Practice Exam
Theoretical Physics 3: QM WS2023/2024

31.01.2024

Exercise 1. General questions (25 points + 10 bonus)

1.1. *(5 p.)* Consider the ground state wave function of the harmonic oscillator in spatial representation

$$\langle x|\psi_0\rangle = A_0\, e^{-\frac{m\omega}{2\hbar}x^2}.$$

Recall

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\, e^{\frac{i}{\hbar}px} \quad \text{and} \quad \int_{-\infty}^{+\infty} e^{-x^2}\mathrm{d}x = \sqrt{\pi}.$$

Compute $\langle p|\psi_0\rangle$.

1.2. *(7 p.)* Consider an operator $\hat{T}(a) \equiv e^{\frac{ia}{\hbar}\hat{p}}$, where $\hat{p}$ is the momentum operator and a is a real parameter.

a) *(2 p.)* Is it an observable? Why?

b) *(5 p.)* Show that $\hat{T}(a)\,\psi(x) = \psi(x+a)$.

1.3. *(7 p.)* Assume $\hat{H}$ is the *time-independent* Hamiltonian.

a) *(2 p.)* Show that the operator $\hat{U}(t-t_0) \equiv e^{-\frac{i}{\hbar}(t-t_0)\hat{H}}$ is unitary.

b) *(5 p.)* Show that the solution to the time-dependent Schrödinger equation is

$$\Psi(x,t) = \hat{U}(t-t_0)\Psi(x,t_0),$$

with $\Psi(x,t_0)$ being a given wave function of the system at time t_0.

1.4. *(6 p.)* Consider two observables $\hat{A}$ and $\hat{B}$.
$\hat{A}$ has two normalized eigenstates $|a_1\rangle$ and $|a_2\rangle$, with eigenvalues a_1 and a_2, respectively.
$\hat{B}$ has two normalized eigenstates $|b_1\rangle$ and $|b_2\rangle$, with eigenvalues b_1 and b_2, respectively.
Assume the eigenstates are related by

$$|a_1\rangle = \frac{3}{5}|b_1\rangle + \frac{4}{5}|b_2\rangle\,, \qquad |a_2\rangle = \frac{4}{5}|b_1\rangle - \frac{3}{5}|b_2\rangle\,.$$

a) *(1 p.)* The observable $\hat{A}$ is measured, and the value a_1 is obtained. What is the state of the system (immediately) after this measurement?

b) *(2 p.)* If afterwards $\hat{B}$ is measured, what are the possible outcomes, and what are their probabilities?

c) *(3 p.)* Right after $\hat{B}$ is measured, $\hat{A}$ is measured again. What is the probability of getting a_1?

1.5. *(Bonus 10 p.) Eigenfunctions and degeneracy.*

a) *(2 p.)* What is the degree of degeneracy for the energy of a one-dimensional free particle?

b) *(3 p.)* Is the ground state of an infinite square well an eigenfunction of momentum? If so, what is its momentum? If not, why not?

c) *(5 p.)* Using the Schrödinger equation, prove that in one dimension there are no degenerate bound states.

Exercise 2 (25 + 5 points). Radial wave function of the Yukawa potential.

A 3-dimensional realistic model for the strong interaction which holds nucleons (protons and neutrons) together within an atomic nucleus is the attractive central potential of the Yukawa type (after H. Yukawa who invented it):

$$V(r) = -V_0 \frac{e^{-r/a}}{r/a}, \tag{1}$$

where $V_0 > 0$ denotes the strength of the interaction, and where the parameter a (of dimension length) is a measure of the range of the interaction (typically of the order 1 - 2 fm, with 1 fm $= 10^{-15}$ m). Consider two nucleons moving around each other in a state of angular momentum $l = 0$. The wave function describing the relative motion of both nucleons satisfies the Schroedinger equation with mass parameter M, which is the reduced mass of the two-nucleon system. To determine the radial wave function, use the following trial function:

$$R(r) = Ce^{-\alpha r}, \tag{2}$$

where the parameters C and α are to be determined in the following.

a) *(5 p.)* Determine the parameter C (as function of α) from the normalization condition on R.

b) *(10 p.)* Show that the average kinetic energy in the above state is given by

$$\langle T \rangle = \frac{\hbar^2}{2M}\alpha^2. \tag{3}$$

Hint: the Laplacian in spherical coordinates is given by

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}. \tag{4}$$

c) *(5 p.)* Show that the average potential energy in the above state is given by

$$\langle V \rangle = -V_0 \frac{4a^3\alpha^3}{(1+2a\alpha)^2}. \tag{5}$$

d) *(5 p.)* The ground state minimizes the total energy. To exploit this fact you are required to do the following steps.
First, introduce the dimensionless variable $p \equiv 2a\alpha$, and express the average total energy in terms of p instead of α.
Then assume that p_0 is the value of the parameter p which minimize the average total energy. Determine the resulting condition for the ground state and express a^2 as function of p_0.
Finally, express the ground state energy as function of *only* V_0 and p_0.

e) *(Bonus 5 p.)* What is the condition on p_0 in order to have a bound state? What does this condition physically means in terms of a (range of the potential) and $1/\alpha$ (range of the wave function)?

Exercise 3. Half-harmonic oscillator (25 points + 5 bonus)

Consider a particle of mass m, which is moving in one dimension in a "half"-harmonic potential $V(x)$

$$V(x) = \begin{cases} \infty, & x < 0; \\ \frac{1}{2}m\omega^2 x^2, & x \geq 0. \end{cases}$$

a) *(5 p.)* Write down the *stationary* Schrödinger equation for $x \geq 0$ using the dimensionless quantities

$$y = \sqrt{\frac{m\omega}{\hbar}}x \quad \text{and} \quad \varepsilon = \frac{E}{\hbar\omega}.$$

b) *(5 p.)* Show that the asymptotic behavior of the solution for large y is given by $e^{-y^2/2}$.

c) *(5 p.)* By separating the asymptotic behavior for $y \to \infty$, we define

$$\psi(y) = h(y)e^{-y^2/2}.$$

Derive the equation for $h(y)$ for $y \geq 0$.

d) *(10 p.)* We know that for the *regular* quantum harmonic oscillator the eigenfunctions of the Hamiltonian are expressed in terms of the Hermite polynomials:

$$\psi_n(y) \propto H_n(y)e^{-y^2/2}, \quad n = 0, 1, 2, \dots \,,$$

where the Hermite polynomials $H_n(y)$ satisfy the differential equation

$$H_n''(y) - 2yH_n'(y) + 2nH_n(y) = 0, \quad n = 0, 1, 2, \dots \,,$$

and can equivalently be defined as

$$H_n(y) = (-1)^n e^{y^2} \frac{\partial^n}{\partial y^n} e^{-y^2}.$$

Deduce the spectrum in the case of the given *"half"*-harmonic potential.

e) *(Bonus 5 p.)* The Hermite polynomials are normalised as

$$\int_{-\infty}^{\infty} dy\, H_n(y)H_m(y)e^{-y^2} = 2^n n! \sqrt{\pi}\delta_{nm}.$$

What are the normalised *ground* state and *first excited* state wave functions of the given *"half"*-harmonic potential?

Exercise 4. Hyperfine splitting (25 points)

The electron and proton are both magnetic dipoles with magnetic moments

$$\begin{aligned} \vec{\mu}_e &= -g_e \frac{e}{2m_e}\vec{\mathbf{s}}_e \,, \quad g_e = 2 \,, \\ \vec{\mu}_p &= +g_p \frac{e}{2m_p}\vec{\mathbf{s}}_p \,, \quad g_p = 5.59 \,. \end{aligned}$$

As in classical electromagnetism, the magnetic field due to a point magnetic dipole of the proton is

$$\vec{\mathbf{B}}_p = \frac{\mu_0}{4\pi r^3}\left[3(\vec{\mu}_p \cdot \vec{\mathbf{e}}_r)\vec{\mathbf{e}}_r - \vec{\mu}_p\right] + \frac{2\mu_0}{3}\delta^{(3)}(\vec{\mathbf{r}})\vec{\mu}_p \,,$$

where $\vec{\mathbf{e}}_r = \vec{\mathbf{r}}/r$.

The electron magnetic dipole moment $\vec{\mu}_e$ interacts with the magnetic field generated by the proton. The corresponding contribution to the Hamiltonian is

$$\hat{H} = -\vec{\mu}_e \cdot \vec{\mathbf{B}}_p \,.$$

Consider $l = 0$ state of Hydrogen, so that the coupling between the proton and the magnetic field generated by the electron's orbital motion is absent. In this case the Hamiltonian

$$\hat{H} = \frac{\mu_0}{8\pi} \frac{g_p e^2}{m_p m_e} \frac{1}{r^3} \left[3(\vec{\mathbf{s}}_p \cdot \vec{\mathbf{e}}_r)(\vec{\mathbf{s}}_e \cdot \vec{\mathbf{e}}_r) - \vec{\mathbf{s}}_p \cdot \vec{\mathbf{s}}_e\right] + \frac{\mu_0 g_p e^2}{3 m_p m_e} \vec{\mathbf{s}}_p \cdot \vec{\mathbf{s}}_e \delta^{(3)}(\vec{\mathbf{r}}) \,,$$

fully describes the magnetic dipole interaction of the proton and electron.

According to the first order perturbation theory, the corresponding energy shift is

$$E = \langle \hat{H} \rangle = E_r + E_c \,,$$

where E_r and E_c are regular and contact contributions corresponding to the first and second terms in Hamiltonian respectively.

a) *(2 p.)* Explain why E_r vanishes for $l = 0$ state.

b) *(7 p.)* Show that the energy shift E_c in first order perturbation theory, is given by

$$E_c = \frac{\mu_0 e^2 g_p}{3 m_e m_p} \frac{1}{2} (\vec{s}^2 - \vec{s}_e^2 - \vec{s}_p^2) |\psi_{n00}(0)|^2,$$

where $\vec{\mathbf{s}} = \vec{\mathbf{s}}_e + \vec{\mathbf{s}}_p$ and ψ is the hydrogen spatial wave function.

c) *(2 p.)* Consider the ground state. How many levels are there? What is the degeneracy of each level?

d) *(7 p.)* Derive the energy shifts for both singlet and triplet ground states. Find the numerical value of the hyperfine splitting between them and the corresponding wavelength.

e) *(7 p.)* Compare the magnitudes of the hyperfine splitting of the ground state and the fine splitting of the $n = 2$ states. What is the main reason that the hyperfine splitting is smaller than the fine splitting? Give an order of magnitude estimate of their ratio.