Exercise sheet 3
Theoretical Physics 3: QM WS2023/2024

15.11.2023

Exercise 0

How much time did you take to complete this homework sheet?

Exercise 1. Harmonic oscillator: ladder operators (35 points)

Consider a quantum harmonic oscillator, the time-independent ground state wave function of which is given by:

$$\psi_0(x) = \sqrt[4]{\frac{m\omega}{\pi\hbar}}e^{-\frac{m\omega x^2}{2\hbar}} \equiv \alpha e^{-\frac{y^2}{2}}$$

Where, for further simplicity, we have introduced $\alpha = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}$ and dimensionless variable $y = \sqrt{\frac{m\omega}{\hbar}}x$.

a) *(5 p.)* Using explicit definition of the raising ladder operator:

$$\hat{a}_+ = \frac{1}{\sqrt{2\hbar\omega m}}(-i\hat{p} + m\omega\hat{x}) \equiv \frac{1}{\sqrt{2}}\left(-\frac{\mathrm{d}}{\mathrm{d}y} + y\right)$$

Derive expression for the first excited state wave function ψ_1 and check its orthogonality to ψ_0.

b) *(20 p.)* Compute $\langle x\rangle, \langle p\rangle, \langle x^2\rangle$ and $\langle p^2\rangle$, for the states ψ_0 and ψ_1 by explicit integration.

c) *(5 p.)* Check the uncertainty principle for these two states.

d) *(5 p.)* Compute expectation values of the kinetic energy $\langle T\rangle$ and the potential energy $\langle V\rangle$. Check these to sum up to $\langle H\rangle$.

Exercise 2. Harmonic oscillator: power series method (40 points)

The quantum harmonic oscillator problem can be solved using the power series method. One starts with the stationary Schrödinger equation ($\psi'' \equiv \frac{d^2\psi}{dx^2}$):

$$-\frac{\hbar^2}{2m}\psi''(x) + \frac{1}{2}m\omega^2 x^2\psi(x) = E\,\psi(x)$$

a) *(10 p.)* To simplify the initial problem, rewrite the equation using the dimensionless quantities

$$y = \sqrt{\frac{m\omega}{\hbar}}x, \quad \varepsilon = E/\hbar\omega$$

Further on, define $\varphi(y) = c\psi(x)$ and find c, such that $\varphi(y)$ is normalized.

b) *(10 p.)* We can now explicitly isolate the asymptotic behaviour of the unknown function:

$$\varphi(y) = h(y)\, e^{-\frac{y^2}{2}}$$

and derive the following equation on $h(y)$:

$$h'' - 2yh' + (2\varepsilon - 1)h = 0$$

At this point, assume that $h(y)$ can be written as an infinite power series in y:

$$h(y) = \sum_{m=0}^{\infty} a_m\, y^m$$

Derive the recurrence relation between the coefficients a_m and show that there are two sets of independent solutions (*even* and *odd*).

c) *(15 p.)* Prove that, in order for the wave function to be finite and normalizable, one has to imply that the infinite series must be "cut off" at some finite integer n: $a_{m>n} = 0$.
(*Hint*: consider Maclaurin expansion of e^{y^2} and compare it to the series behaviour for large y).

d) *(5 p.)* Using the previous conclusion, show that the energy is quantized $E_n = (n + \frac{1}{2})\hbar\omega$.

The obtained polynomials $h_n(y)$ are proportional to *Hermite polynomials* $H_n(y)$. The orthonormal set of solutions of the initial stationary Schrödinger equation then reads:

$$\psi_n(x) = \left(2^n n! \sqrt{\frac{\pi\hbar}{m\omega}}\right)^{-\frac{1}{2}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) e^{-\frac{m\omega x^2}{2\hbar}}, \quad n = 0, 1, 2, ...$$

Exercise 3. Fourier transform (25 points)

We define the (spatial) Fourier transform of a wave function $\Psi(x,t)$, and its corresponding inverse transform as:

$$\Phi(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x,t) e^{-ikx}\, dx$$
$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(k,t) e^{ikx}\, dk$$

Using this definition, calculate the Fourier transforms of the following functions:

a) *(2 p.)* $\Psi(x) = \delta(x)$ and $\Psi(x) = \delta(x - x_0)$

b) *(2 p.)* $\Psi(x) = a = \text{const}$

c) *(4 p.)* $\Psi(x) = \cos(x)$

d) *(7 p.)* $\Psi(x) = \begin{cases} 1 - |x|, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$

e) *(10 p.)* Apply the Fourier transformation to write down the Schrödinger equation representation for the quantum harmonic oscillator in the k-domain.

Hint: use the identity:

$$\int_{-\infty}^{\infty} x e^{ax}\, dx = \frac{\partial}{\partial a} \int_{-\infty}^{\infty} e^{ax}\, dx$$