Exercise sheet 9
Theoretical Physics 3: QM WS2023/2024

10.01.2024

Exercise 0.

How much time did you take to complete this homework sheet?

Exercise 1. Eigenfunctions of angular momentum operators (45 points)

During the last lecture of Chapter 4 you showed that angular momentum in spherical coordinates is written as

$$\vec{L} = -i\hbar \left(\hat{e}_\phi \frac{\partial}{\partial \theta} - \hat{e}_\theta \frac{1}{\sin\theta} \frac{\partial}{\partial \phi} \right).$$

Starting from this we are going to show that the functions we first denoted as f_l^m correspond to the spherical harmonics.

a) *(10 p.)* Express the unit vectors with respect to Cartesian coordinates $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ and read off $\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$.

b) *(10 p.)* Construct L^2 from the above.

c) *(25 p.)* Write down the differential equations that correspond to the eigenvalue equations

$$\hat{L}_z f_l^m(\theta,\phi) = \hbar m f_l^m(\theta,\phi)$$
$$\hat{L}^2 f_l^m(\theta,\phi) = \hbar^2 l(l+1) f_l^m(\theta,\phi)$$

and then show that $f_l^m(\theta,\phi) \equiv Y_{lm}(\theta,\phi)$.
Hint: Use $f_l^m(\theta,\phi) = g(\theta)h(\phi)$. Solve to find $h(\phi)$ and show that $g(\theta)$ satisfies the following equation,

$$\sin\theta \frac{d}{d\theta}\left(\sin\theta \frac{dg}{d\theta}\right) + [l(l+1)\sin^2\theta - m^2]g = 0.$$

That is the exact same equation that the θ-dependent part of $Y_{lm}(\theta,\phi)$ needed to satisfy, so your job for that is done.

Exercise 2. Angular momentum operator (55 points)

a) *(15 p.)* Show that $L_\pm Y_l^m = \hbar\sqrt{l(l+1) - m(m\pm 1)}\, Y_l^{m\pm 1}$.
Hint: consider the norm of $L_\pm Y_l^m$.

b) *(15 p.)* Show that for eigenfunctions of $\hat{L}_z$, we have

$$\langle \hat{L}_x \rangle = \langle \hat{L}_y \rangle = 0; \quad \langle \hat{L}_x^2 \rangle = \langle \hat{L}_y^2 \rangle; \quad \langle \hat{L}_x \hat{L}_y + \hat{L}_y \hat{L}_x \rangle = 0$$

Hint: consider $\langle \hat{L}_\pm \rangle$ and $\langle \hat{L}_\pm^2 \rangle$.

c) *(25 p.)* In the state ψ_{lm} with definite angular momentum l and its z-component m, find the mean values $\langle \hat{L}_x^2 \rangle$, $\langle \hat{L}_y^2 \rangle$ as well as the mean values $\langle \hat{L}_{\tilde{z}} \rangle$ and $\langle \hat{L}_{\tilde{z}}^2 \rangle$ of the angular momentum projection along the $\tilde{z}$-axis making an angle α with the z-axis.
Hint: use $\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 = \hat{L}^2$.

(Bonus) Exercise 3. Degenerate quantum numbers (15 points)

Prove or disprove the following statements:

a) *(10 p.)* If $[\hat{H}, \hat{\vec{L}}] = \vec{0}$, the energy levels do not depend on m (i.e. on the eigenvalues of the projection of one of the components of the angular momentum $\hat{\vec{L}}$).

b) *(5 p.)* If $[\hat{H}, \hat{L}^2] = 0$, the energy levels do not depend on l.