SPIN

⇒ INTRODUCTION

- ORBITAL ANGULAR MOMENTUM L

↳ EIGENSTATES $|\ell m\rangle$ (IN KET NOTATION)

↳ $L^2 |\ell m\rangle = \hbar^2 \ell(\ell+1) |\ell m\rangle$

↳ $L_z |\ell m\rangle = \hbar m |\ell m$

↳ $\ell = 0, 1, 2, \ldots$ INTEGER VALUES !

$m = -\ell, \ldots, +\ell$ ($2\ell+1$ values)

↳ $L_\pm |\ell m\rangle = \hbar \sqrt{\ell(\ell+1) - m(m\pm1)} \, |\ell \, m\pm1\rangle$ (HW)

- INTRINSIC ANGULAR MOMENTUM S (SPIN)

↳ SATISFIES SAME ALGEBRA

$$[S_x, S_y] = i\hbar S_z$$

$$[S_y, S_z] = i\hbar S_x$$

$$[S_z, S_x] = i\hbar S_y$$

↳ S CAN HAVE BOTH ↗ INTEGER ↘ HALF INTEGER VALUES

EIGENSTATES $| s\, s_z \rangle$

$s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$

$s_z = -s, \ldots, +s$

↳ $S^2 | s\, s_z \rangle = \hbar^2 s(s+1) \, | s\, s_z \rangle$

↳ $S_z | s\, s_z \rangle = \hbar\, s_z \, | s\, s_z \rangle$

↳ SPIN IS AN <u>INTRINSIC</u> PROPERTY OF PARTICLE

VISIBLE MATTER IN UNIVERSE IS COMPOSED OF PARTICLES OF SPIN 1/2

↗ LEPTONS (ELECTRON, MUON, TAU)

↘ QUARKS (COME IN 6 'FLAVORS' UP, DOWN, STRANGE, CHARM, BOTTOM, TOP)

↓

MAKE UP PROTONS, NEUTRONS

$\Rightarrow$ SPIN 1/2

- EIGENSTATES $s_z = \pm \frac{1}{2}$

$|\frac{1}{2}\ +\frac{1}{2}\rangle$ ($\uparrow$ SPIN UP)

$|\frac{1}{2}\ -\frac{1}{2}\rangle$ ($\downarrow$ SPIN DOWN)

IN COLUMN VECTOR NOTATION

$$|\tfrac{1}{2}\ +\tfrac{1}{2}\rangle \iff \chi_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle \iff \chi_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- SPIN OPERATOR IN MATRIX NOTATION

$$\leadsto \quad S^2 \chi_\uparrow = \hbar^2 \frac{3}{4} \chi_\uparrow$$

$$S^2 \chi_\downarrow = \hbar^2 \frac{3}{4} \chi_\downarrow$$

$$S^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad 2 \times 2 \text{ MATRIX.}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hbar^2 \frac{3}{4} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\hookrightarrow \quad \begin{pmatrix} a \\ c \end{pmatrix} = \hbar^2 \frac{3}{4} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hbar^2 \frac{3}{4} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\hookrightarrow \quad \begin{pmatrix} b \\ d \end{pmatrix} = \hbar^2 \frac{3}{4} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\therefore \quad \boxed{S^2 = \hbar^2 \frac{3}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

$$\rightsquigarrow \quad S_z \chi_\uparrow = \hbar \frac{1}{2} \chi_\uparrow$$

$$S_z \chi_\downarrow = -\hbar \frac{1}{2} \chi_\downarrow$$

$$\boxed{S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

$$\rightsquigarrow \quad S_\pm |s\, s_z\rangle = \hbar \sqrt{s(s+1) - s_z(s_z \pm 1)}\; |s\; s_z \pm 1\rangle$$

$$S_+ \chi_\uparrow = 0$$

$$S_+ \chi_\downarrow = \hbar \sqrt{\frac{3}{4} + \frac{1}{4}}\; \chi_\uparrow = \hbar \chi_\uparrow$$

$$\Bigg\| \; S_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$S_- \chi_\uparrow = \hbar \sqrt{\frac{3}{4} + \frac{1}{4}} \; \chi_\downarrow = \hbar \chi_\downarrow$$

$$S_- \chi_\downarrow = 0$$

$$S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$\rightsquigarrow$ $$S_\pm = S_x \pm i S_y$$

$$S_x = \frac{1}{2} (S_+ + S_-)$$

$$S_y = -\frac{i}{2} (S_+ - S_-)$$

$$\boxed{S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}$$

$\rightsquigarrow$ NOTATION $\underline{\underline{\bar{S} = \frac{\hbar}{2} \bar{\sigma}}}$

$\bar{\sigma} (\sigma_x, \sigma_y, \sigma_z)$: PAULI SPIN MATRICES. (HERMITIAN)

$$\boxed{\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

- GENERAL SPIN STATE

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix}$$

NORMALIZED $|a|^2 + |b|^2 = 1 \quad (\chi^+ \chi = 1)$

$\rightsquigarrow$ $|a|^2$: PROBABILITY TO FIND χ IN $\chi_\uparrow$ STATE

$|b|^2$: PROBABILITY TO FIND χ IN $\underbrace{\chi_\downarrow}_{\text{EIGENSTATES OF } S_z}$ STATE

$\rightsquigarrow$ WHAT ARE EIGENSTATES, EIGENVALUES OF S_x ?

$$S_x \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

EIGENVALUES $\begin{vmatrix} -\lambda & \frac{\hbar}{2} \\ \frac{\hbar}{2} & -\lambda \end{vmatrix} = 0$

$$\Downarrow$$

$$\lambda^2 = \left(\frac{\hbar}{2}\right)^2$$

$$\lambda = \pm \frac{\hbar}{2}$$

EIGENSTATES :

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \pm \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\beta = \pm \alpha$$

$$S_x \chi_\uparrow^{(x)} = + \frac{\hbar}{2} \chi_\uparrow^{(x)} , \quad \chi_\uparrow^{(x)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\Uparrow$ NORMALIZED

$$S_x \chi_\downarrow^{(x)} = - \frac{\hbar}{2} \chi_\downarrow^{(x)} , \quad \chi_\downarrow^{(x)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\rightsquigarrow$ GENERAL SPINOR

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = \frac{a+b}{\sqrt{2}} \chi_\uparrow^{(x)} + \frac{a-b}{\sqrt{2}} \chi_\downarrow^{(x)}$$

$\frac{1}{2} |a+b|^2$: PROBABILITY TO FIND χ IN $\chi_\uparrow^{(x)}$ STATE

$\frac{1}{2} |a-b|^2$: PROBABILITY TO FIND χ IN $\chi_\downarrow^{(x)}$ STATE

- EXAMPLE : SPIN 1/2 PARTICLE IN SPIN STATE

$$\chi = \frac{1}{\sqrt{6}} \begin{pmatrix} 1+i \\ 2 \end{pmatrix}$$

DETERMINE PROBABILITIES OF MEASURING $+\frac{\hbar}{2}$ & $-\frac{\hbar}{2}$ FOR S_x AND S_z

$\hookrightarrow$ $$\chi = \frac{1}{\sqrt{6}}(1+i)\,\chi_\uparrow + \frac{2}{\sqrt{6}}\,\chi_\downarrow$$

$$P_\uparrow^{(z)} = \frac{1}{6}|1+i|^2 = \frac{1}{3}$$

$$P_\downarrow^{(z)} = \frac{4}{6} = \frac{2}{3}$$

$\hookrightarrow$ $$\chi = \frac{1}{\sqrt{12}}(3+i)\,\chi_\uparrow^{(x)} + \frac{1}{\sqrt{12}}(-1+i)\,\chi_\downarrow^{(x)}$$

$$P_\uparrow^{(x)} = \frac{1}{12}|3+i|^2 = \frac{10}{12} = \frac{5}{6}$$

$$P_\downarrow^{(x)} = \frac{1}{12}|-1+i|^2 = \frac{1}{6}$$

⇒ ELECTRON IN A MAGNETIC FIELD

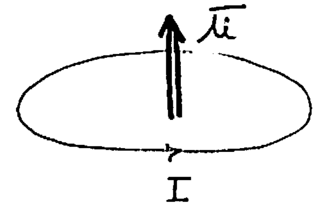
- PARTICLE WITH SPIN IS A MAGNETIC DIPOLE

$\downarrow$

HAS A MAGNETIC MOMENT $\bar{\mu}$

$$\boxed{\bar{\mu} = \underbrace{\gamma}_{\text{PROPORTIONALITY CONSTANT: GYROMAGNETIC RATIO}} \bar{S}}$$

cf. CURRENT LOOP



- PLACE MAGNETIC DIPOLE IN EXTERNAL MAGNETIC FIELD $\bar{B}$

$\downarrow$

INTERACTS SO AS TO ALIGN WITH $\bar{B}$

⇒ INTERACTION ENERGY (HAMILTONIAN)

$$\boxed{H = - \bar{\mu} \cdot \bar{B}}$$

IF $\bar{\mu} \parallel \bar{B}$: H IS MINIMAL $\quad \bar{\mu} \Uparrow \uparrow \bar{B}$

$\bar{\mu}$ ANTI $\parallel$ $\bar{B}$: H IS MAXIMAL $\quad \bar{\mu} \Downarrow \uparrow \bar{B}$

$$\boxed{H = - \gamma \bar{S} \cdot \bar{B}}$$

- LARMOR PRECESSION

↳ TAKE B ALONG Z-AXIS . $\overline{B} = B_0 \, \overline{e}_z$ ← CONSTANT

$$H = -\gamma B_0 S_z = -\gamma B_0 \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

↳ EIGENSTATES OF H ARE SAME AS S_z

$\chi_\uparrow \rightarrow$ EIGENVALUE $E_\uparrow = -\gamma B_0 \frac{\hbar}{2}$ LOWEST ENERGY

$\chi_\downarrow \rightarrow$ EIGENVALUE $E_\downarrow = +\gamma B_0 \frac{\hbar}{2}$

↳ H IS TIME-INDEPENDENT

∴ SOLUTION TO TIME DEP. SCHRÖDINGER EQ.

$$H\chi = i\hbar \frac{\partial \chi}{\partial t}$$

IS SUM OF STATIONARY STATES

$$\chi(t) = a\,\chi_\uparrow\, e^{-\frac{i}{\hbar}E_\uparrow t} + b\,\chi_\downarrow\, e^{-\frac{i}{\hbar}E_\downarrow t}$$

$$= \begin{pmatrix} a\, e^{\frac{i}{2}\gamma B_0 t} \\ b\, e^{-\frac{i}{2}\gamma B_0 t} \end{pmatrix}$$

$$|a|^2 + |b|^2 = 1 \qquad \text{NORMALIZATION}$$

a, b DETERMINED FROM INITIAL CONDITION

TAKE $a = \cos\frac{\alpha}{2}$

$b = \sin\frac{\alpha}{2}$

$$\boxed{\chi(t) = \begin{pmatrix} \cos\frac{\alpha}{2}\, e^{\frac{i}{2}\gamma B_0 t} \\ \sin\frac{\alpha}{2}\, e^{-\frac{i}{2}\gamma B_0 t} \end{pmatrix}}$$

↳ EXPECTATION VALUES OF S_x, S_y, S_z OVER TIME ?

$$\langle S_x \rangle = \frac{\hbar}{2}\, \chi^+(t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \chi(t)$$

$$= \frac{\hbar}{2} \left[\cos\frac{\alpha}{2}\, e^{-\frac{i}{2}\gamma B_0 t} \quad \sin\frac{\alpha}{2}\, e^{\frac{i}{2}\gamma B_0 t} \right] \begin{bmatrix} \sin\frac{\alpha}{2}\, e^{-\frac{i}{2}\gamma B_0 t} \\ \cos\frac{\alpha}{2}\, e^{+\frac{i}{2}\gamma B_0 t} \end{bmatrix}$$

$$= \frac{\hbar}{2} \cdot \sin\frac{\alpha}{2} \cos\frac{\alpha}{2} \underbrace{\left(e^{-i\gamma B_0 t} + e^{i\gamma B_0 t} \right)}_{2\cos\gamma B_0 t}$$

$$\boxed{\langle S_x \rangle = \frac{\hbar}{2} \sin\alpha \cdot \cos(\gamma B_0 t)}$$

$$\langle S_y \rangle = \frac{\hbar}{2} \chi^+(t) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \chi(t)$$

$$= \frac{\hbar}{2} \sin\frac{\alpha}{2} \cos\frac{\alpha}{2} (-i) \underbrace{\left(e^{-i\gamma B_0 t} - e^{i\gamma B_0 t} \right)}_{-2i \sin \gamma B_0 t}$$

$$\boxed{\langle S_y \rangle = -\frac{\hbar}{2} \sin\alpha \; \sin(\gamma B_0 t)}$$

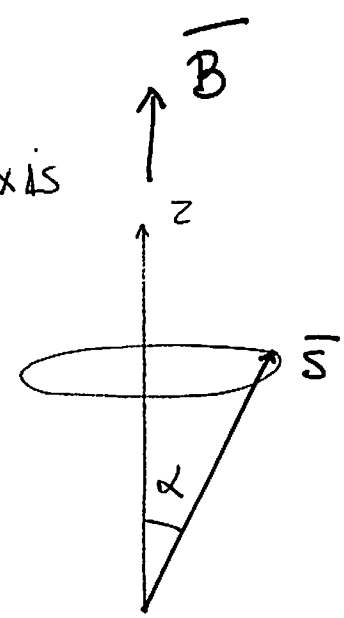
$$\langle S_z \rangle = \frac{\hbar}{2} \chi^+(t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \chi(t)$$

$$= \frac{\hbar}{2} \left(\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} \right)$$

$$\boxed{\langle S_z \rangle = \frac{\hbar}{2} \cos\alpha}$$

$\langle \bar{S} \rangle$ MAKES AN ANGLE α w.r.t. Z-AXIS & PRECESSES AROUND Z-AXIS WITH ANGULAR FREQUENCY $\boxed{\omega = \gamma B_0}$

LARMOR PRECESSION



$\langle \bar{S} \rangle$ EVOLVES ACCORDING TO CLASSICAL EXPECTATIONS

• STERN - GERLACH EXPERIMENT

↳ e^- IN <u>**IN**</u>HOMOGENEOUS MAGN. FIELD

'POTENTIAL ENERGY' $-\bar{\mu}.\bar{B}$

GRADIENT OF POTENTIAL ENERGY

⇓

FORCE $\bar{F} = -\bar{\nabla} V$

FORCE ON MAGNETIC DIPOLE

$$\boxed{\bar{F} = \bar{\nabla}(\bar{\mu}.\bar{B})}$$

$\bar{\mu}$ IS CONSTANT

IF $\bar{B}$ IS CONSTANT

↓

$\bar{F} = 0$

↳ $$\bar{B}(x, y, z) = -\alpha x\, \hat{e}_x + (\alpha z + B_0)\, \hat{e}_z$$

↑ UNIFORM

α SMALL INHOMOGENEITY

$$\bar{\mu}.\bar{B} = -\alpha x\, \gamma S_x + (\alpha z + B_0)\gamma S_z$$

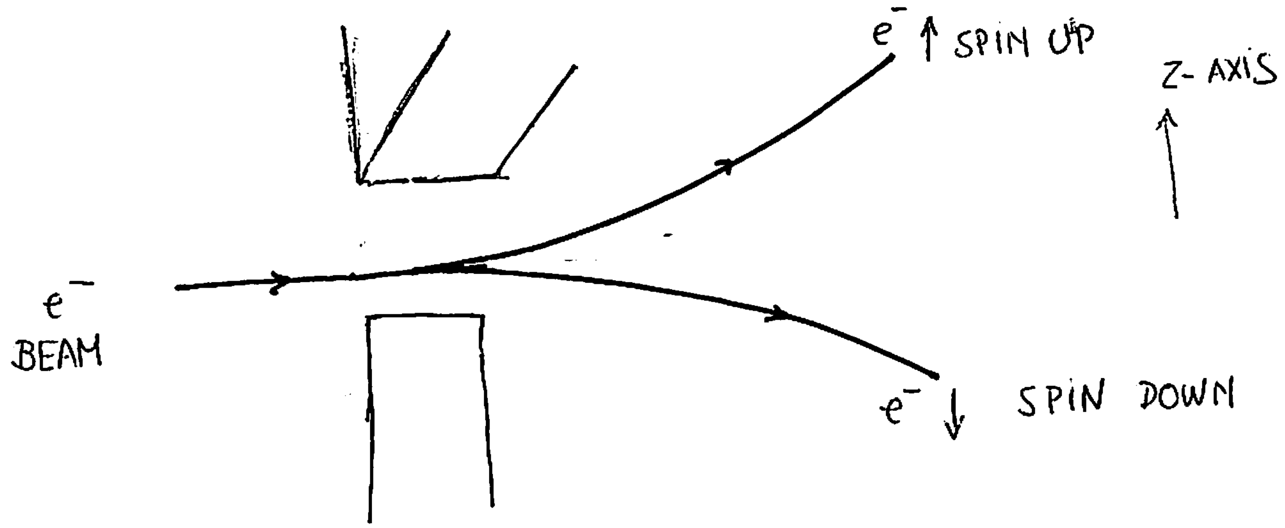
$$\boxed{\bar{F} = +\alpha\gamma\left(-S_x\, \hat{e}_x + S_z\, \hat{e}_z\right)}$$

$\langle S_x \rangle$ AVERAGES TO 0 OVER TIME DUE TO LARMOR PRECESSION AROUND B_0

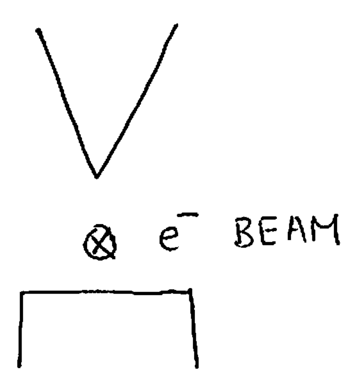
$\therefore$ NET FORCE $\underline{\underline{F_z = \alpha \gamma S_z}}$

DEPENDING OF $S_z = \pm \frac{\hbar}{2}$

THERE IS FORCE ON e^- WHICH PULLS IT IN EITHER ONE OR THE OTHER DIRECTION



FRONT VIEW



↳ EXPERIMENT DONE IN 1922

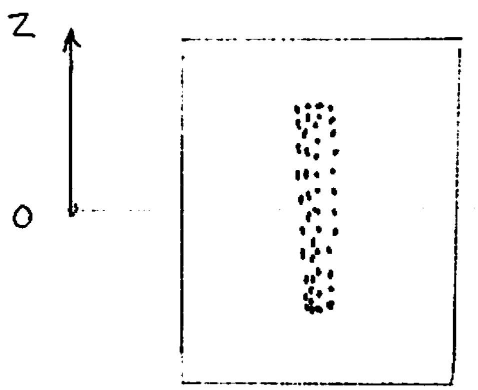
BY STERN & GERLACH (USING ATOMIC BEAMS) SILVER ATOMS

MEASURES z-COMPONENT OF SPIN !

CLASSICAL MECHANICS

ALL e^- HAVE SAME $\bar{S}$

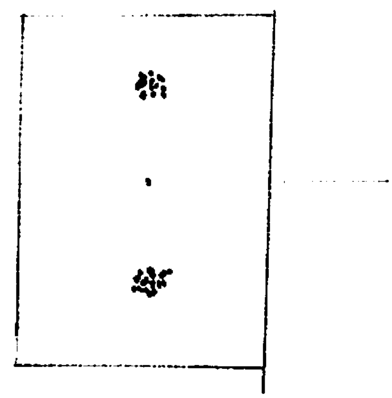
WITH RANDOM ORIENTATION



QUANTUM MECHANICS

ONLY 2 VALUES

OF S_z ALLOWED



BEAM SPLITS

IN 2

⇒ ADDITION OF ANGULAR MOMENTA

↳ 2 SPIN 1/2 PARTICLES

e.g. e^- SPIN 1/2
p SPIN 1/2 IN GROUND STATE $(n=1, \ell=0)$ OF HYDROGEN ATOM
(PROTON)

4 POSSIBILITIES $\underset{e^-\,p}{\uparrow\uparrow}, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow$.

WHAT IS THE TOTAL ANGULAR MOMENTUM OF ATOM ?

$$\bar{S} = \underset{\substack{\uparrow \\ e^- \text{ SPIN}}}{\bar{S}_1} + \underset{\substack{\uparrow \\ p \text{ SPIN}}}{\bar{S}_2}$$ VECTOR ADDITION

$$S_{1z}\, \chi_1 = \hbar\, s_{1z}\, \chi_1$$
$$S_{2z}\, \chi_2 = \hbar\, s_{2z}\, \chi_2$$

↳ $$\left(S_{1z} + S_{2z} \right) \chi_1 \chi_2 = \hbar \underbrace{\left(s_{1z} + s_{2z} \right)}_{s_z} \chi_1 \chi_2$$

$\uparrow\uparrow$ $s_z = +1$

$\left.\begin{matrix}\uparrow\downarrow \\ \downarrow\uparrow\end{matrix}\right\}$ $s_z = 0$ 2 STATES WITH 0 PROJECTION.

$\downarrow\downarrow$ $s_z = -1$

TOTAL SPIN CAN BE 0 OR 1

$\underline{\underline{s=0}}$ $\quad |s\, s_z\rangle = |0\,0\rangle$

$$= \frac{1}{\sqrt{2}}\left\{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right\}$$

SPIN SINGLET (ANTI-SYMMETRIC)

(APPLY SPIN RAISING / LOWERING OPERATOR : HW)

$\underline{\underline{s=1}}$ $\quad |s\, s_z\rangle \Rightarrow |1\,1\rangle = |\uparrow\uparrow\rangle$

$$|1\,0\rangle = \frac{1}{\sqrt{2}}\left\{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right\}$$

$$|1\,-1\rangle = |\downarrow\downarrow\rangle$$

SPIN TRIPLET (SYMMETRIC).

↳ TOTAL SPIN

$$S^2 = (\bar{S}_1 + \bar{S}_2)^2$$

$$= S_1^2 + S_2^2 + 2\bar{S}_1 . \bar{S}_2$$

$$\bar{S}_1 . \bar{S}_2 = \frac{1}{2}\left(S^2 - S_1^2 - S_2^2\right)$$

$$\langle 1\, s_z|\, \bar{S}_1 . \bar{S}_2\, |1\, s_z\rangle = \frac{1}{2}\hbar^2\left(2 - \frac{3}{4} - \frac{3}{4}\right) = \frac{\hbar^2}{4}$$

$$\langle 0\,0|\, \bar{S}_1 . \bar{S}_2\, |0\,0\rangle = \frac{1}{2}\hbar^2\left(0 - \frac{3}{4} - \frac{3}{4}\right) = -\frac{3\hbar^2}{4}$$

↳ COMBINING 2 GENERAL ANGULAR MOMENTA (SPINS)

* $\frac{1}{2}\uparrow$ $\quad$ $\frac{1}{2}\uparrow$

$\nearrow$ $S = 0$ $\quad S_z = 0$

$\searrow$ $S = 1$ $\quad S_z = -1, 0, +1$

* j_1 COMBINE WITH j_2

WHAT IS RESULT ?

e.g. e^- IN $\ell = 1$ STATE OF HYDROGEN ATOM

& e^- HAS INTRINSIC SPIN $s = \frac{1}{2}$

WHAT IS ITS TOTAL ANGULAR MOMENTUM ? $\nearrow \frac{3}{2}$ OR $\searrow \frac{1}{2}$

* GENERAL RESULT :

ALL TOTAL ANGULAR MOMENTA ARE ALLOWED

FROM $\underline{\underline{|j_1 - j_2|, \ldots, j_1 + j_2 - 1, j_1 + j_2}}$

IN STEPS OF 1

* $\bar{J}_1$ $\quad J_1^2 \,|j_1 m_1\rangle = \hbar^2 j_1(j_1+1)\,|j_1 m_1\rangle$

$$J_{1z}\,|j_1 m_1\rangle = \hbar\, m_1\,|j_1 m_1\rangle$$

$\bar{J}_2$ ANALOGOUS J_2^2, J_{2z} HAVE EIGENSTATE $|j_2 m_2\rangle$

$$\bar{J} = \bar{J}_1 + \bar{J}_2$$

↳ J^2, J_z HAS EIGENSTATE $|JM\rangle$

$$J = |j_1 - j_2|, \ldots, j_1 + j_2$$

$$M = -J, \ldots, +J$$

* $$\boxed{|JM\rangle = \sum_{\substack{m_1 \\ m_1+m_2=M}} \sum_{m_2} \underbrace{\langle j_1 m_1, j_2 m_2 | JM\rangle}_{} \,|j_1 m_1\rangle |j_2 m_2\rangle}$$

↓ CLEBSCH - GORDON (CG) COEFFICIENT

<u>NOTATION</u> : LESS USED NOTATION

$$C^{j_1 j_2 J}_{m_1 m_2 M} = \langle j_1 m_1, j_2 m_2 | JM\rangle$$

$*$ EXAMPLE I . $j_1 = 1$ COUPLED WITH $j_2 = \frac{1}{2}$

ORBITAL ANGULAR MOMENTUM OF e^- — SPIN OF e^-

$J: \quad \frac{1}{2}, \frac{3}{2}$

$$\text{e.g.} \quad \left|\tfrac{1}{2}\ +\tfrac{1}{2}\right\rangle = \sum_{m_1}\sum_{\substack{m_2 \\ m_1+m_2=\frac{1}{2}}} \left\langle 1\ m_1, \tfrac{1}{2}\ m_2 \middle| \tfrac{1}{2}\ +\tfrac{1}{2}\right\rangle |1\ m_1\rangle \left|\tfrac{1}{2}\ m_2\right\rangle$$

$$= \left\langle 1\ 0, \tfrac{1}{2}\ +\tfrac{1}{2} \middle| \tfrac{1}{2}\ +\tfrac{1}{2}\right\rangle |1\ 0\rangle \left|\tfrac{1}{2}\ +\tfrac{1}{2}\right\rangle$$

$$+ \left\langle 1\ +1, \tfrac{1}{2}\ -\tfrac{1}{2} \middle| \tfrac{1}{2}\ +\tfrac{1}{2}\right\rangle |1\ +1\rangle \left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle$$

$$\left|\tfrac{1}{2}\ +\tfrac{1}{2}\right\rangle = -\sqrt{\tfrac{1}{3}}\ |1\ 0\rangle \left|\tfrac{1}{2}\ +\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{2}{3}}\ |1\ +1\rangle \left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle$$

$\downarrow$

SUM OF SQUARES OF CG COEFF. IS 1

(BECAUSE LEFT IS NORMALIZED TO 1)

e.g.

$$\left|\tfrac{3}{2}\ +\tfrac{3}{2}\right\rangle = \sum_{m_1}\sum_{m_2} \left\langle 1\ m_1,\ \tfrac{1}{2}\ m_2 \,\middle|\, \tfrac{3}{2}\ +\tfrac{3}{2}\right\rangle \left|1\ m_1\right\rangle \left|\tfrac{1}{2}\ m_2\right\rangle$$

$$m_1 + m_2 = +\tfrac{3}{2}$$

$$= \underbrace{\left\langle 1\ +1,\ \tfrac{1}{2}\ +\tfrac{1}{2} \,\middle|\, \tfrac{3}{2}\ +\tfrac{3}{2}\right\rangle}_{1} \left|1\ +1\right\rangle \left|\tfrac{1}{2}\ +\tfrac{1}{2}\right\rangle$$

$\left|\tfrac{3}{2}\ +\tfrac{1}{2}\right\rangle = \ldots$ (WORK IT OUT YOURSELF).

* ALSO WORKS IN REVERSE WAY

$$|j_1\, m_1\rangle\, |j_2\, m_2\rangle = \sum_J \sum_M \langle j_1\, m_1,\ j_2\, m_2 \,|\, J M\rangle\ |J M\rangle$$

$$J = |j_1 - j_2|, \ldots j_1 + j_2$$

e.g.

$$\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle |1\ \ 0\rangle = \sqrt{\tfrac{3}{5}}\ \left|\tfrac{5}{2}\ \ \tfrac{1}{2}\right\rangle$$

$$+ \sqrt{\tfrac{1}{15}}\ \left|\tfrac{3}{2}\ \ \tfrac{1}{2}\right\rangle$$

$$+ -\sqrt{\tfrac{1}{3}}\ \left|\tfrac{1}{2}\ \ \tfrac{1}{2}\right\rangle$$