CHAPTER 3 : FORMALISM

⇒ 3.1 HILBERT SPACE

- QUANTUM THEORY
  - ↗ WAVE FUNCTIONS ⇒ DESCRIBE STATE OF SYSTEM
  - ↘ OPERATORS ⇒ DESCRIBE OBSERVABLES e.g. ENERGY, MOMENTUM, ...

- ↳ STATE VECTOR $|\alpha\rangle$ LIVES IN N-DIM SPACE

$$|\alpha\rangle \Leftrightarrow a = \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix}$$

INNER PRODUCT $\langle\beta|\alpha\rangle = b^\dagger a = \sum_{i=1}^{N} b_i^* a_i$

↳ OPERATORS ⇒ REPRESENTED BY MATRICES IN N-DIM SPACE

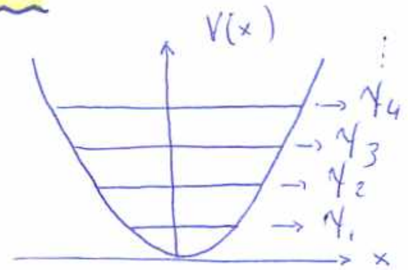
$$|\beta\rangle = \hat{T}\,|\alpha\rangle$$

$$\Updownarrow$$

$$b = T\,a \qquad T = \begin{pmatrix} T_{11} & \cdots & T_{1N} \\ \vdots & & \\ T_{N1} & \cdots & T_{NN} \end{pmatrix}$$

- HILBERT SPACE

↳ e.g. H.O.



VECTORS IN Q.M. ARE SQUARE INTEGRABLE FUNCTIONS $f(x)$

PHYSICAL STATE OF SYSTEM : $\int_{-\infty}^{+\infty} dx \; |\Psi(x,t)|^2 = 1$

↳ $f(x)$ IS SQUARE INTEGRABLE ON INTERVAL $[a,b]$

$$\Updownarrow$$

$$\int_a^b dx \; |f(x)|^2 < \infty$$

↳ SET OF ALL SQUARE INTEGRABLE FUNCTIONS ON A SPECIFIED INTERVAL IS CALLED A HILBERT SPACE (∞ DIMENSIONAL !)

- INNER PRODUCT OF 2 FUNCTIONS

↳ $$\boxed{\langle f | g \rangle \equiv \int_a^b dx \; f^*(x) \, g(x)}$$

IF BOTH f & g ARE SQUARE-INTEGRABLE

↳ INNER PRODUCT EXISTS

PROOF : SCHWARZ INEQUALITY

$$|\langle f | g \rangle|^2 \leq \langle f | f \rangle \langle g | g \rangle$$

↳ $\langle g | f \rangle = \langle f | g \rangle^*$ ✓ VERIFIED FROM DEFINITION

↳ $\langle f | f \rangle \geq 0$ & REAL

$\langle f | f \rangle = 0 \iff f(x) = 0$

∴ $\langle f | g \rangle$ SATISFIES ALL PROPERTIES OF AN INNER PRODUCT

- ORTHOGONALITY, NORMALIZATION

SET OF FUNCTIONS $\{f_n\}$ IS ORTHONORMAL

$$\Updownarrow$$

$$\langle f_n | f_m \rangle = \delta_{nm}$$

e.g. STATIONARY STATES $\{\psi_n\}$ OF H.O.

- COMPLETENESS

↳ SET OF FUNCTIONS IS COMPLETE

$$\Updownarrow$$

ANY OTHER FUNCTION IN HILBERT SPACE CAN BE EXPRESSED AS A LINEAR COMBINATION OF THEM

$$f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$$

e.g. STATIONARY STATES $\{\psi_n\}$ OF H.O.

↳ IF $\{f_n\}$ ARE ORTHONORMAL

$$c_n = \langle f_n | f \rangle$$

⇒ 3.2 OBSERVABLES

- HERMITIAN OPERATORS

↳ EXPECTATION VALUE OF OBSERVABLE $Q(x, p)$

e.g. $H(x,p) = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$

$$\langle Q \rangle = \int dx \; \Psi^* \hat{Q} \Psi$$

$$= \langle \Psi | \hat{Q} \Psi \rangle$$

↳ MEASUREMENT ⇒ REAL OUTCOME

$\langle Q \rangle = \langle Q \rangle^*$ REAL EXPECTATION VALUES.

↳ $\langle \Psi | \hat{Q} \Psi \rangle^* = \langle \hat{Q} \Psi | \Psi \rangle$

$$= \langle \Psi | \hat{Q} \Psi \rangle$$

OPERATORS $\hat{Q}$ SATISFYING

$$\langle \hat{Q} \Psi | \Psi \rangle = \langle \Psi | \hat{Q} \Psi \rangle \quad \forall \; \Psi(x)$$

ARE HERMITIAN OPERATORS

∴ OBSERVABLES ARE REPRESENTED BY HERMITIAN OPERATORS

↳ $\langle \hat{Q} f | g \rangle = \langle f | \hat{Q} g \rangle \quad \forall \; f(x), g(x)$

SHOW THAT THIS IS EQUIVALENT TO ABOVE

↳ EXAMPLE : MOMENTUM OPERATOR $\hat{p} = -i\hbar \frac{\partial}{\partial x}$

$$\langle f | \hat{p} g \rangle = \int_{-\infty}^{+\infty} dx \; f^*(x) \, (-i\hbar) \frac{d}{dx} g(x)$$

↓ INTEGRATION BY PARTS

$$= -i\hbar \cancel{f^* g \Big|_{-\infty}^{+\infty}} + i\hbar \int_{-\infty}^{+\infty} dx \left(\frac{d}{dx} f^*(x) \right) g(x)$$

f & g ARE
SQUARE-INTEGRABLE

$$= \int_{-\infty}^{+\infty} dx \left(-i\hbar \frac{d}{dx} f(x) \right)^* g(x)$$

$$= \langle \hat{p} f | g \rangle$$

■

$\left(-i\hbar \frac{d}{dx} \right)$ IS HERMITIAN

($\frac{d}{dx}$ is NOT !)

DETERMINATE STATES

↳ IN DETERMINATE STATE : EVERY MEASUREMENT OF OBSERVABLE Q RETURNS THE SAME VALUE

e.g. STATIONARY STATES ARE DETERMINATE STATES OF HAMILTONIAN

$$\langle \psi_n | \hat{H} \psi_n \rangle = \underset{\substack{\uparrow \\ \text{ENERGY VALUE}}}{E_n}$$

↳ STANDARD DEVIATION σ OF Q IS ZERO IN A DETERMINATE STATE $|\hat{Q}\psi\rangle = q|\psi\rangle$

$$\langle \hat{Q} \rangle = \langle \psi | \hat{Q} \psi \rangle = q$$

$$\sigma^2 = \langle (\hat{Q} - \langle \hat{Q} \rangle)^2 \rangle$$

$$= \langle \psi | (\hat{Q} - q)^2 \psi \rangle$$

$$= \langle (\hat{Q} - q)\psi | (\hat{Q} - q)\psi \rangle$$

$$\Big\} \quad |\hat{Q}\psi\rangle = q|\psi\rangle$$

$$= 0$$

DETERMINATE STATE IS EIGENFUNCTION OF $\hat{Q}$

$$\hat{Q}|\psi\rangle = q|\psi\rangle$$

↳ EIGENVALUES OF HERMITIAN OPERATOR ARE <u>REAL</u>

COLLECTION OF ALL EIGENVALUES OF OPERATOR ⇒ <u>SPECTRUM</u>

IF 2 OR MORE LIN. INDEP EIGENFUNCTIONS SHARE SAME EIGENVALUE ⇒ SPECTRUM IS <u>DEGENERATE</u>

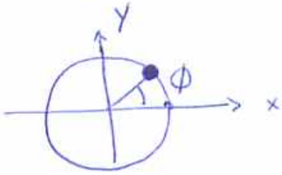
↳ e.g. ENERGY SPECTRUM

EIGENVALUES OF $\hat{H}$

$$\hat{H}\,|\psi_n\rangle = E_n\,|\psi_n\rangle$$

(TIME INDEP. SCHRÖDINGER EQ.)

↳ <u>EXAMPLE</u> $\hat{Q} = i\dfrac{\partial}{\partial\phi}$



$$\phi \in [0, 2\pi] \Rightarrow f(\phi) = f(\phi + 2\pi) \quad (*)$$

⇒ <u>HERMITIAN</u>

$$\langle f\,|\,\hat{Q} g\rangle = \int_0^{2\pi} d\phi\; f^*(\phi)\; i\frac{\partial}{\partial\phi}\, g(\phi)$$

$$= \underbrace{\cancel{i f^* g\Big|_0^{2\pi}}}_{(*)} - i\int_0^{2\pi} d\phi \left(\frac{\partial f}{\partial\phi}\right)^* g(\phi)$$

$$= \int_0^{2\pi} d\phi \left(i\frac{\partial}{\partial\phi} f\right)^* g$$

$$= \langle \hat{Q} f\,|\,g\rangle \quad \blacksquare \qquad \forall f, g$$

$\Rightarrow$ EIGENVALUES

$$i \frac{d}{d\phi} f(\phi) = q\, f(\phi)$$

$$f(\phi) = A\, e^{-iq\phi}$$

$$f(\phi) = f(\phi + 2\pi) \Rightarrow e^{-iq2\pi} = 1$$

$$\Downarrow$$

$$\underline{\underline{q = 0, \pm 1, \pm 2, \ldots}}$$

SPECTRUM OF $\hat{Q}$

⇒ 3.3 EIGENFUNCTIONS OF HERMITIAN OPERATOR

SPECTRUM ↗ DISCRETE e.g. H.O., INFINITE SQUARE WELL
↳ EIGENSTATES NORMALIZABLE

SPECTRUM ↘ CONTINUOUS e.g. $V(x) = 0$
↳ EIGENSTATES <u>NOT</u> NORMALIZABLE

- DISCRETE SPECTRUM OF HERMITIAN OPERATOR

↳ EIGENVALUES ARE <u>REAL</u>

$$\hat{Q}\,|f\rangle = q\,|f\rangle$$

$\hat{Q}$ is HERMITIAN $\quad \langle f|\hat{Q}f\rangle = \langle \hat{Q}f|f\rangle$

$$q\,\langle f|f\rangle = q^*\,\langle f|f\rangle$$

$$q = q^* \quad \blacksquare$$

↳ EIGENFUNCTIONS BELONGING TO DISTINCT EIGENVALUES ARE <u>ORTHOGONAL</u> $\;(\hat{Q} = \hat{Q}^+)$

$$\hat{Q}\,|f\rangle = q\,|f\rangle$$

$$\hat{Q}\,|g\rangle = q'\,|g\rangle$$

$$\langle f|\hat{Q}g\rangle = \langle \hat{Q}f|g\rangle$$

$$q'\langle f|g\rangle = q^*\langle f|g\rangle$$

$$= q\,\langle f|g\rangle$$

IF $q \neq q'$ ⇓ $\langle f|g\rangle = 0 \quad \blacksquare$

IF <u>NOT</u> USE GRAM SCHMIDT ORTHOGONALIZATION PROCEDURE TO CONSTRUCT ORTHOG. EIGENF.

↳ IN FINITE DIM. VECTOR SPACE

EIGENVECTORS OF HERMITIAN OPERATOR ARE COMPLETE

i.e. SPAN THE WHOLE SPACE

↳ FOR HILBERT SPACE (∞ DIM)

AXIOM : EIGENFUNCTIONS OF AN OBSERVABLE OPERATOR ARE COMPLETE :

ANY FUNCTION IN HILBERT SPACE CAN BE WRITTEN AS A LIN. COMBINATION OF THEM

- CONTINUOUS SPECTRUM

↳ EIGENFUNCTIONS NOT NORMALIZABLE

↳ HOW TO UNDERSTAND PROPERTIES SUCH AS REALITY (OF EIGENVALUES), ORTHOGONALITY, COMPLETENESS

↳ EXAMPLE : EIGENVALUES & EIGENFUNCTIONS OF MOMENTUM OPERATOR

$$\underbrace{-i\hbar \frac{d}{dx}}_{\hat{p}} f_p(x) = p \, f_p(x)$$

↳ EIGENFUNCTION

$$\Downarrow$$

$$f_p(x) = A \, e^{\frac{i}{\hbar} p x} \qquad \left(\Rightarrow \lambda = \frac{2\pi\hbar}{p} \right)$$

(NOT SQUARE INTEGRABLE)

$\Rightarrow f_p$ IS NOT A FUNCTION IN HILBERT SPACE

$\Rightarrow$ BUT, WE CAN UNDERSTAND ORTHONORMALITY IN FOLLOWING SENSE:

$$\langle f_{p'} | f_p \rangle = \int_{-\infty}^{+\infty} dx \; f_{p'}^*(x) \, f_p(x) = |A|^2 \int_{-\infty}^{+\infty} dx \; e^{-\frac{i}{\hbar}(p'-p)x}$$

$$= |A|^2 \, 2\pi\hbar \, \delta(p-p')$$

CHOOSE $\underline{\underline{A = \frac{1}{\sqrt{2\pi\hbar}}}}$

$$f_p(x) = \frac{1}{\sqrt{2\pi\hbar}} \, e^{\frac{i}{\hbar} p x}$$

$$\langle f_{p'} | f_p \rangle = \delta(p-p')$$

REPLACES KRONECKER DELTA IN CONTINUOUS CASE

CALLED: DIRAC ORTHONORMALITY

$\Rightarrow$ EIGENFUNCTIONS ARE COMPLETE $\sum \longrightarrow \int$

$$f(x) = \int_{-\infty}^{+\infty} dp \; c(p) \, f_p(x)$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dp \; c(p) \, e^{\frac{i}{\hbar} p x} \qquad \text{(FOURIER TF.)}$$

$$\underline{\underline{\langle f_{p'} | f \rangle}} = \int_{-\infty}^{+\infty} dp \; c(p) \underbrace{\langle f_{p'} | f_p \rangle}_{\delta(p-p')} = \underline{\underline{c(p')}}$$

↳ EXAMPLE : EIGENVALUES & EIGENFUNCTIONS OF POSITION OPERATOR

$$\hat{x} \; g_y(x) = y \; g_y(x)$$

$$\underline{\underline{g_y(x) = A\,\delta(x-y)}}$$

$$\langle g_{y'} | g_y \rangle = \int dx \; g_{y'}^*(x) \, g_y(x)$$

$$= |A|^2 \int dx \; \delta(x-y') \, \delta(x-y)$$

$$= |A|^2 \; \delta(y-y')$$

CHOOSE $A = 1$

$$g_y(x) = \delta(x-y)$$

$$\langle g_{y'} | g_y \rangle = \delta(y-y')$$

⇒ 'COMPLETENESS'

$$f(x) = \int_{-\infty}^{+\infty} dy \; c(y) \, g_y(x)$$

$$= \int_{-\infty}^{+\infty} dy \; c(y) \, \delta(x-y) = c(x)$$

$$c(y) = f(y)$$

⇒ 3.4 STATISTICAL INTERPRETATION

- OBSERVABLE $Q(x, p)$ HERMITIAN $Q^{+} = Q$

  ↓

  MEASUREMENT GIVES ONE OF THE REAL EIGENVALUES OF Q.

↳ DISCRETE SPECTRUM (q_n)

PROBABILITY TO FIND EIGENVALUE q_n

$$\hookrightarrow |c_n|^2 \Rightarrow \quad c_n = \langle f_n | \Psi \rangle$$

$$\Psi(x,t) = \sum_n c_n f_n(x)$$

↳ CONTINUOUS SPECTRUM $(q(z))$

PROBABILITY TO FIND EIGENVALUE IN RANGE dz

$$\hookrightarrow |c(z)|^2 dz \Rightarrow c(z) = \langle f_z | \Psi \rangle$$

∴ BY DOING THE MEASUREMENT ↝ MEASURING A PARTICULAR EIGENVALUE

⇓

THE WAVE FUNCTION COLLAPSES TO THE CORRESPONDING EIGENSTATE

- TOTAL PROBABILITY / EXPECTATION VALUES

↳ $\sum_n |c_n|^2 = 1$ (TOTAL PROBABILITY)

↳ $\Psi = \sum_n c_n f_n$

$$\langle \Psi | \Psi \rangle = \sum_n \sum_{n'} c_n^* c_{n'} \underbrace{\langle f_n | f_{n'} \rangle}_{\delta_{nn'}}$$

$$= \sum_n |c_n|^2$$

$$= 1$$ (NORMALIZATION OF W.F.)

↳ $\langle Q \rangle = \langle \Psi | \hat{Q} \Psi \rangle$

$$= \sum_n \sum_{n'} c_n^* c_{n'} \langle f_n | \underbrace{\hat{Q} f_{n'}}_{q_{n'} | f_{n'} \rangle} \rangle$$

$$= \sum_n \sum_{n'} c_n^* c_{n'} \, q_{n'} \, \delta_{nn'}$$

$$\langle Q \rangle = \sum_n q_n |c_n|^2$$

- MOMENTUM SPACE WAVE FUNCTION

$$f_p(x) = \frac{1}{\sqrt{2\pi\hbar}}\, e^{\frac{i}{\hbar} p x}$$ EIGENFUNCTIONS OF $\hat{p}$

$$\hat{p}\, f_p(x) = p\, f_p(x)$$

$$C(p) = \langle f_p | \Psi \rangle$$ PROB. AMPL. TO FIND Ψ IS MOMENTUM EIGENSTATE

$$C(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dx\, e^{-\frac{i}{\hbar} p x}\, \Psi(x,t)$$

$C(p) \equiv \Phi(p,t)$ IS CALLED MOMENTUM SPACE WAVE FUNCTION.

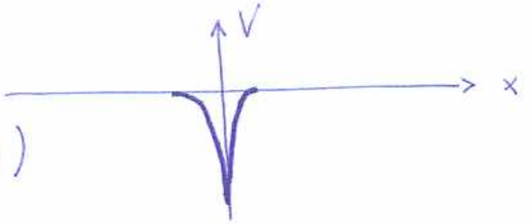
(FOURIER TF OF POSITION SPACE WF $\Psi(x,t)$)

$$\boxed{\begin{aligned} \Phi(p,t) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dx\, e^{-\frac{i}{\hbar} p x}\, \Psi(x,t) \\ \Psi(x,t) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dp\, e^{\frac{i}{\hbar} p x}\, \Phi(p,t) \end{aligned}}$$

$\therefore$ PROBABILITY THAT A MEASUREMENT OF MOMENTUM YIELDS A VALUE BETWEEN p AND $p + dp$

$$|\Phi(p,t)|^2\, dp$$

- EXAMPLE: MOM. SPACE W.F. OF PARTICLE OF MASS m IN δ - FUNCTION POT.



$$V(x) = -\alpha\, \delta(x)$$

↳ BEFORE $\qquad E = -\dfrac{\hbar^2 K^2}{2m} \qquad K = \dfrac{m\alpha}{\hbar^2}$

↳ BOUND STATE ENERGY

$$\Psi(x,t) = \sqrt{K}\, e^{-K|x|}\, e^{-\frac{i}{\hbar}Et}$$

↳ $$\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dx\, e^{-\frac{i}{\hbar}px}\, \Psi(x,t)$$

$$= \frac{\sqrt{K}}{\sqrt{2\pi\hbar}}\, e^{-\frac{i}{\hbar}Et} \int_{-\infty}^{+\infty} dx\, e^{-\frac{i}{\hbar}px}\, e^{-K|x|}$$

$$= \frac{\sqrt{K}}{\sqrt{2\pi\hbar}}\, e^{-\frac{i}{\hbar}Et} \left\{ \underbrace{\int_{-\infty}^{0} dx\, e^{(K-\frac{i}{\hbar}p)x}}_{\frac{1}{K-\frac{i}{\hbar}p}\, e^{(K-\frac{i}{\hbar}p)x}\Big|_{-\infty}^{0}} + \underbrace{\int_{0}^{+\infty} dx\, e^{(-K-\frac{i}{\hbar}p)x}}_{\frac{1}{-K-\frac{i}{\hbar}p}\, e^{-(K+\frac{i}{\hbar}p)x}\Big|_{0}^{+\infty}} \right\}$$

$$= \frac{\sqrt{K}}{\sqrt{2\pi\hbar}}\, e^{-\frac{i}{\hbar}Et} \left\{ \frac{1}{K-\frac{i}{\hbar}p} + \frac{1}{K+\frac{i}{\hbar}p} \right\}.$$

$$\Phi(p,t) = \frac{\sqrt{K}}{\sqrt{2\pi\hbar}} \, e^{-\frac{i}{\hbar}Et} \, \frac{2K}{K^2 + \frac{p^2}{\hbar^2}}$$

$$\boxed{\Phi(p,t) = \sqrt{\frac{2}{\pi}} \, \frac{(\hbar K)^{3/2}}{p^2 + (\hbar K)^2} \, e^{-\frac{i}{\hbar}Et}}$$

$\hookrightarrow$ PROBABILITY TO FIND PARTICLE WITH MOMENTUM $> \hbar K$

$$P(p > \hbar K) = \int_{\hbar K}^{+\infty} dp \, |\Phi(p,t)|^2$$

$$= \frac{2}{\pi} (\hbar K)^3 \int_{\hbar K}^{+\infty} dp \, \frac{1}{\left(p^2 + (\hbar K)^2\right)^2}$$

$\downarrow$ HELP: $2p_0^3 \int dp \, \frac{1}{(p^2+p_0^2)^2} = \frac{p p_0}{p^2+p_0^2} + \operatorname{atan}\left(\frac{p}{p_0}\right)$

$$P(p > \hbar K) = \frac{1}{\pi} \left[\frac{p \, \hbar K}{p^2 + (\hbar K)^2} + \operatorname{atan}\left(\frac{p}{\hbar K}\right) \right]_{\hbar K}^{\infty}$$

$$= \frac{1}{\pi} \left[\frac{\pi}{2} - \frac{1}{2} - \frac{\pi}{4} \right]$$

$$= \frac{1}{4} - \frac{1}{2\pi} \simeq 0.09$$

⇒ 3.5 THE UNCERTAINTY PRINCIPLE

- PROOF OF UNCERTAINTY PRINCPLE

↳ OBSERVABLE A ($\hat{A}$ IS HERMITIAN)

$$\langle \hat{A} \rangle$$

$$\sigma_A^2 = \langle (\hat{A} - \langle \hat{A} \rangle)^2 \rangle$$

$$= \langle \Psi | (\hat{A} - \langle \hat{A} \rangle)^2 \Psi \rangle$$

$\hat{A}$ HERMITIAN

$$= \langle (\hat{A} - \langle \hat{A} \rangle) \Psi \,|\, (\hat{A} - \langle \hat{A} \rangle) \Psi \rangle$$

$$\equiv \langle f | f \rangle \quad \text{WITH } |f\rangle = |(\hat{A} - \langle \hat{A} \rangle) \Psi \rangle$$

↳ OBSERVABLE B ($\hat{B}$ IS HERMITIAN)

$$\langle \hat{B} \rangle$$

$$\sigma_B^2 \equiv \langle g | g \rangle \quad \text{WITH } |g\rangle = |(\hat{B} - \langle \hat{B} \rangle) \Psi \rangle$$

↳ SCHWARTZ INEQUALITY

$$\sigma_A^2 \sigma_B^2 = \langle f | f \rangle \langle g | g \rangle \geq |\langle f | g \rangle|^2$$

↳ $\forall z$: COMPLEX NUMBER

$$|z|^2 = (\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2 \geq (\mathrm{Im}\, z)^2 = \left[\frac{1}{2i} (z - z^*) \right]^2$$

3.19

LET $z = \langle f | g \rangle$

$$\frac{1}{2i}(z - z^*) = \frac{1}{2i}\left(\langle f | g \rangle - \langle f | g \rangle^* \right)$$

$$= \frac{1}{2i}\left(\langle f | g \rangle - \langle g | f \rangle \right)$$

$$\hookrightarrow \langle f | g \rangle = \langle (\hat{A} - \langle \hat{A} \rangle)\Psi \,|\, (\hat{B} - \langle \hat{B} \rangle)\Psi \rangle$$

$$= \langle \Psi | (\hat{A} - \langle \hat{A} \rangle)(\hat{B} - \langle \hat{B} \rangle)\Psi \rangle$$

$$= \langle \Psi | \left(\hat{A}\hat{B} - \langle \hat{A} \rangle \hat{B} - \hat{A} \langle \hat{B} \rangle + \langle \hat{A} \rangle \langle \hat{B} \rangle \right) \Psi \rangle$$

$$= \langle \Psi | \hat{A}\hat{B}\Psi \rangle - \langle \hat{A} \rangle \langle \Psi | \hat{B}\Psi \rangle - \langle \hat{B} \rangle \langle \Psi | \hat{A}\Psi \rangle + \langle \hat{A} \rangle \langle \hat{B} \rangle \underbrace{\langle \Psi | \Psi \rangle}_{1}$$

$$= \langle \Psi | \hat{A}\hat{B}\Psi \rangle - 2\langle \hat{A} \rangle \langle \hat{B} \rangle + \langle \hat{A} \rangle \langle \hat{B} \rangle$$

$$= \langle \Psi | \hat{A}\hat{B}\Psi \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle$$

$$\langle g | f \rangle = \langle \Psi | \hat{B}\hat{A}\Psi \rangle - \langle \hat{B} \rangle \langle \hat{A} \rangle$$

$$\hookrightarrow \langle f | g \rangle - \langle g | f \rangle = \langle \Psi | (\hat{A}\hat{B} - \hat{B}\hat{A})\Psi \rangle$$

$$= \langle \Psi | [\hat{A}, \hat{B}]\Psi \rangle$$

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$$ COMMUTATOR

$\hookrightarrow \quad \sigma_A^2 \, \sigma_B^2 \geqslant \left[\frac{1}{2i} \left(\langle f|g \rangle - \langle g|f \rangle \right) \right]^2$

$$\boxed{\sigma_A^2 \, \sigma_B^2 \geqslant \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2}$$

UNCERTAINTY PRINCIPLE

$\hookrightarrow$ NOTE COMMUTATOR OF 2 HERMITIAN OPERATORS IS ANTI-HERMITIAN !

$$[\hat{A}, \hat{B}]^{+} = (\hat{A}\hat{B} - \hat{B}\hat{A})^{+}$$

$$= (\hat{B}^{+}\hat{A}^{+} - \hat{A}^{+}\hat{B}^{+}) \quad \hat{A}^{+} = \hat{A}, \; \hat{B}^{+} = \hat{B}$$

$$= \hat{B}\hat{A} - \hat{A}\hat{B}$$

$$= -[\hat{A}, \hat{B}]$$

$\rightsquigarrow$ EIGENVALUES OF ANTI-HERMITIAN OPERATOR ARE IMAGINARY

$\rightsquigarrow \quad \frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle$ IS REAL NUMBER !

• POSITION-MOMENTUM UNCERTAINTY : $\sigma_x \sigma_p$

$\hat{A} = \hat{x}$ POSITION OPERATOR

$\hat{B} = \hat{p}$ MOMENTUM OPERATOR

$= -i\hbar \frac{d}{dx}$

$$[\hat{A}, \hat{B}] = [\hat{x}, \hat{p}] = i\hbar$$

$$\sigma_x^2 \sigma_p^2 \geq \left(\frac{1}{2i} i\hbar\right)^2 = \left(\frac{\hbar}{2}\right)^2$$

$$\boxed{\sigma_x \sigma_p \geq \frac{\hbar}{2}}$$

EQUIVALENT NOTATION

$$\Delta x \, \Delta p \geq \frac{\hbar}{2}$$

ORIGINAL UNCERTAINTY PRINCIPLE OF HEISENBERG

↳ PAIR OF OBSERVABLES A, B FOR WHICH

$[\hat{A}, \hat{B}] \neq 0 \Rightarrow$ A, B ARE <u>IN</u>COMPATIBLE OBSERVABLES.

↓

THEY DO <u>NOT</u> HAVE

A COMPLETE SET OF SHARED EIGENFUNCTIONS

↳ COMPATIBLE OBSERVABLES

$[\hat{A}, \hat{B}] = 0$ ↳ DO HAVE A COMPLETE SET OF SHARED EIGENFUNCTIONS

↳ (SEE PROBLEM 3.15 !)

MINIMUM UNCERTAINTY WAVE PACKET

↳ GROUND STATE OF H.O.

$$\sigma_x \sigma_p = \frac{\hbar}{2} \rightarrow \text{HITS THE UNCERTAINTY LIMIT}$$

↳ EQUALITY MEANS :

1) SCHWARTZ INEQUALITY BECOMES EQUALITY

$$\langle f | f \rangle \langle g | g \rangle = |\langle f | g \rangle|^2$$

$$\Updownarrow$$

$$g(x) = c\, f(x)$$

2) $$|z|^2 = |\langle f | g \rangle|^2 = \left(\mathrm{Im} \langle f | g \rangle\right)^2$$

$$\Updownarrow$$

$\langle f | g \rangle$ IS PURELY IMAGINARY $\Rightarrow c = ia$

$\therefore \quad g(x) = ia\, f(x) \qquad a$ IS REAL

↳ $f(x) \Rightarrow |(\hat{x} - \langle \hat{x} \rangle)\Psi\rangle$

$g(x) \Rightarrow |(\hat{p} - \langle \hat{p} \rangle)\Psi\rangle$

$g(x) = ia\, f(x)$

$$\left(-i\hbar \frac{d}{dx} - \langle p \rangle\right)\Psi = ia\,(x - \langle x \rangle)\Psi$$

SOLUTION

TRY: $\Psi(x) = A e^{\frac{i}{\hbar}\langle p \rangle x} \phi(x)$

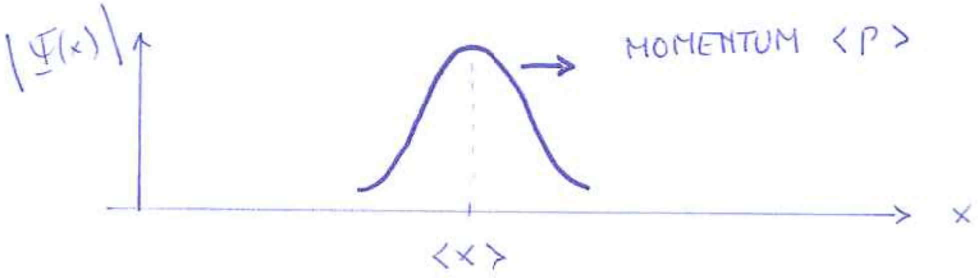
$$A e^{\frac{i}{\hbar}\langle p \rangle x}\left(-i\hbar \frac{d}{dx}\right)\phi = ia A e^{\frac{i}{\hbar}\langle p \rangle x}\left(x - \langle x \rangle\right)\phi$$

$$\downarrow$$

$$\frac{d}{dx}\phi(x) = -\frac{a}{\hbar}\left(x - \langle x \rangle\right)\phi(x)$$

SOLUTION $\phi(x) = e^{-\frac{a}{2\hbar}\left(x - \langle x \rangle\right)^2}$

$$\therefore \boxed{\Psi(x) = A e^{\frac{i}{\hbar}\langle p \rangle x} \cdot e^{-\frac{a}{2\hbar}\left(x - \langle x \rangle\right)^2}}$$



↳ CALLED: MINIMUM UNCERTAINTY GAUSSIAN WAVEPACKET

ENERGY - TIME UNCERTAINTY PRINCIPLE

$\hookrightarrow \frac{d}{dt} \langle \hat{Q} \rangle$

$$= \frac{d}{dt} \langle \Psi | \hat{Q} \Psi \rangle$$

$$= \langle \frac{\partial \Psi}{\partial t} | \hat{Q} \Psi \rangle + \langle \Psi | \frac{\partial \hat{Q}}{\partial t} \Psi \rangle + \langle \Psi | \hat{Q} \frac{\partial \Psi}{\partial t} \rangle$$

$$\Big\downarrow \quad i\hbar \frac{\partial \Psi}{\partial t} = \hat{H} \Psi$$

$$= \frac{i}{\hbar} \langle \hat{H} \Psi | \hat{Q} \Psi \rangle + \langle \Psi | \frac{\partial \hat{Q}}{\partial t} \Psi \rangle - \frac{i}{\hbar} \langle \Psi | \hat{Q} \hat{H} \Psi \rangle$$

$$= \frac{i}{\hbar} \langle \Psi | \hat{H} \hat{Q} \Psi \rangle + \langle \Psi | \frac{\partial \hat{Q}}{\partial t} \Psi \rangle - \frac{i}{\hbar} \langle \Psi | \hat{Q} \hat{H} \Psi \rangle$$

$$= \frac{i}{\hbar} \langle [\hat{H}, \hat{Q}] \rangle + \langle \frac{\partial \hat{Q}}{\partial t} \rangle$$

$$\boxed{\frac{d}{dt} \langle \hat{Q} \rangle = \frac{i}{\hbar} \langle [\hat{H}, \hat{Q}] \rangle + \langle \frac{\partial \hat{Q}}{\partial t} \rangle}$$

↳ GENERALIZED UNCERTAINTY PRINCIPLE

$\hat{A} = \hat{H}$

$\hat{B} = \hat{Q}$ WHICH DOES <u>NOT</u> DEPEND ON TIME $\frac{\partial \hat{Q}}{\partial t} = 0$

$$\sigma_H^2 \, \sigma_Q^2 \geq \left(\frac{1}{2i} \langle [\hat{H}, \hat{Q}] \rangle \right)^2$$

$$= \left(\frac{\hbar}{2}\right)^2 \left(\frac{d\langle Q \rangle}{dt} \right)^2$$

$$\sigma_H \, \sigma_Q \geq \frac{\hbar}{2} \left| \frac{d\langle Q \rangle}{dt} \right|$$

$$\sigma_H \equiv \Delta E$$

AND $\boxed{\Delta t \equiv \frac{\sigma_Q}{\left| \frac{d\langle Q \rangle}{dt} \right|}}$

$\boxed{\Delta E \, \Delta t \geq \frac{\hbar}{2}}$ ENERGY - TIME UNCERTAINTY PRINCIPLE

MEANING: $\sigma_Q = \left| \frac{d\langle Q \rangle}{dt} \right| \Delta t$

↑ RATE OF CHANGE OF $\langle Q \rangle$ PER UNIT OF TIME

Δt : TIME IT TAKES FOR EXPECTATION VALUE $\langle Q \rangle$ TO CHANGE BY ONE STANDARD DEVIATION

↳ EXAMPLE :

- FOR STATIONARY STATE :

ENERGY IS UNIQUELY DETERMINED $\Rightarrow \Delta E = 0$

$$\Downarrow$$

$$\Delta t = \infty$$

(ALL OBSERVABLES ARE CONSTANT IN TIME $\frac{d}{dt}\langle Q \rangle = 0$)

- LINEAR COMBINATION OF 2 STATIONARY STATES.

$$\Psi(x,t) = a\,\psi_1(x)\,e^{-\frac{i}{\hbar}E_1 t} + b\,\psi_2(x)\,e^{-\frac{i}{\hbar}E_2 t}$$

a, b, ψ_1, ψ_2 REAL

$$|\Psi(x,t)|^2 = a^2\psi_1^2 + b^2\psi_2^2 + 2ab\,\psi_1\psi_2\cos\left(\frac{E_2 - E_1}{\hbar}t\right)$$

PERIOD OF OSCILLATION $\tau = \dfrac{2\pi\hbar}{E_2 - E_1}$

$\Delta E = E_2 - E_1$ (ROUGHLY)

$\Delta t = \tau$ (ROUGHLY)

$$\therefore \quad \Delta E\,\Delta t \sim 2\pi\hbar = (4\pi)\frac{\hbar}{2} \geq \frac{\hbar}{2}$$

(see PROBLEM 3.18 !)

⇒ 3.6 DIRAC NOTATION

- STATE $|\Psi\rangle$

  $|x\rangle$ POSITION EIGEN STATE

  $\Psi(x,t) = \langle x|\Psi\rangle$

  ↳ W.F. IN COORDINATE SPACE

- $|p\rangle$ MOMENTUM EIGENSTATE $\hat{p}\,|p\rangle = p\,|p\rangle$

  $\Phi(p,t) = \langle p|\Psi\rangle$

  ↳ W.F. IN MOMENTUM SPACE

- $|n\rangle$ STATIONARY STATE $\hat{H}\,|n\rangle = E_n\,|n\rangle$

  $c_n(t) = \langle n|\Psi\rangle$

- OPERATORS : TRANSFORM ONE VECTOR INTO ANOTHER

  $$|\beta\rangle = \hat{Q}\,|\alpha\rangle$$

  BASIS $|e_n\rangle$

  $$|\alpha\rangle = \sum_n a_n\,|e_n\rangle$$

  $$|\beta\rangle = \sum_n b_n\,|e_n\rangle$$

  $$Q_{nm} \equiv \langle e_n|\,\hat{Q}\,|e_m\rangle$$

DIRAC's BRA & KET NOTATION

↳

$$|\alpha\rangle \iff \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

⇓

'KET'

$$\langle\alpha| \iff \left(a_1^* \; \dots \; a_n^* \right)$$

⇓

'BRA'

$$|\alpha\rangle = \sum_n a_n |e_n\rangle$$

$$\langle\alpha| = \sum_n a_n^* \langle e_n|$$

↳ PROJECTION OPERATOR $\hat{P} \equiv |\alpha\rangle\langle\alpha|$

$$\hat{P}|\beta\rangle = \langle\alpha|\beta\rangle\,|\alpha\rangle$$

↳ IF $\{|e_n\rangle\}$ IS ORTHONORMAL BASIS (DISCRETE)

$$\boxed{\begin{array}{l} \langle e_n|e_m\rangle = \delta_{nm} \\ \sum_n |e_n\rangle\langle e_n| = 1 \end{array}}$$

⇒ COMPLETENESS

$$\sum_m |e_m\rangle \underbrace{\langle e_m|\alpha\rangle}_{a_m}$$

$$= \sum_m a_m |e_m\rangle$$

$$= |\alpha\rangle$$

$$\therefore \quad \sum_m |e_m\rangle\langle e_m| = 1$$

↳ IF $\{|e_z\rangle\}$ IS DIRAC ORTHONORMALIZED BASIS

$$\boxed{\begin{gathered} \langle e_z|e_{z'}\rangle = \delta(z-z') \\ \int dz\, |e_z\rangle\langle e_z| = 1 \end{gathered}}$$

$$|\Psi\rangle = \int dz\, |e_z\rangle \underbrace{\langle e_z|\Psi\rangle}_{\Psi(z)}$$

$$\underbrace{\langle e_{z'}|\Psi\rangle}_{\Psi(z')} = \int dz\, \underbrace{\langle e_{z'}|e_z\rangle}_{\delta(z-z')} \Psi(z) \overset{!}{=} \Psi(z')$$

- EIGENSTATES / EIGENVALUES OF OPERATOR (OBSERVABLE)

$\Updownarrow$

MATRIX DIAGONALIZATION

$$\hat{Q}\,|\psi\rangle = \lambda\,|\psi\rangle$$

$$|\psi\rangle = \sum_n a_n\,|\psi_n\rangle$$

$$\hat{Q}\,|\psi\rangle = \lambda\,|\psi\rangle$$

$$\Updownarrow$$

$$\sum_n a_n \hat{Q}\,|\psi_n\rangle = \lambda \sum_n a_n\,|\psi_n\rangle$$

$\downarrow$ INNER PRODUCT WITH $|\psi_m\rangle$

$$\sum_n a_n \underbrace{\langle \psi_m|\hat{Q}|\psi_n\rangle}_{Q_{mn}} = \lambda \sum_n a_n \underbrace{\langle \psi_m|\psi_n\rangle}_{\delta_{nm}}$$

$$\sum_n Q_{mn}\, a_n = \lambda\, a_m$$

$$\Big(\quad Q \quad\Big)\Big(a \Big) = \lambda \Big(a \Big)$$

MATRIX DIAGONALIZATION.