Exercise sheet 7
Theoretical Physics 3: QM WS2023/2024

13.12.2023

Exercise 0.

How much time did you take to complete this homework sheet?

Exercise 1. Laplace operator in spherical coordinates (40 points)

a) *(15 p.)* Calculate the Jacobian for spherical coordinates and then extract the volume element

$$\mathrm{d}V = r^2 \sin\theta \ \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi.$$

b) *(20 p.)* Show that the Laplace operator $\Delta \equiv \boldsymbol{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ in three dimensions in spherical coordinates takes the form:

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

c) *(5 p.)* Show that the radial term can also be written as:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2}r$$

Exercise 2. Operators and Dirac notation (60 points)

a) *(5 p.)* Consider the ladder operators for the quantum harmonic oscillator problem. Show that:

$$\hat{a}_+ = (\hat{a}_-)^\dagger$$

b) *(10 p.)* Show that for any observable $\hat{q}$ with *nondegenerate* spectrum:

$$\hat{q} = \sum_q q \left|q\right\rangle \left\langle q\right|$$

Where $\hat{q}\left|q\right\rangle = q\left|q\right\rangle$, and in the case of continuous spectrum $\sum_q \to \int \mathrm{d}q$.
Hint: since the set of eigenfunctions is complete and orthonormal, one can use:

$$\sum_q \left|q\right\rangle\left\langle q\right| = \hat{1}$$

c) *(10 p.)* We already know from the lecture that the eigenfunctions of the momentum operator are given by:

$$\langle x|p\rangle \equiv f_p(x) = \frac{1}{\sqrt{2\pi\hbar}}\, e^{\frac{i}{\hbar}px}$$

Show that:

$$\langle p|\hat{x}|\psi\rangle = i\hbar\frac{\partial}{\partial p}\left\langle p|\psi\right\rangle$$

d) *(10 p.)* Show that:

$$\langle\psi|\hat{x}|\psi\rangle = \int_{-\infty}^{+\infty} \mathrm{d}p\, \phi^*(p) \left[i\hbar\frac{\partial}{\partial p}\right] \phi(p)$$

Where:

$$\phi(p) \equiv \langle p|\psi\rangle$$

e) *(10 p.)* Show that:

$$\langle x|\hat{p}|x'\rangle = -i\hbar\frac{\partial}{\partial x}\delta(x - x')$$

f) *(15 p.)* Recall the infinite square well stationary state wave function:

$$\langle x|n\rangle = \begin{cases} \sqrt{\frac{2}{a}}\sin\left(\frac{\pi n x}{a}\right) & 0 < x < a \\ 0 & \text{otherwise} \end{cases}$$

Compute $\langle p|n\rangle$.