Exercise sheet 5
Theoretical Physics 3: WS2023/2024

29.11.2023

Exercise 0.

How much time did you take to complete this homework sheet?

Exercise 1. Commutators (20 points + 20 bonus)

The commutator of two matrices $A,\ B$ is defined as $[A, B] = AB - BA$.

a) *(5 p.)* Explain why, in general, $e^A e^B \neq e^{A+B}$. In which cases does the equality sign hold?

b) *(15 p.)* Consider the same case $[A, [A, B]] = [B, [A, B]] = 0$, and prove the Glauber formula

$$e^A\, e^B = e^{A+B}\, e^{\frac{1}{2}[A,B]}.$$

Hint: Consider the function $F(t) = e^{tA}\, e^{tB}$. Show that it has to satisfy the differential equation $\frac{\mathrm{d}F(t)}{\mathrm{d}t} = (A+B+t[A, B])F(t)$. Then solve the equation by noting that $(A+B)$ and $[A, B]$ commute, and, hence, can be treated as mere numbers.

c) *(Bonus 20 p.)* Using the method of mathematical induction, show that:

$$[A, B]_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} A^{n-k} B A^k \tag{1}$$

where the multi-commutator $[A, B]_j$ is defined recursively as $[A, B]_j = A[A, B]_{j-1} - [A, B]_{j-1}A$.
Hint: In the second step of induction, you assume that Eq. (1) holds for n and try to show it for $n + 1$. Starting from Eq. (1) for $n + 1$, show that it can be expressed as

$$[A, B]_{n+1} = A \left[\sum_{k=0}^{n+1} (-1)^k \binom{n}{k} A^{n-k} B A^k \right] + \left[\sum_{k=0}^{n+1} (-1)^k \binom{n}{k-1} A^{n+1-k} B A^{k-1} \right] A.$$

Then, you will need a clever change of a summation variable to end up with the definition of the multi-commutator for $n + 1$, thus concluding the proof.

Exercise 2. Potential barrier (30 points)

Consider the potential barrier:

$$V(x) = V_0(\theta(x) - \theta(x - a))$$

where $\theta(x)$ is the Heavyside step function. Consider and incoming wave from $x \to -\infty$ with wave-function $\psi_E(x) = e^{ikx}$ where $k = (2mE/\hbar^2)^{1/2}$.

a) *(15 p.)* Determine the values of the energy E (if they exist), for a particle of mass m, such that the reflected wave is absent (total transmission). Assume that $E < V_0$.

b) *(10 p.)* Repeat the calculation of (a) assuming that $E > V_0$.

c) *(5 p.)* Do any values of the energy E exist for which the transmitted wave is absent (total reflection)?

Exercise 3. Finite square well potential (20 points)

A particle of mass m is moving in a finite square potential well:

$$V(x) = \begin{cases} -\frac{\alpha}{2a} & \text{for } -a \le x \le a \\ 0 & \text{for } x > |a| \end{cases}$$

The energy levels are determined by the condition:

$$z \tan z = \sqrt{z_0^2 - z^2}$$

Where:

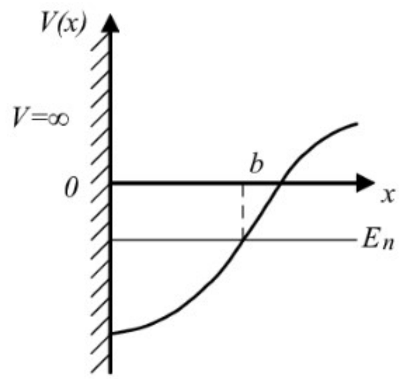
$$z = \frac{a}{\hbar}\sqrt{2m\left(E + \frac{\alpha}{2a}\right)}; \qquad z_0 = \frac{a}{\hbar}\sqrt{\frac{m\alpha}{a}}$$

a) *(10 p.)* Considering the limit $a \to 0$ and assuming that E is finite in this limit, show that you recover the unique bound state of the δ potential well $E = -\frac{m\alpha^2}{2\hbar^2}$.

b) *(10 p.)* What should be the value of $m\alpha a/\hbar^2$ in order for the system to have precisely n bound states?

Exercise 4. WKB approximation (30 points)

Consider the potential $V(x)$ shown on the figure below:



To the right of the right turning point b, the wave function has the semi-classical form:

$$\Psi(x) = \frac{C}{2\sqrt{|p(x)|}} \exp\left(-\frac{1}{\hbar}\int_b^x |p(x)|dx\right); \quad x > b$$

In the region $x < b$ the corresponding wave function is then:

$$\Psi(x) = \frac{C}{\sqrt{p(x)}} \sin\left(\frac{1}{\hbar}\int_x^b p(x)dx + \frac{\pi}{4}\right); \quad x < b$$

a) *(5 p.)* Derive the following relation for the energy levels:

$$\frac{1}{\hbar}\int\limits_{0}^{b}\sqrt{2m[E_n - V(x)]}dx = \pi\left(n + \frac{3}{4}\right)$$

b) *(5 p.)* From now on consider the specific potential:

$$V(x) = \begin{cases} -\frac{\alpha}{x^2} & \text{for } x \geq a \\ \infty & \text{for } x < a \end{cases} \qquad \alpha > 0$$

Under which conditions can the semi-classical approximation be applied? For which values of x?

c) *(10 p.)* Using the result of a) and performing the integration, derive the relation for the energy levels E_n.

d) *(10 p.)* Find the explicit form for the upper energy levels, defined by the condition $|E_n| \ll \alpha/a^2$ expanding the result of c) by the small parameter $|E_n|a^2/\alpha$. Keep the leading terms only. How does the distance between energy levels change with growing n?