^U \approx t^aA_\mu^a + i^2A_\mu^a\theta^b \cdot f^{bac}t^c + \frac{1}{g}t^a\partial_\mu\theta^a \tag{51}$$

Rename the indices in the second term to write:

$$A_\mu^U \approx t^a\left(A_\mu^a - A_\mu^c\theta^b \cdot f^{bca} + \frac{1}{g}\partial_\mu\theta^a\right) = A_\mu + \frac{1}{g}t^a\left(-gA_\mu^c\theta^b \cdot f^{bca} + \delta_{ab}\partial_\mu\right)\theta^b \tag{52}$$

And finally:

$$A_\mu^U \approx A_\mu + \frac{1}{g}t^aD_\mu^{ab}\theta^b = A_\mu + \frac{1}{g}D_{r;\mu}\theta \tag{53}$$

Where we introduced an adjoint covariant derivative:

$$D_{r;\mu}^{ab} = \delta_{ab}\partial_\mu - gA_\mu^c \cdot f^{bca} \tag{54}$$

$\bar{c}D_{r;\mu}c$ means the following:

$$\bar{c}_a\left(\delta_{ab}\partial_\mu - gA_\mu^c \cdot f^{bca}\right)c_b \tag{55}$$

VII. SOME PATH INTEGRAL ALGEBRA

We deal with:

$$\int\int \delta\left(A_\mu - A_\mu'^U\right)\delta\left(\partial^\mu A_\mu'\right)\,\mathcal{D}A_\mu'\mathcal{D}U = \frac{1}{J\left[A_\mu\right]} \tag{56}$$

Now assume that the gauge transformation $U = e^{i\theta}$ is infinitely small. Then:

$$A_\mu'^U \approx A_\mu' + \frac{1}{g}D_{\mu;r}\left(A'\right)\theta \tag{57}$$

And we get:

$$\int\int \delta\left(A_\mu - A'_\mu - \frac{1}{g}D_{\mu;r}\left(A'\right)\theta\right)\delta\left(\partial^\mu A'_\mu\right)\mathcal{D}A'_\mu d\theta = \frac{1}{J\left[A_\mu\right]} \tag{58}$$

Note that we have:

$$A_\mu - A'_\mu - \frac{1}{g}D_{\mu;r}\left(A'\right)\theta = 0 \tag{59}$$

We can insert the identity $A'_\mu = A_\mu - \frac{1}{g}D_{\mu;r}\left(A'\right)\theta$ in the definition of $D_{\mu;r}\left(A'\right)\theta$. If we neglect θ^2, then it gives:

$$A_\mu - A'_\mu - \frac{1}{g}D_{\mu;r}\left(A\right)\theta \approx 0 \tag{60}$$

So we write:

$$\int\int \delta\left(A_\mu - A'_\mu - \frac{1}{g}D_{\mu;r}\left(A\right)\theta\right)\delta\left(\partial^\mu A'_\mu\right)\mathcal{D}A'_\mu d\theta = \frac{1}{J\left[A_\mu\right]} \tag{61}$$

Integrating over the field we obtain:

$$\int \delta\left[\partial^\mu\left(D_{\mu;r}\left(A\right)\theta\right)\right]d\theta = \int \delta\left[\left(\delta_{ab}\partial^2 - g\partial^\mu A^c_\mu\cdot f^{bca} - gA^c_\mu\cdot f^{bca}\partial^\mu\right)\theta\right]d\theta = \frac{1}{J\left[A_\mu\right]} \tag{62}$$

Note that the term $\partial^\mu A_\mu$ was dropped (that is the gauge condition). Finally:

$$\frac{1}{J\left[A_\mu\right]} = \left|\frac{1}{\mathrm{Det}\left[\delta_{ab}\partial^2 - g\partial^\mu A^c_\mu\cdot f^{bca} - gA^c_\mu\cdot f^{bca}\partial^\mu\right]}\right| \tag{63}$$

Normally there stands just the modulus:

$$\delta(g(x)) = \sum_k \frac{\delta(x-x_k)}{|g'(x_k)|} \tag{64}$$

But in case if we deal with matrices we get the modulus of the determinant. The modulus itself, however, is unimportant, since the measure of the functional integral is defined up to a certain factor. In other words, we don't care if it is plus or minus - the relevant part is the determinant itself.

VIII. GHOSTS

We can write the determinant in the following form:

$$J\left[A_\mu\right] = \int\int e^{iS_g[\bar{c};c]}\mathcal{D}\bar{c}\mathcal{D}c; \qquad S_g = \int\left(\partial_\mu\bar{c}\right)\times\left[D^\mu_r\left(A\right)\right]c\,d^nq \tag{65}$$

In the last step we used the integration by parts.

c and $\bar{c}$ are so-called ghost and antighost fields. They are scalar fields under Lorentz transformation, but at the same time they are Grassmann numbers (spin-statistic theorem is violated). Note that S_g arises from J, which is gauge invariant, meaning that the ghost term is gauge invariant as well.

The number of ghost fields is obviously equal to the number of generators, not to the order n of the rotation group. Consequently, ghosts reside not in the fundamental representation but rather in what is known as the adjoint representation.

Finally, in case of QED $f^{bca} = 0$ and ghosts are non-interacting (but still present in the theory).
Thus we have in total:

$$\int\int\int \delta\left(\partial A\right) e^{iS[A]+iS_g[\bar{c};c]}\, \mathcal{D}A\mathcal{D}\bar{c}\mathcal{D}c \tag{66}$$

The delta function in this integral fixes the Lorenz gauge condition. Let's now change it to the generalized Lorenz gauge condition:

$$PI = \int\int\int \delta\left(\partial A - \eta\left(q\right)\right) e^{iS[A]+iS_g[\bar{c};c]}\, \mathcal{D}A\mathcal{D}\bar{c}\mathcal{D}c \tag{67}$$

Where $\eta\left(q\right)$ is an arbitrary function. This step is legal, because the integration measure $\mathcal{D}A$ is invariant under the gauge transformation (the path integral anyway goes over all possible fields) and the action $S + S_g$ is invariant as well.

This means that the path integral does not actually depend on the function η. Let's now multiply the equation above with a factor:

$$\exp\left\{-\frac{i}{2\xi}\int \eta^2\left(q\right)\, d^n q\right\} \tag{68}$$

And integrate over $\mathcal{D}\eta$. Physical results do not depend on the particular choice of ξ. The left side gives just a numerical factor, which is unimportant and thus can be consumed by the normalization. On the right side this integration cancels the delta function and gives the well-known gauge-fixing term. We finally obtain:

$$PI = \int\int\int e^{iS[A]+iS_g[\bar{c};c]+iS_a}\, \mathcal{D}A\mathcal{D}\bar{c}\mathcal{D}c; \qquad S_a = -\frac{1}{2\xi}\int \left(\partial A\right)^2 d^n q \tag{69}$$

The quadratic part of the gauge action is now invertible and well-defined. The quadratic part is especially easy to write if we choose $\xi = 1$:

$$\propto A_\mu \partial^2 A^\mu \tag{70}$$

Now using this we can develop the diagram method and build the perturbation theory.

Homework 5 (35 points)

Perform the Faddeev-Popov quantization in the gauge $A^3 = 0$ (Arnowitt-Fickler) and prove that ghosts are non-interacting.

Homework 6* (25 bonus points)

Prove Furry's theorem, which states that in QED an arbitrary diagram with an odd number of photons is equal to zero, i.e:

$$\langle 0|T\left[\hat{A}\left(q_1\right)...\hat{A}\left(q_{2n+1}\right)\right]|0\rangle = 0 \tag{71}$$

Give a comment on the non-abelian case.

Hint: one of the easiest ways to do it is to note that the determinant of the matter field is an even function of the photon field.

IX. GRIBOV AMBIGUITY

In the previous sections we learned how to fix the gauge and integrate only over the part of the $\mathcal{D}A$ which corresponds to it. However, an important question arises: Is the gauge condition truly unique? Could there be multiple field configurations that satisfy it? If so, does this imply that we might overcount when calculating the path integral?

Vladimir Gribov was the first to raise this question. He demonstrated that while this issue does not impact perturbative calculations, it becomes crucial in the non-perturbative regime. To address the potential overcounting, Gribov proposed 2 possible resolutions:

1) Restricting the path integral to what is known as the first Gribov region - the region containing the first possible copy of the gauge field configuration. However, it remains unclear how to evaluate the path integral with such boundary conditions.

2) Use specific gauges, which are uniquely defined and do not have copies. This approach is more rigorous, but much more complicated technically and also remains underexplored.

X. GRAVITY

Gravity can be understood as a gauge theory, but one that is constructed around local coordinate transformations (referred to as diffeomorphisms) rather than internal spaces. This concept can be explained using the same language as Yang-Mills theories, which involve concepts like covariant derivatives and other tools - just make the Lorentz transformations local.

The Christoffel symbols $\Gamma^{\alpha}_{\mu\nu}$ play the role of a gauge field:

$$\nabla_\mu T_\alpha = \partial_\mu T_\alpha - \Gamma^{\beta}_{\mu\alpha} T_\beta \tag{72}$$

Which is not Lorentz/gauge invariant by itself, but can be used to construct the manifestly invariant object. The Riemann tensor $R^{\mu}{}_{\nu\alpha\beta}$ is an analogue of the field strength tensor:

$$[\nabla_\alpha \nabla_\beta]^{\mu}_{\nu} = R^{\mu}{}_{\nu\alpha\beta} = \partial_\alpha \Gamma^{\mu}_{\beta\nu} - \partial_\beta \Gamma^{\mu}_{\alpha\nu} + \Gamma^{\mu}_{\alpha\sigma}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\mu}_{\beta\sigma}\Gamma^{\sigma}_{\alpha\nu} \tag{73}$$

The path integral has the most natural and convenient form if written in terms of metrics:

$$\int \mathcal{D}g_{\mu\nu} g^{-5/2} \exp\{iS_g\}; \qquad S_g = \int R\sqrt{g}\, d^4q \tag{74}$$

Just like before, the gauge must be fixed. The most commonly used gauge is the de Donder condition:

$$\partial_\nu \left(\underbrace{\sqrt{-g} g^{\mu\nu}}_{h^{\mu\nu}} \right) = 0 \tag{75}$$

Homework 7* (100 bonus points)

Perform the Faddeev-Popov procedure and prove the statements below.

1) The de Donder condition gives rise to the ghost term of the form:

$$\int \exp\left\{ i \int \bar{\theta}_\mu A^{\mu\nu} \theta_\nu \, d^4 q \right\} \mathcal{D}\bar{\theta}_\mu \mathcal{D}\theta_\nu \tag{76}$$

Where denoted:

$$A^{\mu\nu}\theta_\nu = \partial_\nu \left(h^{\nu\lambda} \partial_\lambda \theta^\mu \right) - \partial_\lambda \left(\partial_\nu h^{\mu\nu} \theta^\lambda \right) \tag{77}$$

I.e. ghosts are represented by some anticommuting vectors.

2) The gauge-fixing term can be written as:

$$\exp\left\{ \frac{i}{4} \int \partial_\alpha h^{\mu\alpha} \eta_{\mu\nu} \partial_\beta h^{\nu\beta} \, d^4 x \right\} \tag{78}$$

With $\eta_{\mu\nu}$ being Minkowski tensor.

Hint: N.P. Konopleva, V.P. Popov: Gauge Fields.
L. D. Landau, E. M. Lifshitz: The Classical Theory of Fields.