APPENDIX : LINEAR ALGEBRA

⇒ A.1 VECTORS

- VECTOR SPACE

VECTOR SPACE CONSISTS OF VECTORS $|\alpha\rangle, |\beta\rangle, |\gamma\rangle, \ldots$

SCALARS a, b, c

WHICH IS CLOSED UNDER OPERATIONS OF VECTOR ADDITION & SCALAR MULTIPLICATION

↳ i.e. WHEN WE PERFORM THESE OPERATIONS ON ANY MEMBER OF VECTOR SPACE, WE STAY WITHIN VECTOR SPACE

① VECTOR ADDITION

↳ SUM OF 2 VECTORS IS VECTOR.

$$|\alpha\rangle + |\beta\rangle = |\gamma\rangle$$

↳ ASSOCIATIVE

$$\left(|\alpha\rangle + |\beta\rangle\right) + |\gamma\rangle = |\alpha\rangle + \left(|\beta\rangle + |\gamma\rangle\right)$$

↳ NULL VECTOR $|0\rangle$

$$|\alpha\rangle + |0\rangle = |\alpha\rangle$$

↳ $\forall\ |\alpha\rangle$: $\exists$ INVERSE $|-\alpha\rangle$

$$|\alpha\rangle + |-\alpha\rangle = |0\rangle$$

↳ COMMUTATIVE

$$|\alpha\rangle + |\beta\rangle = |\beta\rangle + |\alpha\rangle$$

② SCALAR MULTIPLICATION

$$a|\alpha\rangle = |\gamma\rangle$$ ANOTHER VECTOR

↳ DISTRIBUTIVE w.r.t. VECTOR ADDITION

$$a\left(|\alpha\rangle + |\beta\rangle\right) = a|\alpha\rangle + a|\beta\rangle$$

↳ DISTRIBUTIVE w.r.t. SCALAR ADDITION

$$(a+b)|\alpha\rangle = a|\alpha\rangle + b|\alpha\rangle$$

↳ ASSOCIATIVE w.r.t. SCALAR MULTIPLICATION

$$a\left(b|\alpha\rangle\right) = (a.b)|\alpha\rangle$$

↳ $0|\alpha\rangle = |0\rangle$

$1|\alpha\rangle = |\alpha\rangle$

↳ $|-\alpha\rangle = (-1)|\alpha\rangle \equiv -|\alpha\rangle$

LINEAR COMBINATIONS - BASIS VECTORS

↳ LINEAR COMBINATION OF VECTORS $|\alpha\rangle, |\beta\rangle, |\gamma\rangle, \dots$

$$a|\alpha\rangle + b|\beta\rangle + c|\gamma\rangle + \cdots$$

↳ VECTOR $|\lambda\rangle$ IS LINEARLY INDEPENDENT OF $|\alpha\rangle, |\beta\rangle, |\gamma\rangle, \dots$

IF IT CANNOT BE WRITTEN AS LINEAR COMBINATION

↳ SET OF LINEARLY INDEPENDENT VECTORS

THAT SPANS WHOLE VECTOR SPACE IS A BASIS

(e.g. VECTORS IN 3 DIM: $\hat{\imath}, \hat{\jmath}, \hat{k}$: BASIS).

EVERY OTHER VECTOR CAN BE WRITTEN AS

LINEAR COMBINATION OF BASIS VECTORS.

DIMENSION OF SPACE : # BASIS VECTORS

↳ BASIS : $|e_1\rangle, |e_2\rangle, \dots, |e_n\rangle$

$\forall$ VECTOR : $|\alpha\rangle = a_1|e_1\rangle + a_2|e_2\rangle + \cdots + a_n|e_n\rangle$

$|\alpha\rangle \Leftrightarrow (a_1, \dots, a_n)$ COMPONENTS OF VECTOR

(e.g. $\bar{a} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$)

(a_x, a_y, a_z) COMP. OF 3-DIM VECTOR

↳ $|\alpha\rangle + |\beta\rangle \Leftrightarrow (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$

$c|\alpha\rangle \Leftrightarrow (ca_1, ca_2, \dots, ca_n)$

$|0\rangle \Leftrightarrow (0, 0, \dots, 0)$

$|-\alpha\rangle \Leftrightarrow (-a_1, -a_2, \dots, -a_n)$

⇒ A.2 INNER PRODUCTS

↳ IN 3-DIM : DOT PRODUCT OF 2 VECTORS

$\vec{a}\ (a_x, a_y, a_z)$

$\vec{b}\ (b_x, b_y, b_z)$

$$\vec{a}.\vec{b} = a_x b_x + a_y b_y + a_z b_z$$

↳ GENERALIZATION TO n-DIM VECTOR SPACE

INNER PRODUCT OF 2 VECTORS $|\alpha\rangle$ AND $|\beta\rangle$

$$\langle\alpha|\beta\rangle$$

↳ PROPERTIES

$$\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$$

$$\langle\alpha|\alpha\rangle \geq 0$$

$$\langle\alpha|\alpha\rangle = 0 \Leftrightarrow |\alpha\rangle = |0\rangle$$

$$\langle\alpha|\left(b|\beta\rangle + c|\gamma\rangle\right) = b\langle\alpha|\beta\rangle + c\langle\alpha|\gamma\rangle$$

↳ VECTOR SPACE + INNER PRODUCT : INNER PRODUCT SPACE

↳ NORM OF VECTOR (REAL)

$$\| \alpha \| \equiv \sqrt{\langle \alpha | \alpha \rangle} \qquad (\text{"LENGTH"})$$

(IN 3 DIM $\quad |\bar{r}| = \sqrt{\bar{r} \cdot \bar{r}}$)

UNIT VECTOR $| e_1 \rangle$

$$\| e_1 \| = 1 \qquad (\text{CALLED TO BE NORMALIZED})$$

↳ 2 VECTORS ARE ORTHOGONAL

IF $\quad \langle \alpha_i | \alpha_j \rangle = 0 \qquad (i \neq j)$

↳ SET IS ORTHONORMAL

IF $\quad \langle \alpha_i | \alpha_j \rangle = \delta_{ij}$

↳ CONVENIENT TO WORK WITH ORTHONORMAL BASIS

$$|e_1\rangle, \ldots, |e_n\rangle$$

$$\langle e_i | e_j \rangle = \delta_{ij}$$

$$|\alpha\rangle = a_1 |e_1\rangle + \ldots + a_n |e_n\rangle$$

$$|\beta\rangle = b_1 |e_1\rangle + \ldots + b_n |e_n\rangle$$

$$\langle e_i | \alpha \rangle = a_i$$

$$\langle \alpha | e_i \rangle = a_i^*$$

$$\langle \alpha | \beta \rangle = \langle \alpha | \left(b_1 | e_1 \rangle + \dots + b_n | e_n \rangle \right)$$

$$\boxed{\langle \alpha | \beta \rangle = a_1^* b_1 + \dots + a_n^* b_n}$$

SEE LATER:
FORMALLY INTRODUCE NOTATION (DUE TO P.A.M DIRAC)

$$\langle \alpha | = a_1^* \langle e_1 | + \dots + a_n^* \langle e_n |$$

"BRA"

$$| \beta \rangle = b_1 | e_1 \rangle + \dots + b_n | e_n \rangle$$

"KET"

$$\langle \alpha | \beta \rangle = a_1^* b_1 + \dots + a_n^* b_n$$

BRACKET

↳ NORM OF VECTOR

NORM SQUARED $\langle \alpha | \alpha \rangle = |a_1|^2 + \dots + |a_n|^2$

$\hookrightarrow$ "ANGLE" OF VECTOR

- 3 DIM: $\cos\theta = \dfrac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|}$

- GENERALIZATION

$\langle\alpha|\beta\rangle$ COMPLEX NUMBER

BUT $$\boxed{|\langle\alpha|\beta\rangle|^2 \leq \langle\alpha|\alpha\rangle \cdot \langle\beta|\beta\rangle}$$

SCHWARTZ INEQUALITY

$\hookrightarrow$ DEFINE ANGLE: $\cos\theta \equiv \sqrt{\dfrac{\langle\alpha|\beta\rangle\langle\beta|\alpha\rangle}{\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle}}$

⇒ A.3 MATRICES

- LINEAR TRANSFORMATION $\hat{T}$

↳ $\hat{T}$ TRANSFORMS ONE VECTOR $|\alpha\rangle$ INTO ANOTHER VECTOR $|\alpha'\rangle$

$$|\alpha'\rangle = \hat{T}\,|\alpha\rangle$$

↳ OPERATION IS LINEAR

$$\hat{T}\left(a|\alpha\rangle + b\,|\beta\rangle\right) = a\left(\hat{T}|\alpha\rangle\right) + b\left(\hat{T}\,|\beta\rangle\right)$$

↳ OPERATION ON BASIS VECTORS (ORTHONORMAL)

e.g. $\hat{T}\,|e_1\rangle = T_{11}\,|e_1\rangle + T_{21}\,|e_2\rangle + \dots + T_{n1}|e_n\rangle$

$$\therefore \boxed{\hat{T}\,|e_j\rangle = \sum_{i=1}^{n} T_{ij}\,|e_i\rangle} \qquad j = 1 \dots n$$

↳ $T_{ij} = \langle e_i|\,\hat{T}\,|e_j\rangle$

↳ OPERATION ON ARBITRARY VECTOR

$$|\alpha\rangle = a_1\,|e_1\rangle + \dots + a_n|e_n\rangle$$

$$= \sum_{j=1}^{n} a_j\,|e_j\rangle$$

$$\hat{T}\,|\alpha\rangle = \sum_{j=1}^{n} a_j\,\hat{T}\,|e_j\rangle$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{n} a_j\,T_{ij}\,|e_i\rangle$$

$$\hat{T}\,|\alpha\rangle = |\alpha'\rangle = \sum_{i=1}^{n} a_i' \,|e_i\rangle$$

$$\Big\| \quad a_i' = \sum_{j=1}^{n} T_{ij}\, a_j$$

- MATRIX

↳ n^2 ELEMENTS $(T_{11}, T_{12}, \dots, T_{1n}, T_{21}, T_{22}, \dots, \dots, T_{nn})$ UNIQUELY CHARACTERIZE A LINEAR TRANSFORMATION $\hat{T}$

$$T_{ij} = \langle e_i | \hat{T} | e_j \rangle$$

↓

FORM ELEMENTS OF A $n \times n$ MATRIX

n ROWS n COLUMNS

$$T = \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & & & \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{pmatrix}$$

↳ SUM $\underbrace{(\hat{S} + \hat{T})}_{\hat{U}}\,|\alpha\rangle = \hat{S}\,|\alpha\rangle + \hat{T}\,|\alpha\rangle$

$$\Updownarrow$$

MATRICES : $U = S + T$

$$U_{ij} = S_{ij} + T_{ij}$$

↳ PRODUCT OF LIN. TF.

$$|\alpha\rangle \xrightarrow{\hat{T}} |\alpha'\rangle \xrightarrow{\hat{S}} |\alpha''\rangle$$

$$|\alpha\rangle \xrightarrow{\hat{U}} |\alpha''\rangle$$

$$|\alpha'\rangle = \hat{T}\,|\alpha\rangle$$

$$|\alpha''\rangle = \hat{S}\,|\alpha'\rangle = \hat{S}\,\hat{T}\,|\alpha\rangle$$

$$|\alpha''\rangle = \hat{U}\,|\alpha\rangle$$

↳ $\hat{U} = \hat{S}\,\hat{T}$

IN COMPONENTS.

$$a''_i = \sum_{j=1}^{n} S_{ij}\, a'_j$$

$$= \sum_{j=1}^{n} S_{ij} \sum_{k=1}^{n} T_{jk}\, a_k.$$

$$= \sum_{k=1}^{n} \left(\sum_{j=1}^{n} S_{ij}\, T_{jk} \right) a_k$$

$$= \sum_{k=1}^{n} U_{ik}\, a_k.$$

$$U_{ik} = \sum_{j=1}^{n} S_{ij}\, T_{jk}$$

$$\Updownarrow$$

$U = S\,T$ MATRIX RELATION

MATRIX MULTIPLICATION

$$U = S\,T$$

U_{ij} ELEMENT : ROW i OF S

MULTIPLY WITH COLUMN j OF T

$$\begin{pmatrix} U_{11} & \cdots & U_{1j} & \cdots & U_{1n} \\ \vdots & & & & \\ U_{i1} & \cdots & U_{ij} & \cdots & U_{in} \\ \vdots & & & & \\ U_{n1} & \cdots & U_{nj} & \cdots & U_{nn} \end{pmatrix} = \begin{pmatrix} S_{11} & \cdots & & S_{1n} \\ \vdots & & & \vdots \\ \boxed{S_{i1} \quad S_{i2} \quad \cdots \quad S_{in}} & & & \\ \vdots & & & \vdots \\ S_{n1} & & & S_{nn} \end{pmatrix} \begin{pmatrix} T_{11} & \cdots & T_{1j} & \cdots & T_{1n} \\ \vdots & & \vdots & & \\ T_{n1} & \cdots & T_{nj} & \cdots & T_{nn} \end{pmatrix}$$

$$U \qquad = \qquad S \qquad\qquad T$$

↳ COLUMN MATRIX

VECTOR $|\alpha\rangle \iff a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

↳ COMPONENTS OF VECTOR.

$$|\alpha'\rangle = \hat{T}\,|\alpha\rangle$$

$$\Updownarrow$$

IN MATRIX NOTATION : $a' = T\,a$

$$\begin{pmatrix} a'_1 \\ \vdots \\ a'_n \end{pmatrix} = \begin{pmatrix} T_{11} & \cdots & T_{1n} \\ \vdots & & \\ T_{n1} & \cdots & T_{nn} \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

PROPERTIES OF MATRICES

↳ TRANSPOSE $\tilde{T}$ OF T

$$\tilde{T}_{ij} = T_{ji}$$ COLUMNS & ROWS INTERCHANGED

$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \implies \tilde{a} = (a_1 \dots a_n)$$

COLUMN MATRIX

ROW MATRIX.

↳ SQUARE MATRIX IS SYMMETRIC

IF $\tilde{T} = T$

ANTI-SYMM. IF $\tilde{T} = -T$

↳ COMPLEX CONJUGATE T^* OF MATRIX T

$$T^* = \begin{pmatrix} T_{11}^* & \dots & T_{1n}^* \\ \vdots & & \\ T_{n1}^* & \dots & T_{nn}^* \end{pmatrix}$$

$$a^* = \begin{pmatrix} a_1^* \\ \vdots \\ a_n^* \end{pmatrix}$$

↳ <u>REAL</u> MATRIX , <u>IMAGINARY</u> MATRIX

REAL $T^* = T$

IMAG $T^* = -T$

↳ <u>HERMITIAN CONJUGATE</u> T^+ OF MATRIX T

(ALSO CALLED ADJOINT)

$$\boxed{T^+ \equiv \tilde{T}^*}$$

$$T^+ = \begin{pmatrix} T_{11}^* & T_{21}^* & \cdots & T_{m1}^* \\ T_{12}^* & T_{22}^* & & \\ \vdots & & \ddots & \\ T_{1m}^* & & & T_{mm}^* \end{pmatrix}$$

$$a^+ = \begin{pmatrix} a_1^* & a_2^* & \cdots & a_m^* \end{pmatrix}$$
$$= \tilde{a}^*$$

↳ <u>HERMITIAN MATRIX</u>

T is <u>HERMITIAN</u> if $\boxed{T^+ = T}$

↳ <u>ANTI-HERMITIAN</u>

T is ANTI-HERMITIAN if $T^+ = -T$

↳ EXAMPLES OF HERMITIAN MATRICES

2×2 $\quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

3×3 $\quad \begin{pmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{pmatrix}$

↳ INNER PRODUCT

$$\langle \alpha | \beta \rangle = \sum_{i=1}^{n} a_i^* b_i$$

$$\boxed{\langle \alpha | \beta \rangle = a^{+} b}$$

$$= (a_1^* \dots a_n^*) \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

↳ MATRIX MULTIPLICATION IN GENERAL DOES NOT COMMUTE

i.e. $ST \neq TS$

DIFFERENCE $\Rightarrow$ COMMUTATOR $\boxed{[S, T] \equiv ST - TS}$

↳ TRANSPOSE OF PRODUCT

$$\widetilde{(ST)} = \tilde{T} \tilde{S}$$

PROOF: $(ST)_{ij} = \sum_{k=1}^{n} S_{ik} T_{kj}$

$$(\widetilde{ST})_{ij} = (ST)_{ji} = \sum_{k=1}^{n} S_{jk} T_{ki}$$

$$= \sum_{k=1}^{n} \widetilde{T}_{ik} \widetilde{S}_{kj}$$

$$= (\widetilde{T}\widetilde{S})_{ij}$$

↳ HERMITIAN CONJUGATE

$$(ST)^{+} = T^{+} S^{+}$$

↳ UNIT MATRIX $I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

$$I_{ij} = \delta_{ij}$$

↳ INVERSE T^{-1} OF SQUARE MATRIX T

$$T^{-1} T = T T^{-1} = I$$

MATRIX HAS INVERSE $\Longleftrightarrow$ IT HAS A DETERMINANT $\neq 0$
$\det T \neq 0$

MATRIX WHICH HAS NO INVERSE IS SINGULAR

↳ DETERMINANT $\det T$ e.g. $\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \rightarrow \det T = T_{11} T_{22} - T_{12} T_{21}$

↳ INVERSE OF PRODUCT

$$(ST)^{-1} = T^{-1} S^{-1}$$

PROOF $(ST)^{-1}(ST) = I$

$$= T^{-1} \underbrace{S^{-1} S}_{I} T$$

$$= T^{-1} T \overset{!}{=} I$$

↳ <u>UNITARY</u> MATRIX U

$$\boxed{U^{+} = U^{-1}}$$

↳ UNITARY TRANSFORMATIONS PRESERVE INNER PRODUCTS !

MATRIX NOTATION

$$|\alpha'\rangle = \hat{U} |\alpha\rangle \iff a' = U a$$

$$|\beta'\rangle = \hat{U} |\beta\rangle \iff b' = U b$$

$$\langle \beta' | \alpha' \rangle = b'^{+} a'$$

$$= (b^{+} U^{+}) \cdot (U a)$$

$$= b^{+} \underbrace{U^{-1} U}_{I} a$$

$$= b^{+} a = \langle \beta | \alpha \rangle \quad \blacksquare \text{ QED}$$

⇒ A.4 CHANGE OF BASIS

↳ OLD BASIS $|e_i\rangle$ $i=1\dots n$

NEW BASIS $|f_i\rangle$ $i=1\dots n$

$$|e_j\rangle = \sum_{i=1}^{n} S_{ij}\,|f_i\rangle \qquad j=1\dots n$$

↳ $$a^e = \begin{pmatrix} a_1^e \\ \vdots \\ a_n^e \end{pmatrix}$$

↳ COMPONENTS OF VECTOR w.r.t. BASIS e

$$a^f = \begin{pmatrix} a_1^f \\ \vdots \\ a_n^f \end{pmatrix}$$

↳ COMPONENTS OF VECTOR w.r.t. BASIS f

$$a^f = S\,a^e$$

↳ MATRIX T ALSO CHANGES WHEN CHANGING BASIS

BASIS e: TRANSFORMATION $a^{e\prime} = T^e a^e$

$a^e = S^{-1} a^f$ RELATION BETWEEN ORIGINAL VECTORS IN BOTH BASES

$a^{f\prime} = S\,a^{e\prime}$ RELATION BETWEEN TRANSFORMED VECTORS IN BOTH BASES

$$a^{f'} = S \cdot T^e a^e$$

$$= S T^e S^{-1} a^f$$

$$\equiv T^f a^f \quad \text{TRANSFORMATION IN BASIS } f$$

$$\underline{\underline{T^f = S T^e S^{-1}}}$$

T^e, T^f ARE '<u>SIMILAR</u>' MATRICES

i.e. REPRESENT SAME LINEAR TRANSFORMATION

w.r.t. DIFFERENT BASES

↳ IF BASIS e IS ORTHONORMAL

BASIS f $(a^f = S a^e)$ IS ORTHONORMAL

$$\Updownarrow$$

MATRIX S IS UNITARY

<u>PROOF</u> $|\alpha\rangle = \sum_i a_i |e_i\rangle$ $\qquad a^e = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}$

$|\beta\rangle = \sum_i b_i |e_i\rangle$ $\qquad b^e = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

↳ IN ORTHONORMAL BASIS $\langle \alpha | \beta \rangle = a^{e+} b^e$

↳ IF $\{|f_i\rangle\}$ IS ORTHONORMAL $\langle \alpha | \beta \rangle = a^{f+} b^f$

$$b^f = S b^e \quad , \quad a^f = S a^e \Rightarrow a^{f+} = a^{e+} S^+$$

↳ DETERMINANT OF MATRIX IS LEFT UNCHANGED BY CHANGE OF BASIS

$$T^f = S T^e S^{-1}$$

$$\det T^f = \det (S T^e S^{-1})$$

$$\downarrow \qquad \det(AB) = \det A \,.\, \det B$$

$$= \det S \,.\, \det T^e \,.\, \det S^{-1}$$

$$= \underbrace{\det(\underbrace{S S^{-1}}_{I})}_{1} \,.\, \det T^e$$

$$= \det T^e$$

↳ TRACE OF MATRIX IS LEFT UNCHANGED BY CHANGE OF BASIS

$$\mathrm{Tr}(T) = \sum_{i=1}^{n} T_{ii} \qquad \text{(SUM OF DIAGONAL ELEMENTS)}$$

SHOW $\mathrm{Tr}(T_1 T_2) = \mathrm{Tr}(T_2 T_1)$ (HOMEWORK PROBLEM)

$$T^f = S T^e S^{-1}$$

$$\mathrm{Tr}(T^f) = \mathrm{Tr}(S T^e S^{-1})$$

$$= \mathrm{Tr}(S^{-1} S T^e)$$

$$= \mathrm{Tr}(T^e)$$

⇒ A.5 EIGENVECTORS & EIGENVALUES

- VECTOR $|\alpha\rangle$ WHICH TRANSFORMS INTO A SCALAR MULTIPLE OF ITSELF

$$\hat{T}|\alpha\rangle = \lambda|\alpha\rangle$$

SUCH A VECTOR $|\alpha\rangle$ IS CALLED AN EIGENVECTOR OF $\hat{T}$

λ IS CALLED ITS EIGENVALUE

- IN MATRIX NOTATION

$$Ta = \lambda a \qquad (a \text{ IS NON-ZERO})$$

$$(T - \lambda \underbrace{I}_{\text{IDENTITY MATRIX}})\, a = \underbrace{0}_{\text{ZERO MATRIX}}$$

$a \neq 0 \Rightarrow$ ONLY POSSIBLE IF $T - \lambda I$ IS SINGULAR

$$\Downarrow$$

$$\det(T - \lambda I) = 0$$

↓

POLYNOMIAL OF DEGREE n FOR EIGENVALUES. (CHARACTERISTIC EQUATION)

↓

HAS n COMPLEX ROOTS

(IF SOME EIGENVALUES ARE SAME ⇒ CALLED DEGENERATE

- EXAMPLE

DETERMINE EIGENVALUES & EIGENVECTORS OF

$$M = \begin{pmatrix} 2 & 0 & -2 \\ -2i & i & 2i \\ 1 & 0 & -1 \end{pmatrix}$$

↳ $\det(M - \lambda I) = 0$

$$\Updownarrow$$

$$\begin{vmatrix} 2-\lambda & 0 & -2 \\ -2i & i-\lambda & 2i \\ 1 & 0 & -1-\lambda \end{vmatrix} = 0$$

$$\Updownarrow$$

$$(2-\lambda)(i-\lambda)(-1-\lambda) + 2(i-\lambda) = 0$$

$$\Updownarrow$$

$$\underline{\underline{(i-\lambda)\,\lambda\,(\lambda-1) = 0}}$$

ROOTS: $\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = i$ EIGENVALUES

↳ EIGENVALUE $\lambda_1 = 0$ $\Rightarrow$ EIGENVECTOR $a^{(1)}$

$$\begin{pmatrix} 2 & 0 & -2 \\ -2i & i & 2i \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0 \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$$\begin{cases} 2a_1 - 2a_3 = 0 \\ -2i\,a_1 + i\,a_2 + 2i\,a_3 = 0 \\ a_1 - a_3 = 0 \end{cases}$$

$$a_1 = a_3 \quad , \quad a_2 = 0$$

WE CAN TAKE ANY VALUE OF a_1 $\sim$ MULTIPLE IS STILL EIGENVECTOR

$\downarrow$

CHOOSE $a_1 = 1$

$$\lambda_1 = 0 \quad \Longleftrightarrow \quad a^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

EIGENVALUE EIGENVECTOR

$\hookrightarrow$ EIGENVALUE $\lambda_2 = 1$ $\Rightarrow$ EIGENVECTOR $a^{(2)}$

$$\begin{pmatrix} 2 & 0 & -2 \\ -2i & i & 2i \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$$\begin{cases} a_1 - 2a_3 = 0 \\ -2i\,a_1 + (i-1)\,a_2 + 2i\,a_3 = 0 \\ a_1 - 2a_3 = 0 \end{cases}$$

$$a_1 = 2a_3 \quad \Rightarrow \quad a_2 = \frac{2i}{i-1}\,a_3 = (1-i)\,a_3$$

CHOOSE $a_3 = 1$

$$\lambda_2 = 1 \iff a^{(2)} = \begin{pmatrix} 2 \\ 1-i \\ 1 \end{pmatrix}$$

↳ EIGENVALUE $\lambda_3 = i$ ⇒ EIGENVECTOR $a^{(3)}$

$$\begin{pmatrix} 2 & 0 & -2 \\ -2i & i & 2i \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = i \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$$\begin{cases} (2-i)\, a_1 - 2a_3 = 0 \\ -2i\, a_1 + 2i\, a_3 = 0 \\ a_1 - (1+i)\, a_3 = 0 \end{cases}$$

SOLUTION $a_1 = a_3 = 0$, a_2 UNDETERMINED

CHOOSE $a_2 = 1$

$$\lambda_3 = i \iff a^{(3)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

- USE EIGENVECTORS AS <u>BASIS</u> (ONLY POSSIBLE IF THEY SPAN THE WHOLE VECTOR SPACE)

$$\hat{T}\,|f_1\rangle = \lambda_1\,|f_1\rangle$$
$$\hat{T}\,|f_2\rangle = \lambda_2\,|f_2\rangle$$
$$\vdots$$
$$\hat{T}\,|f_n\rangle = \lambda_n\,|f_n\rangle$$

IN THIS BASIS $\quad T = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

$\downarrow$

T IS DIAGONAL

& NORMALIZED EIGENVECTORS ARE

$$f_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad f_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \quad \cdots \quad f_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

↳ CHANGE BASIS TO MAKE $\hat{T}$ DIAGONAL

$\hat{T}$ IS CALLED DIAGONALIZABLE

↳ $\quad T^f = S\,T^e\,S^{-1}$

T^f ⇑ IN NEW BASIS DIAGONAL

T^e ⇑ IN OLD BASIS

CHOOSE $\underline{\underline{(S^{-1})_{ij} = \left(a^{(j)}\right)_i}}$

↳ COLUMNS ARE EIGENVECTORS IN OLD BASIS.

- PREVIOUS EXAMPLE

$$S^{-1} = \begin{array}{ccc} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1-i & 1 \\ 1 & 1 & 0 \end{pmatrix} \\ \Uparrow \quad\; \Uparrow \quad\; \Uparrow \\ a^{(1)} \quad a^{(2)} \quad a^{(3)} \end{array} \Rightarrow S = \begin{pmatrix} -1 & 0 & 2 \\ 1 & 0 & -1 \\ i-1 & 1 & 1-i \end{pmatrix}$$

$$M S^{-1} = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 1-i & i \\ 0 & 1 & 0 \end{pmatrix}$$

$$S M S^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix}$$

- IN PRACTISE : EVERY HERMITIAN MATRIX / EVERY UNITARY MATRIX IS DIAGONALIZABLE

2 MATRICES CAN BE SIMULTANEOUSLY DIAGONALIZED BY SAME SIMILARITY MATRIX S

$\Updownarrow$

BOTH MATRICES COMMUTE

SHOW (HOMEWORK PROBLEM)

⇒ A.6 EIGENVALUES & EIGENVECTORS OF HERMITIAN MATRIX

$$\hat{T}^{+} = \hat{T}$$

- EIGENVALUES OF $\hat{T}$ ARE **REAL**

$$\hat{T}\,|\alpha\rangle = \lambda\,|\alpha\rangle \qquad |\alpha\rangle \neq |0\rangle$$

$$\langle\alpha|\hat{T}|\alpha\rangle = \lambda\,\langle\alpha|\alpha\rangle$$

NOTE: $\langle\beta|\hat{T}|\alpha\rangle = b^{+}\,T\,a$

$$\langle\alpha|\hat{T}^{+}|\beta\rangle = a^{+}\,T^{+}\,b$$

$$= \left(b^{+}\,T\,a\right)^{+}$$

$$\boxed{\langle\alpha|\hat{T}^{+}|\beta\rangle = \langle\beta|\hat{T}|\alpha\rangle^{*}}$$

FOR $|\beta\rangle = |\alpha\rangle$

$$\hat{T} = \hat{T}^{+}$$

$$\langle\alpha|\hat{T}|\alpha\rangle = \langle\alpha|\hat{T}|\alpha\rangle^{*}$$

$$\lambda\langle\alpha|\alpha\rangle = \lambda^{*}\langle\alpha|\alpha\rangle$$

$$\lambda = \lambda^{*} \Longrightarrow \lambda \text{ IS REAL}$$

- EIGENVECTORS OF $\hat{T}$ BELONGING TO DIFFERENT EIGENVALUES ARE ORTHOGONAL

$$\hat{T}\,|\alpha\rangle = \lambda\,|\alpha\rangle \qquad \lambda \neq \mu$$

$$\hat{T}\,|\beta\rangle = \mu\,|\beta\rangle$$

$$(1) \quad \langle \alpha | \hat{T} | \beta \rangle = \mu \langle \alpha | \beta \rangle$$

$$\langle \beta | \hat{T} | \alpha \rangle = \lambda \langle \beta | \alpha \rangle$$

$$= \langle \alpha | \hat{T}^{+} | \beta \rangle^{*} = \lambda \langle \beta | \alpha \rangle$$

$\Updownarrow$ TAKE $*$

$$(2) \quad \langle \alpha | \hat{T} | \beta \rangle = \lambda^{*} \langle \beta | \alpha \rangle^{*} = \lambda^{*} \langle \alpha | \beta \rangle$$

$$(1) - (2) \implies (\mu - \lambda^{*}) \langle \alpha | \beta \rangle = 0$$

λ IS REAL $\lambda = \lambda^{*}$

AND $\lambda \neq \mu$

$\Updownarrow$

$$\langle \alpha | \beta \rangle = 0$$

ORTHOGONAL ■

- EIGENVECTORS OF $\hat{T}$ SPAN WHOLE VECTOR SPACE