Exercise sheet 6
Theoretical Physics 3: QM WS2023/2024

06.12.2023

Exercise 0.

How much time did you take to complete this homework sheet?

Exercise 1. Energy-time uncertainty relation (50 points)

Consider a particle in the infinite square well:

$$V(x) = \begin{cases} 0, & 0 \leq x \leq l \\ +\infty, & \text{otherwise} \end{cases}$$

Which has as it's initial wave function a mixture of the first and second excited stationary states:

$$\Psi(x,0) = A\left(\Psi_2(x) + \Psi_3(x)\right)$$

a) *(40 p.)* Calculate σ_H, σ_x and $d\langle x\rangle/dt$.

Hint: You should end up with the following expressions,

$$\begin{aligned} \sigma_H^2 &= \frac{1}{4} \cdot \frac{25\hbar^4\pi^4}{4m^2l^4} \\ \sigma_x^2 &= \frac{l^2}{4}\left[\frac{1}{3} - \frac{13}{36\pi^2} - \left(\frac{96}{25\pi^2}\right)^2 \cos^2\left(\frac{(E_3 - E_2)\,t}{\hbar}\right)\right] \\ \frac{d\langle x\rangle}{dt} &= \frac{24\hbar}{5ml}\sin\left(\frac{(E_3 - E_2)\,t}{\hbar}\right). \end{aligned}$$

b) *(10 p.)* Check that energy-time uncertainty relation holds.

Hint: Energy-time uncertainty relation:

$$\sigma_H^2\sigma_x^2 \geq \frac{\hbar^2}{4}\left(\frac{d\langle x\rangle}{dt}\right)^2.$$

Exercise 2. Coherent states (20 points + 10 bonus)

a) *(5 p.)* Prove that eigenfunctions $|\alpha\rangle$ of annihilation operator $\hat{a}$ can be written in the following form:

$$|\alpha\rangle = e^{-|\alpha|^2/2} e^{\alpha\hat{a}^\dagger}|0\rangle$$

b) *(15 p.)* Harmonic oscillator Hamiltonian can be written in the form:

$$\hat{E} = \hbar\omega \left(\hat{n} + \frac{1}{2} \right); \quad \hat{n} = \hat{a}^{\dagger}\hat{a}$$

Calculate the expectation value of $\hat{n}$ in a state $|\alpha\rangle$, σ_n and additionally:

$$P(n) = |\langle \alpha | n \rangle|^2$$

Which can be interpreted as a probability to observe a coherent state $|\alpha\rangle$ with an energy E_n.

c) *(10 Bonus p.)* Show that these functions form a set of normalized but non-orthogonal states. *Hint*: use the Baker-Campbell-Hausdorff formula:

$$e^{\hat{A}} e^{\hat{B}} = e^{[\hat{A}, \hat{B}]} e^{\hat{B}} e^{\hat{A}}$$

Exercise 3. Hermitian operators (30 points)

In this exercise, you are going to prove some theorems which are very important for quantum mechanics.

a) *(5 p.)* **Theorem I:** If two operators $\hat{A}$ and $\hat{B}$ commute, and if $|\psi\rangle$ is an eigenvector of $\hat{A}$, then $\hat{B}|\psi\rangle$ is also an eigenvector of $\hat{A}$ with the same eigenvalue.

b) *(5 p.)* **Theorem II:** If two observables A and B commute, and if $|\psi_1\rangle$ and $|\psi_2\rangle$ are two eigenvectors of $\hat{A}$ with different eigenvalues, then the matrix element $\langle\psi_1| \hat{B} |\psi_2\rangle$ vanishes.

c) *(15 p.)* **Theorem III:** If two observables A and B commute, one can construct an orthonormal basis made of eigenvectors common to both $\hat{A}$ and $\hat{B}$. Consider only the case where the spectra of A and B are discrete.
Hint: Since A is an observable, there is at least one orthonormal basis made of eigenvectors of $\hat{A}$:

$$\hat{A} \left| u_n^i \right\rangle = a_n \left| u_n^i \right\rangle; \qquad n = 1, 2, \ldots; \qquad i = 1, 2, \ldots, g_n$$

Where g_n is the degree of degeneracy of eigenvalue a_n and with $\langle u_n^i | u_m^j \rangle = \delta_{nm}\delta_{ij}$. Discuss the matrix elements $\langle u_n^i | \hat{B} | u_m^j \rangle$.

d) *(5 p.)* Show the reciprocal of the theorem III.