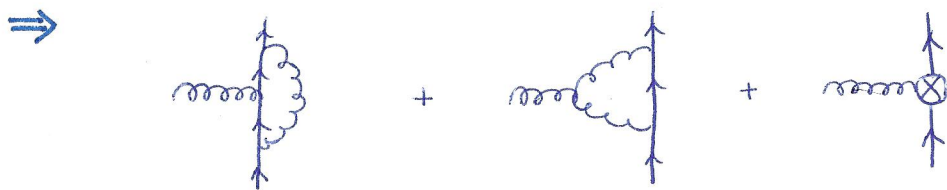


QCD RUNNING COUPLING CONSTANT AND ASYMPTOTIC FREEDOM



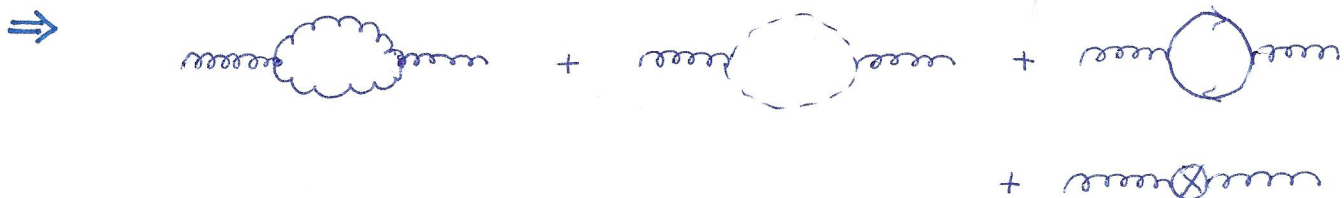
↳
$$Z_2 = 1 - \frac{g^2}{(4\pi)^2} \cdot \frac{1}{\epsilon} \cdot \frac{4}{3} \zeta + O(g^4)$$



$$Z_{1F} = Z_g Z_2 Z_3^{1/2}$$

↳
$$Z_{1F} = 1 - \frac{g^2}{(4\pi)^2} \cdot \frac{1}{\epsilon} \cdot \left[\frac{g}{4} + \frac{25}{12} \zeta \right] + O(g^4)$$

$\frac{13}{2}$ FOR $\zeta = 1$



↳
$$Z_3 = 1 - \frac{g^2}{(4\pi)^2} \cdot \frac{1}{\epsilon} \cdot \left\{ \left(-\frac{13}{2} + \frac{2}{3} N_f \right) + \frac{3}{2} \zeta \right\} + O(g^4)$$

$$\Rightarrow Z_g = Z_{1F} Z_2^{-1} Z_3^{-1/2}$$

$$Z_g = \left[1 - \frac{g^2}{(4\pi)^2} \cdot \frac{1}{\epsilon} \left(\frac{g}{4} + \frac{25}{12} \zeta \right) + O(g^4) \right]$$

$$\cdot \left[1 + \frac{g^2}{(4\pi)^2} \cdot \frac{1}{\epsilon} \left(\frac{4}{3} \zeta \right) + O(g^4) \right]$$

$$\cdot \left[1 + \frac{g^2}{(4\pi)^2} \cdot \frac{1}{\epsilon} \left(-\frac{13}{4} + \frac{1}{3} N_f + \frac{3}{4} \zeta \right) + O(g^4) \right]$$

$$= 1 - \frac{g^2}{(4\pi)^2} \cdot \frac{1}{\epsilon} \left(\frac{11}{2} - \frac{1}{3} N_f \right) + O(g^4)$$

↓

NOTE THAT THE ζ DEPENDENCE

DROPS OUT !



CONSEQUENCE OF GAUGE INVARIANCE

$$Z_g = 1 - \frac{g^2}{(4\pi)^2} \cdot \frac{1}{\epsilon} \left(\frac{11}{2} - \frac{1}{3} N_f \right) + O(g^4)$$

$$g_B = Z_g g$$

$$\Rightarrow g_B = Z_g \cdot g$$

IN D DIMENSIONS

$$[L_{ggg}] = D$$

$\int d^D x \mathcal{L}$ IS DIMENSIONLESS

↳ MASS DIMENSION



$$\left\{ \begin{aligned} [g_B \bar{\psi} \gamma^\mu A_{a\mu} \psi] &= D \\ [g] &= \frac{D-1}{2} \\ [A_a^\mu] &= \frac{D-2}{2} \end{aligned} \right.$$

$$\rightarrow [g_B] = 2 - \frac{D}{2} = \epsilon$$

∴ IN D - DIMENSIONS

$$g_B = \mu^\epsilon Z_g g$$

μ : MASS SCALE

BARE (UNPHYSICAL)
COUPLING CONSTANT
IS IN DEPENDENT OF μ

RENORMALIZED COUPLING CONSTANT
||
PHYSICAL COUPLING CONSTANT
AT MASS SCALE μ

⇒ β - FUNCTION

↳ DESCRIBES HOW THE PHYSICAL (RENORMALIZED) COUPLING CONSTANT VARIES WITH MASS SCALE

$$\beta(g) \equiv + \mu \frac{\partial g}{\partial \mu} = \frac{\partial g}{\partial \ln \mu}$$

$$g_B = \mu^\epsilon \cdot Z_g \cdot g$$

↓
TAKE DERIVATIVE OF BOTH SIDES W.R.T. $\ln \mu$

$$\mu \frac{\partial}{\partial \mu}$$

$$0 = \epsilon \mu^\epsilon Z_g g + \mu^\epsilon \left(\mu \frac{\partial Z_g}{\partial \mu} \right) g + \mu^\epsilon Z_g \left(\mu \frac{\partial g}{\partial \mu} \right)$$

USE $Z_g = 1 - \frac{g^2}{(4\pi)^2} \frac{1}{2\epsilon} \beta_0$

WITH $\beta_0 \equiv 11 - \frac{2}{3} N_f$

$$\mu \frac{\partial Z_g}{\partial \mu} = - \frac{1}{\epsilon} \frac{\beta_0}{(4\pi)^2} g \cdot \underbrace{\left(\mu \frac{\partial g}{\partial \mu} \right)}_{\beta(g)}$$

⇓

$$0 = \mu^\epsilon Z_g \left\{ \epsilon g + \frac{g}{Z_g} \mu \frac{\partial Z_g}{\partial \mu} + \beta(g) \right\}$$

⇓

$$0 = \epsilon g + \frac{g}{Z_g} \left(-\frac{1}{\epsilon} \frac{\beta_0}{(4\pi)^2} g \beta(g) \right) + \beta(g)$$

$$-\epsilon g = \beta(g) \left[1 - \frac{1}{\epsilon} \frac{\beta_0}{(4\pi)^2} \frac{g^2}{Z_g} \right]$$

⇓ TO ORDER g^2

$$-\epsilon g = \beta(g) \left[1 - \frac{1}{\epsilon} \frac{\beta_0}{(4\pi)^2} g^2 + O(g^4) \right]$$

⇓

$$\beta(g) = -\epsilon g \left[1 + \frac{1}{\epsilon} \frac{\beta_0}{(4\pi)^2} g^2 + O(g^4) \right]$$

⇓ LIMIT $\epsilon \rightarrow 0$

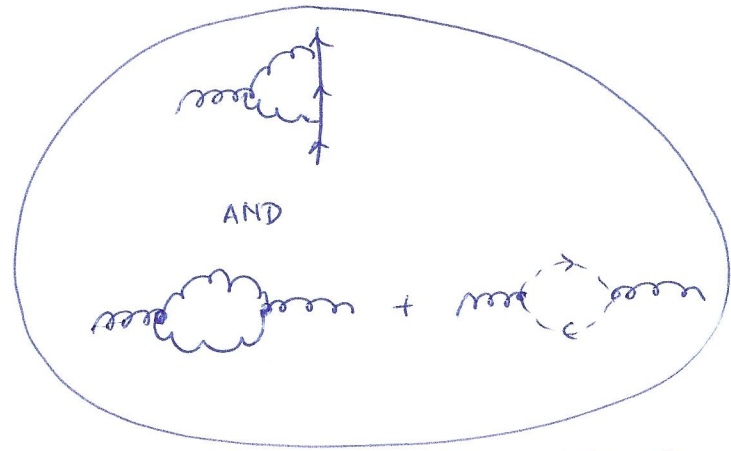
$$\beta(g) = -\beta_0 \frac{1}{(4\pi)^2} g^3$$

⇒ ASYMPTOTIC FREEDOM

$$\beta(g) \equiv \mu \frac{\partial g}{\partial \mu} = -\beta_0 \frac{1}{(4\pi)^2} \cdot g^3$$

$$\beta_0 = 11 - \frac{2}{3} N_f$$

+ SIGN GIVEN BY



FROM

cf. QED
POLARIZATION OF
VACUUM

DUE TO NON-ABELIAN NATURE OF
THEORY

$$\beta_0 > 0 \quad \text{FOR} \quad N_f < \frac{33}{2}$$

IN NATURE $N_f = 6$ (u, d, s, c, b, t)

$$\beta_0 > 0 \quad (\text{QCD})$$

COUPLING CONSTANT DECREASES AT $\mu \uparrow$, i.e. AT
SHORT DISTANCES QUARKS BEHAVE AS FREE PARTICLES

ASYMPTOTIC FREEDOM (POLITZER, GROSS, WILCZEK)
1973



↳ IN QED WE HAVE AN OPPOSITE BEHAVIOR

N_f TERM DOMINATES



$\beta_0 < 0$ (QED)

COUPLING CONSTANT INCREASES AT SHORT DISTANCES.

(" INFRARED SLAVERY ")

⇒ RUNNING COUPLING CONSTANT IN QCD

$$\beta(g) = \mu \frac{\partial g}{\partial \mu} = -\beta_0 \frac{1}{(4\pi)^2} g^3$$

↓

$$\frac{dg}{g^3} = -\frac{\beta_0}{(4\pi)^2} d \ln \mu$$

$$d\left(-\frac{1}{2} \frac{1}{g^2}\right) = -\frac{\beta_0}{(4\pi)^2} d \ln \mu$$

INTEGRATE BETWEEN MASS SCALE μ_1 & MASS SCALE μ_2

$$-\frac{1}{2} \frac{1}{g^2(\mu_2)} + \frac{1}{2} \frac{1}{g^2(\mu_1)} = -\frac{\beta_0}{(4\pi)^2} \ln \frac{\mu_2}{\mu_1}$$

$$\underbrace{\frac{1}{g^2(\mu_1)} - \frac{\beta_0}{(4\pi)^2} \ln \mu_1^2}_{\text{DEPENDS ONLY ON } \mu_1} = \underbrace{\frac{1}{g^2(\mu_2)} - \frac{\beta_0}{(4\pi)^2} \ln \mu_2^2}_{\text{DEPENDS ONLY ON } \mu_2}$$

DEPENDS ONLY
ON μ_1

DEPENDS ONLY
ON μ_2

↓ HENCE BOTH SIDES
HAVE TO BE EQUAL
TO A CONSTANT

$$\equiv -\frac{\beta_0}{(4\pi)^2} \ln \Lambda_{\text{QCD}}^2$$

|| Λ_{QCD} IS FUNDAMENTAL PARAMETER
(MASS SCALE) IN QCD

AT SOME MASS SCALE Q

$$\frac{1}{g^2(Q)} - \frac{\beta_0}{(4\pi)^2} \ln Q^2 = - \frac{\beta_0}{(4\pi)^2} \ln \Lambda_{QCD}^2$$

↓

$$\frac{1}{g^2(Q)} = + \frac{\beta_0}{(4\pi)^2} \ln \left(\frac{Q^2}{\Lambda_{QCD}^2} \right)$$

DEFINE

$$\alpha_s(Q) \equiv \frac{g^2(Q^2)}{4\pi}$$

$$= \frac{4\pi}{\beta_0 \ln(Q^2 / \Lambda_{QCD}^2)}$$

QCD RUNNING
COUPLING CONSTANT

EXPERIMENT $\Lambda_{QCD} \approx 0.2 \text{ GeV}$

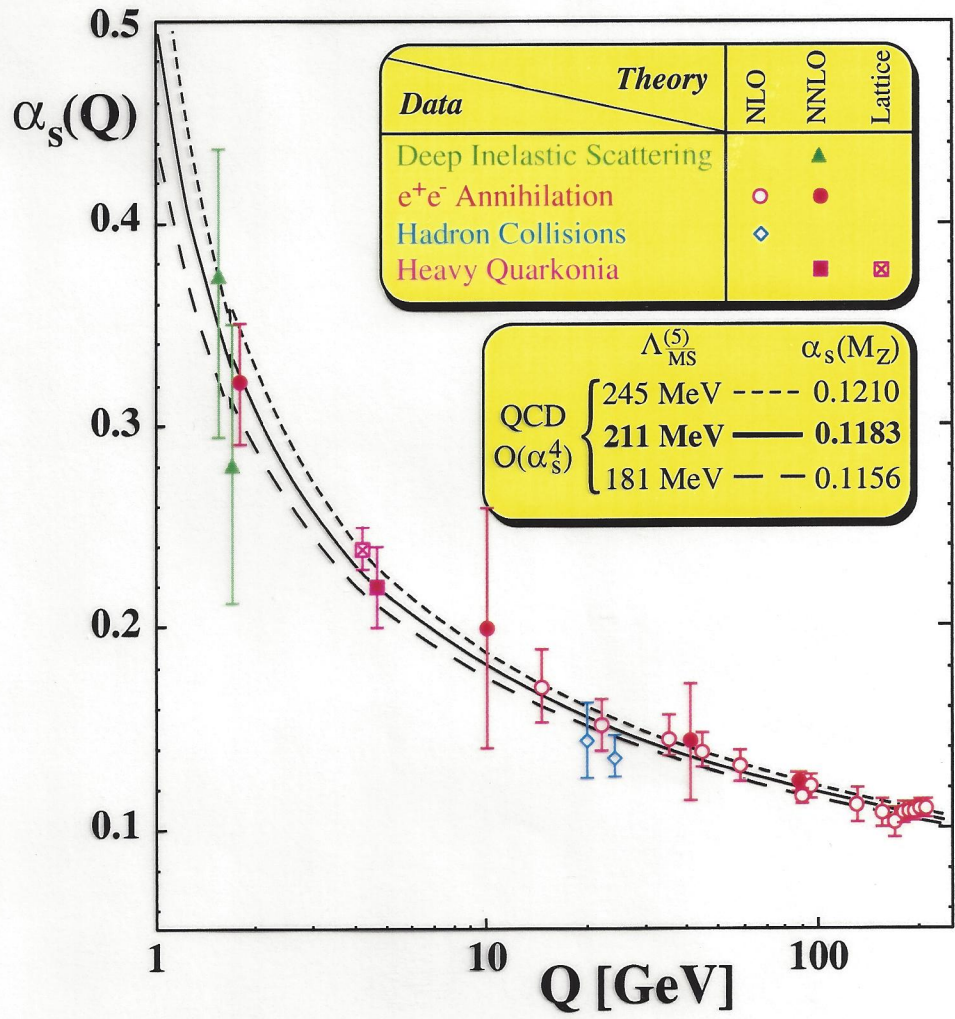
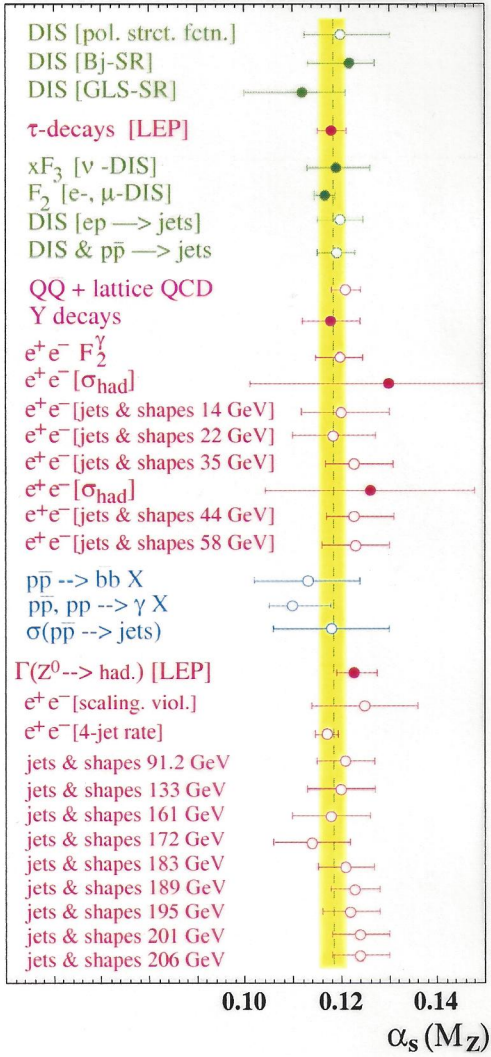
→ $Q \uparrow \Rightarrow \alpha_s \downarrow$

AT LARGE SCALES
↓
PERTURBATION THEORY

→ AT SMALL SCALES (LARGE DISTANCES) $\alpha_s \uparrow$

WHEN $Q = \Lambda_{QCD} \rightarrow$ FORMULA HAS POLE (LANDAU POLE)
 α_s DIVERGES

QCD : running coupling constant



⇒ RUNNING COUPLING CONSTANT IN QED

• β -FUNCTION IN QED

$$e_B = \mu^\epsilon Z_e e$$



VACUUM POLARIZATION

$$\beta(e) \equiv \mu \frac{\partial e}{\partial \mu}$$

$$Z_e = 1 + \frac{e^2}{(4\pi)^2} \frac{1}{2\epsilon} \cdot \frac{4}{3}$$

$$\beta(e) = \frac{e^3}{(4\pi)^2} \cdot \frac{4}{3}$$

> 0 INCREASES FOR SHORTER DISTANCES

$$\mu \frac{\partial e}{\partial \mu} = \frac{e^3}{(4\pi)^2} \cdot \frac{4}{3}$$



$$-\frac{1}{2} d\left(\frac{1}{e^2}\right) = d\left(\frac{4}{3} \cdot \frac{1}{(4\pi)^2} \ln \mu\right)$$



INTEGRATE BETWEEN LOW MASS SCALE m (MASS OF e^-) AND LARGE MASS SCALE Q

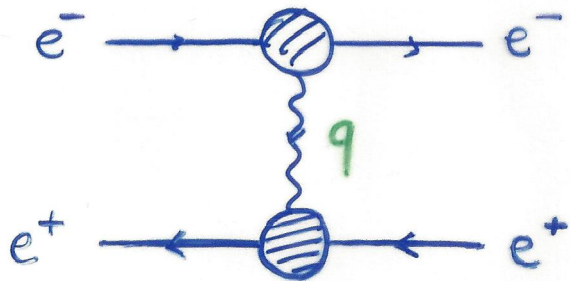
$$\frac{1}{e^2(Q^2)} - \frac{1}{e^2(m^2)} = -\frac{4}{3} \frac{1}{(4\pi)^2} \ln \frac{Q^2}{m^2}$$

$$\alpha_{em}(Q^2) \equiv \frac{e^2(Q^2)}{4\pi} = \alpha_{em}(m^2) \left[1 + \frac{4}{3} \frac{\alpha_{em}}{4\pi} \ln \frac{Q^2}{m^2} \right]$$

↳ FINE STRUCTURE 'CONSTANT'

$$\frac{1}{137}$$

QED COUPLING CONSTANT



$$q^2 < 0$$

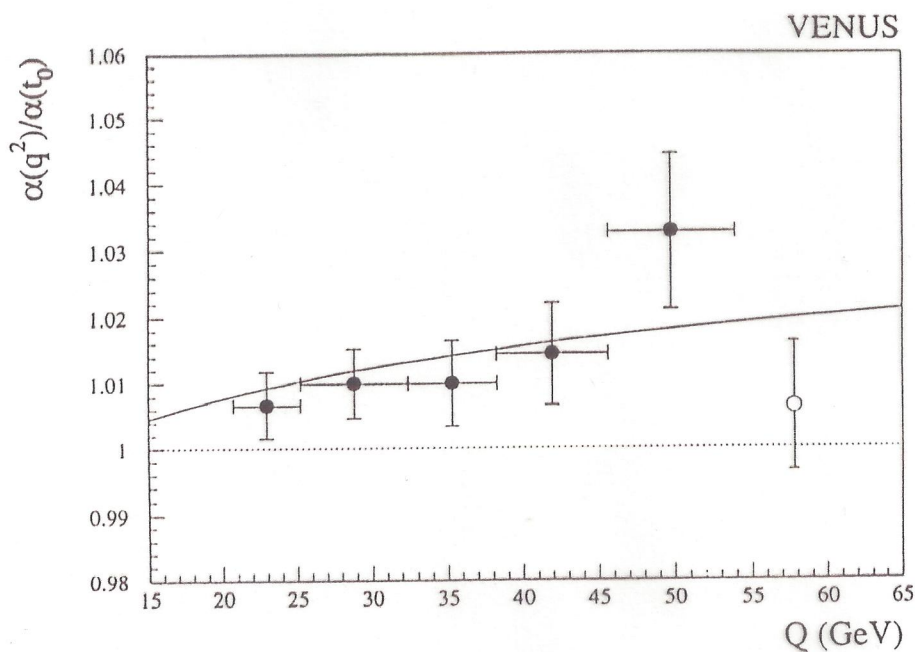
$$q^2 \equiv -Q^2$$

BHABHA SCATTERING



$$\alpha_{em}(Q^2) = \frac{e^2(Q^2)}{4\pi}$$

TRISTAN
e⁺e⁻ COLLIDER (KEK)



t_0 : LOW SCALE

$$\alpha(0) \approx \frac{1}{137}$$