Exercise sheet 10
Theoretical Physics 3: QM WS2023/2024

17.01.2024

Exercise 0.

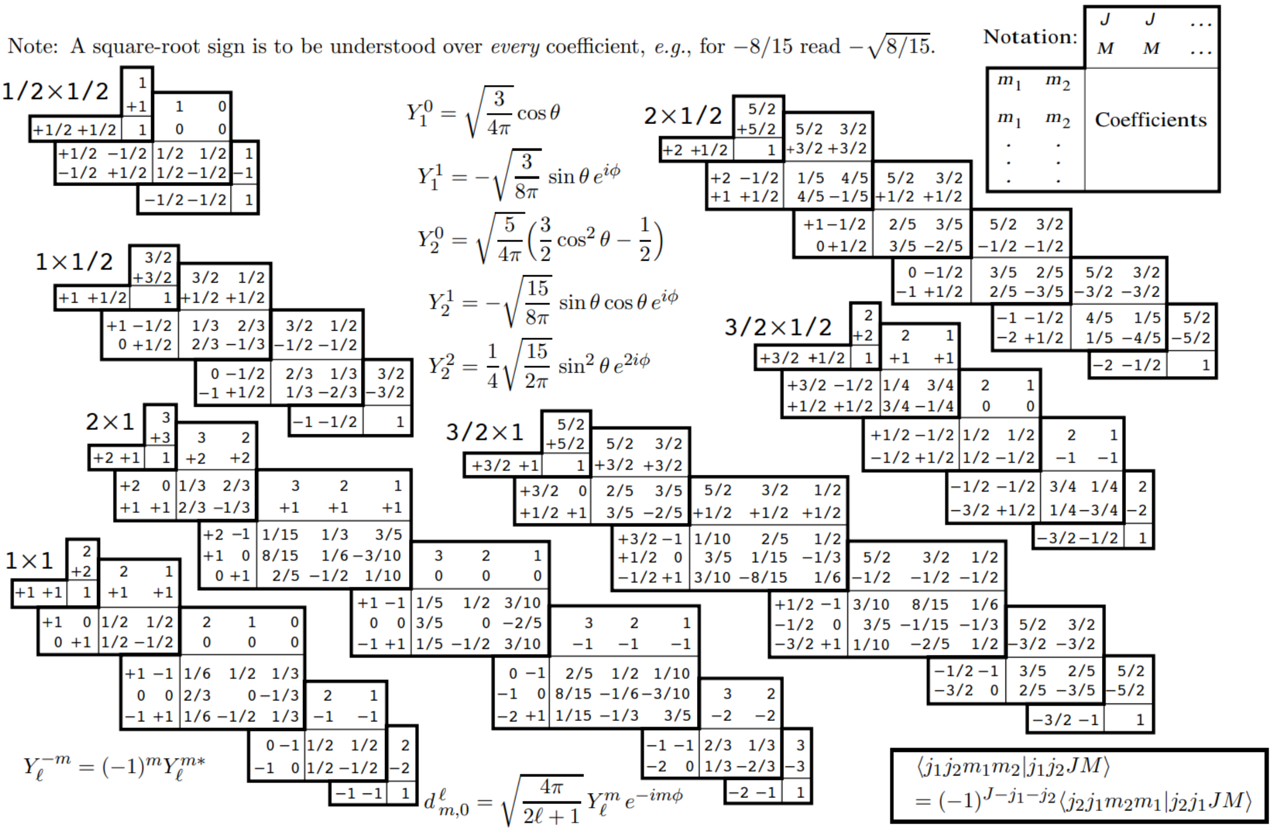
How much time did you take to complete this homework sheet?

Exercise 1. Clebsch-Gordan coefficients (45 points)

In this exercise we will practice how to couple two angular momenta j_1 and j_2, using the Clebsch-Gordan Table.
Recall that the coupled states which are characterized by the total angular momentum J and its projection M can be expanded via the completeness relation in the uncoupled basis:

$$|JM\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1 m_1\rangle|j_2 m_2\rangle\langle j_1 m_1|\langle j_2 m_2|JM\rangle$$

The expansion coefficients, $(\langle j_1 m_1|\langle j_2 m_2|)\,|JM\rangle$, are the Clebsch-Gordan coefficients which can be found in the following table:



a) *(25 p.)* Write down all the possible states $|JM\rangle$ in the basis $|j_1 m_1\rangle\,|j_2 m_2\rangle$ for the compositions $\frac{1}{2}\otimes 1$ and $1\otimes 1$ (the symbol $\otimes$ stands for the coupling of two angular momenta).

b) *(20 p.)* Check explicitly that the decompositions of the state $|\frac{5}{2},+\frac{1}{2}\rangle$ in the basis $|\frac{1}{2}m_1\rangle\,|1m_2\rangle\,|1m_3\rangle$ obtained from $(\frac{1}{2}\otimes 1)\otimes 1$ and $\frac{1}{2}\otimes(1\otimes 1)$ are the same.

Exercise 2. Spin 1 matrices (30 points)

a) *(15 p.)* Derive the spin matrices S_x, S_y, S_z in the basis $|s, s_z\rangle$ for $s=1$.

b) *(15 p.)* Find the eigenvalues and the normalized eigenvectors of S_x and S_y in that basis.

Hint: The general relation $S_\pm|s,s_z\rangle = \hbar\sqrt{s(s+1)-s_z(s_z\pm 1)}|s,s_z\pm 1\rangle$ can we useful.

Exercise 3. Spin operator (25 points)

a) *(15 p.)* Reduce an arbitrary function of the argument $a+\mathbf{b}\boldsymbol{\sigma}$ to a linear function:

$$f\left(a+\mathbf{b}\boldsymbol{\sigma}\right) = A + \mathbf{B}\boldsymbol{\sigma}$$

By writing the coefficients A and $\mathbf{B}$ explicitly ($\boldsymbol{\sigma}$ stands for Pauli matrices).

Hint: use the rotational invariance and act on the eigenstates of σ_3.

b) *(10 p.)* One of the most important properties of Pauli matrices is the expansion of the exponential form:

$$e^{i\alpha(\mathbf{n}\boldsymbol{\sigma})} = I_2\cos\alpha + i\left(\mathbf{n}\boldsymbol{\sigma}\right)\sin\alpha$$

Where I_2 denotes a unit 2×2 matrix and $\mathbf{n}$ is a unit vector in an arbitrary direction. Prove this formula using the result of the part a).

(Bonus) Exercise 4. Entanglement (20 points)

Consider the system of two spins in the state $|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\right)$. Calculate the expectation values $\langle S_{1z}S_{2z}\rangle$, $\langle S_{2z}\rangle$ and $\langle S_{2z}\rangle$ in this state. Do the same for the state $|\Psi'\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$. Give an interpretation of the results in terms of entanglement.