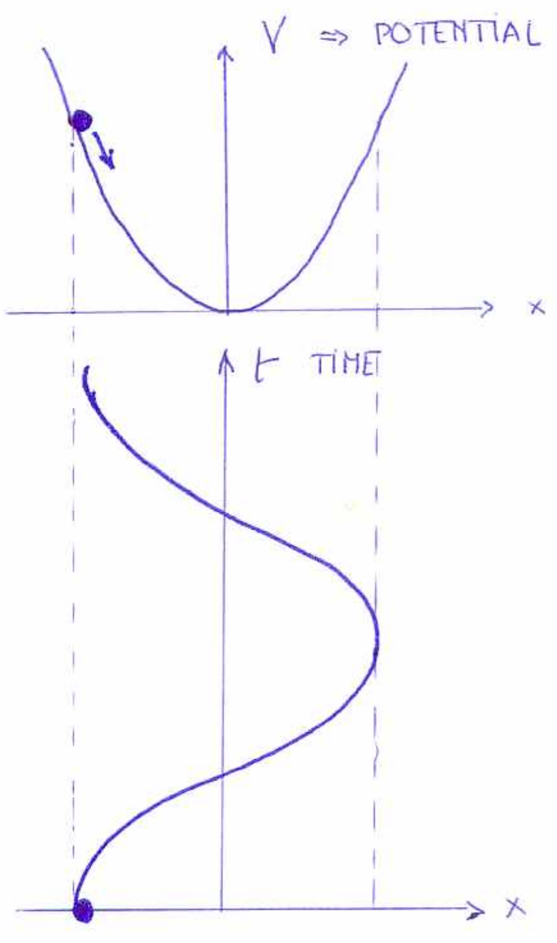
CHAPTER 1 : WAVE FUNCTION

1.1 SCHRÖDINGER EQUATION

- CLASSICAL MECHANICS

e.g. (HARMONIC OSCILLATOR)



$x(t)$ SOLUTION OF NEWTON'S EQ.

$$m \cdot \frac{d^2x}{dt^2} = F = -\frac{\partial V}{\partial x}$$

MASS FORCE POTENTIAL

SOLVE NEWTON'S EQ. , INITIAL CONDITIONS

$\Downarrow$ $\quad x(0)\,,\ v(0)$

POSITION $x(t)$

$\Downarrow$

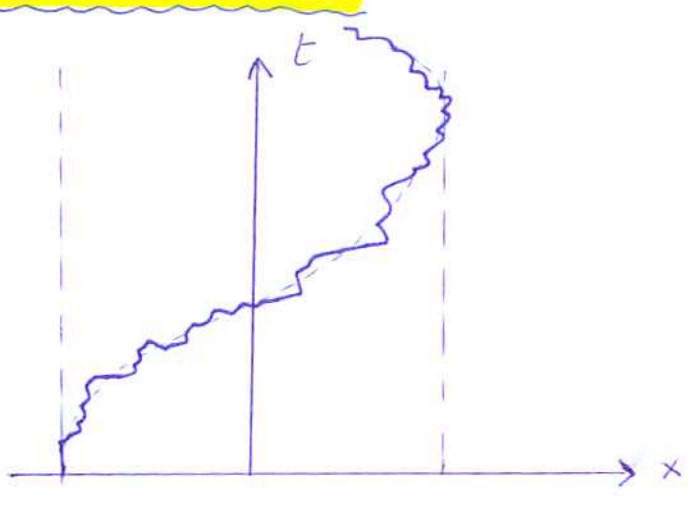
VELOCITY $v(t) = \dfrac{dx}{dt}$

$\Downarrow$

MOMENTUM $p = m v$

KINETIC ENERGY $T = \dfrac{1}{2} m v^2 = \dfrac{p^2}{2m}$

- **QUANTUM MECHANICS**



→ POSITION CAN NOT BE DETERMINED TO ARBITRARY ACCURACY

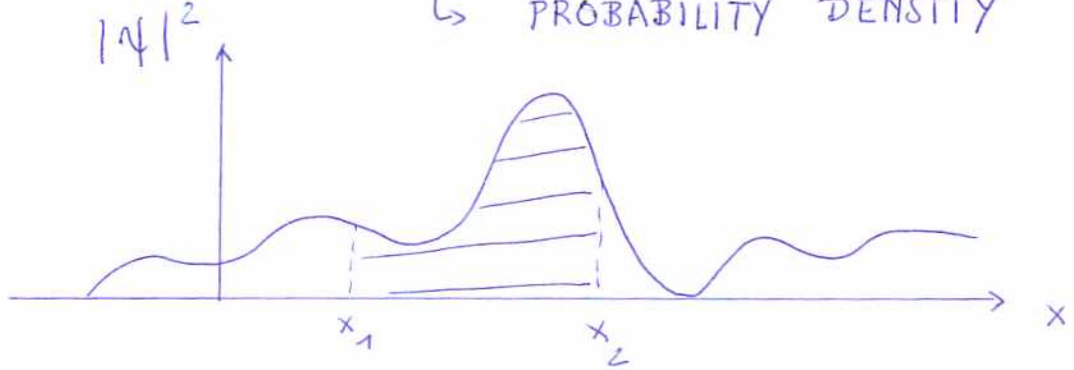
→ FLUCTUATIONS (ON MICROSCOPIC SCALE) AROUND CLASSICAL PATH / TRAJECTORY

→ WE CAN ONLY SPEAK OF PROBABILITY TO FIND $\left\{\begin{array}{l}\text{PARTICLE}\\ \text{OBJECT}\end{array}\right.$ AT A GIVEN TIME t AT A GIVEN POSITION x

→ BASIC CONCEPT : WAVE FUNCTION $\psi(x,t)$

INTERPRETATION : $|\psi(x,t)|^2 = \psi^*(x,t) \cdot \psi(x,t)$

↳ PROBABILITY DENSITY



$$\int_{x_1}^{x_2} dx \, |\psi(x,t)|^2$$ IS PROBABILITY TO FIND $\left|\begin{array}{l}\text{PARTICLE}\\ \text{OBJECT}\end{array}\right.$ AT TIME t

BETWEEN x_1 AND x_2.

→ WAVE FUNCTION SATISFIES SCHRÖDINGER EQ.

$$\boxed{-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi = i\hbar\frac{\partial \psi}{\partial t}}$$

KINETIC ENERGY — POTENTIAL ENERGY — TOTAL ENERGY

SOLUTION $\left[+ \begin{cases}\text{BOUNDARY} \\ \text{INITIAL}\end{cases} \text{CONDITIONS} \quad \psi(x,0) \right]$

↳ $\psi(x,t)$

$$\hbar = \frac{h}{2\pi} = 1.054\,572 \; 10^{-34} \; J.s$$

h : PLANCK's CONSTANT $\Bigg\{$ RECALL : LIGHT OF FREQUENCY ν

↓

ENERGY PACKETS $E = h\nu$ (PHOTON)

→ SMALLNESS OF h : FOR MACROSCOPIC OBJECTS

FLUCTUATIONS AROUND CLASSICAL TRAJECTORY

VERY TINY ⇒ TO GOOD APPROXIMATION :

WE CAN APPLY NEWTON's EQ.

1.3 PROBABILITY

- DISCRETE VARIABLES

GROUP OF PEOPLE N = TOTAL # PERSONS

$j = 0, 1, 2, \ldots$ AGE OF PERSON IN GROUP

$N(j)$ = # PERSONS WITH AGE j

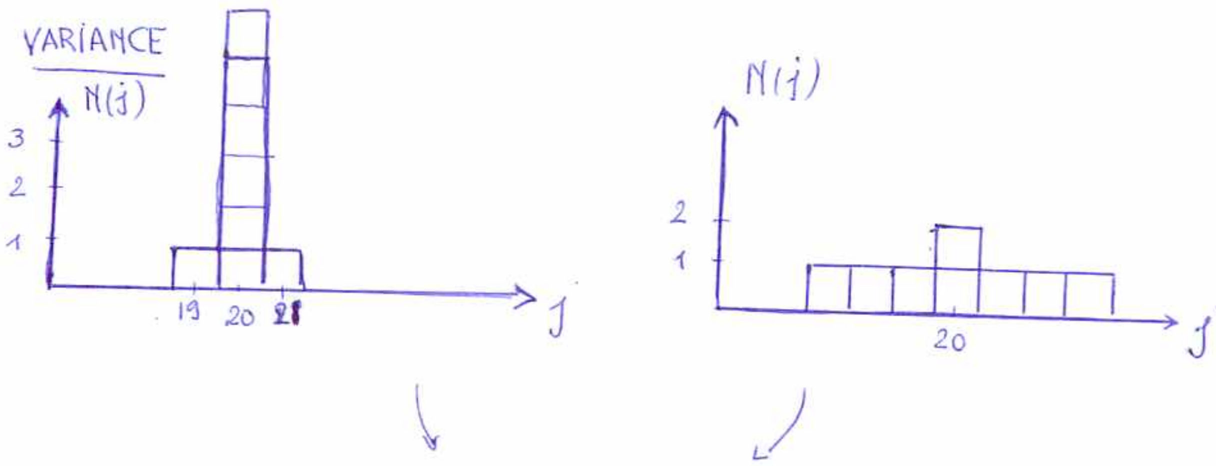
→ PROBABILITY TO FIND PERSON WITH AGE j

$$P(j) = \frac{N(j)}{N}$$

→ AVERAGE AGE

$$\langle j \rangle = \sum_{j=0}^{\infty} j\, P(j) = \frac{1}{N} \sum_{j=0}^{\infty} j\, N(j)$$

→ VARIANCE



2 DISTRIBUTIONS HAVE SAME AVERAGE

BUT DIFFERENT SPREAD

$\Downarrow$

MEASURE OF SPREAD : VARIANCE σ^2

$$\sigma^2 \equiv \langle (j - \langle j \rangle)^2 \rangle$$

$$= \langle j^2 - 2j\langle j \rangle + \langle j \rangle^2 \rangle$$

$$= \langle j^2 \rangle - 2\langle j \rangle^2 + \langle j \rangle^2$$

$$= \langle j^2 \rangle - \langle j \rangle^2$$

→ STANDARD DEVIATION

$$\sigma = \sqrt{\sigma^2} = \sqrt{\langle j^2 \rangle - \langle j \rangle^2}$$

• CONTINUOUS VARIABLES

e.g. HEIGHT OF PERSON x

→ PROBABILITY DENSITY $\rho(x)$

* INFINITESIMAL INTERVAL

PROBABILITY TO FIND SOMEONE WITH HEIGHT BETWEEN x AND $x + dx$

$\hookrightarrow \rho(x)\, dx$

* FINITE INTERVAL

PROBABILITY TO FIND SOMEONE WITH HEIGHT BETWEEN a AND b

$$\hookrightarrow P_{ab} = \int_a^b dx \; \rho(x)$$

* NORMALIZATION

$$1 = \int_{-\infty}^{+\infty} dx \; \rho(x)$$

→ AVERAGE

$$\langle x \rangle = \int_{-\infty}^{+\infty} dx \; x \, \rho(x)$$

$$\langle f(x) \rangle = \int_{-\infty}^{+\infty} dx \; f(x)\, \rho(x).$$

→ VARIANCE

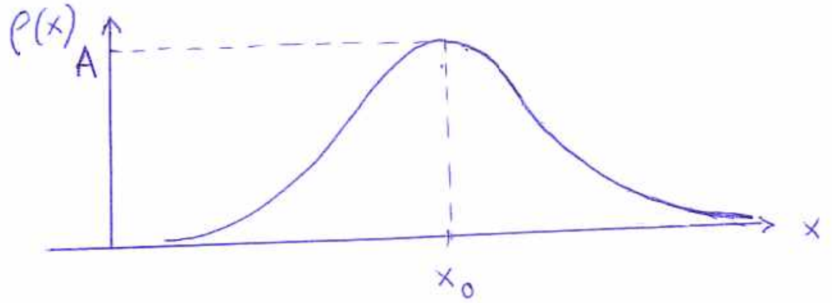
$$\sigma^2 = \langle (x - \langle x \rangle)^2 \rangle$$

$$= \int_{-\infty}^{+\infty} dx \, (x - \langle x \rangle)^2 \, \rho(x)$$

$$= \langle x^2 \rangle - \langle x \rangle^2$$

→ EXAMPLE : GAUSSIAN DISTRIBUTION.

* $\rho(x) = A \, e^{-\lambda (x - x_0)^2}$



* NORMALIZATION

$$\int_{-\infty}^{+\infty} dx \, \rho(x) = 1$$

$$1 = A \int_{-\infty}^{+\infty} dx \, e^{-\lambda (x - x_0)^2} = A \underbrace{\int_{-\infty}^{+\infty} dx' \, e^{-\lambda x'^2}}_{= \sqrt{\frac{\pi}{\lambda}}}$$

$x' = x - x_0$

$dx' = dx$

$\sqrt{\frac{\pi}{\lambda}}$ GAUSSIAN INTEGRAL

$$1 = A \cdot \sqrt{\frac{\pi}{\lambda}}$$

$$\Downarrow$$

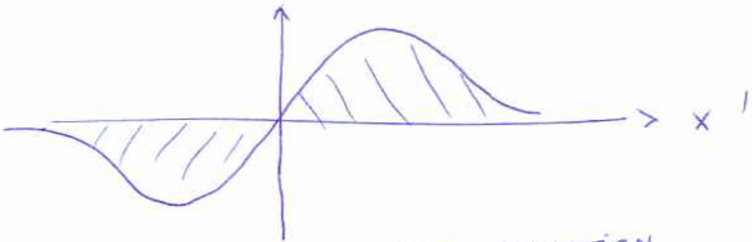
$$\underline{\underline{A = \sqrt{\frac{\lambda}{\pi}}}}$$

* AVERAGE

$$\langle x \rangle = A \int_{-\infty}^{+\infty} dx \quad x \quad e^{-\lambda (x - x_0)^2}$$

$$\underset{x' = x - x_0}{=} A \int_{-\infty}^{+\infty} dx' \left(x' + x_0 \right) e^{-\lambda x'^2}$$

$$= A \left\{ \underbrace{\int_{-\infty}^{+\infty} dx' \, x' \, e^{-\lambda x'^2}}_{0} + x_0 \underbrace{\int_{-\infty}^{+\infty} dx' \, e^{-\lambda x'^2}}_{\sqrt{\frac{\pi}{\lambda}}} \right\}$$



ODD FUNCTION
INTEGRATED BETWEEN
SYMMETRIC INTEGRATION BOUNDS

$$= A x_0 \sqrt{\frac{\pi}{\lambda}}$$

$$\underline{\underline{\langle x \rangle = x_0}}$$

* VARIANCE

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$$

$$= \langle (x - \langle x \rangle)^2 \rangle$$

$$= \langle (x - x_0)^2 \rangle$$

$$= A \int_{-\infty}^{+\infty} dx \; (x - x_0)^2 \; e^{-\lambda (x - x_0)^2}$$

$$x' = x - x_0$$

$$= A \int_{-\infty}^{+\infty} dx' \; x'^2 \; e^{-\lambda x'^2}$$

$$= A \; \frac{1}{2} \sqrt{\frac{\pi}{\lambda}} \; \frac{1}{\lambda}$$

$$\underline{\underline{\sigma^2 = \frac{1}{2\lambda}}}$$

$$\underline{\underline{\sigma = \frac{1}{\sqrt{2\lambda}}}} \quad , \quad \lambda = \frac{1}{2\sigma^2}$$

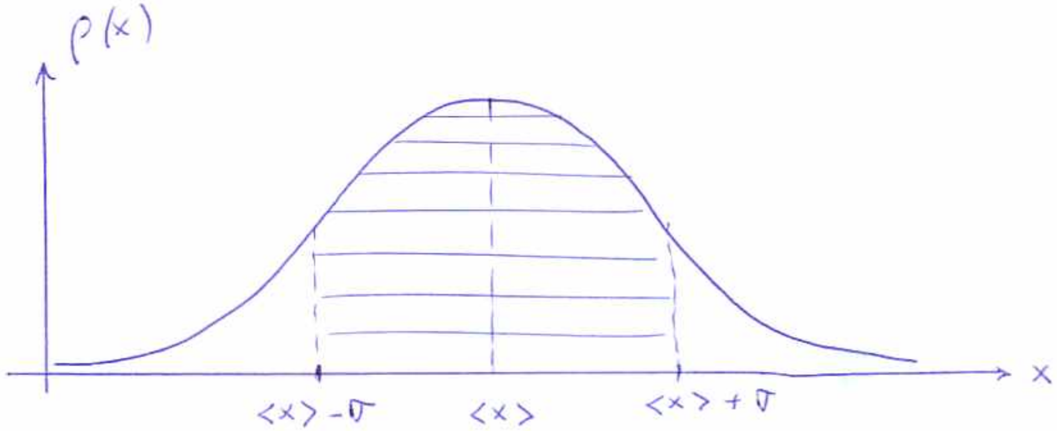
MATH HELP:

$$\int_{-\infty}^{+\infty} dx \; e^{-\lambda x^2} = \sqrt{\frac{\pi}{\lambda}}$$

$$\int_{-\infty}^{+\infty} dx \; x^2 \, e^{-\lambda x^2} = -\frac{d}{d\lambda} \int_{-\infty}^{+\infty} dx \; e^{-\lambda x^2}$$

$$= -\frac{d}{d\lambda} \sqrt{\frac{\pi}{\lambda}}$$

$$= \frac{1}{2} \sqrt{\frac{\pi}{\lambda}} \; \frac{1}{\lambda}$$



$$\rho(x) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma} e^{-\frac{1}{2\sigma^2}(x - \langle x \rangle)^2}$$

PROBABILITY $x \in [\langle x \rangle - \sigma, \ \langle x \rangle + \sigma]$

$P_{1\sigma}$ "1σ" DEVIATION FROM AVERAGE

$$P_{1\sigma} = A \int_{\langle x \rangle - \sigma}^{\langle x \rangle + \sigma} dx \cdot e^{-\frac{1}{2\sigma^2}(x - \langle x \rangle)^2}$$

$\downarrow \quad x' = x - \langle x \rangle$

$$= A \int_{-\sigma}^{+\sigma} dx' \ e^{-\frac{1}{2\sigma^2} x'^2}$$

$\downarrow \quad A = \sqrt{\frac{\lambda}{\pi}} = \frac{1}{\sqrt{2\pi}\,\sigma}$

$P_{1\sigma} = 0.68 \quad (68\%)$

$$\boxed{\begin{aligned} P_{1\sigma} &= 68\% \\ P_{2\sigma} &= 95\% \\ P_{3\sigma} &= 99.7\% \end{aligned}}$$

1.4 NORMALIZATION

→ WAVE FUNCTION $\Psi(x,t)$: INTERPRET AS "PROBABILITY AMPLITUDE"

SOLUTION OF SCHRÖDINGER EQUATION

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi = i\hbar\frac{\partial \Psi}{\partial t}$$

→ PROBABILITY DENSITY

$$\rho(x,t) = |\Psi(x,t)|^2 = \Psi^*\Psi$$

↳ PROBABILITY TO FIND PARTICLE AT POINT x AT TIME t

Ψ is COMPLEX

$\Psi = |\Psi|\, e^{i\phi}$ $\qquad |\Psi|$: AMPLITUDE

ϕ : PHASE

$\mathrm{Re}\,\Psi = |\Psi| \cos\phi$

$\mathrm{Im}\,\Psi = |\Psi| \sin\phi$

→ NORMALIZATION

$|\Psi(x,t)|^2\, dx$: PROB. TO FIND PARTICLE BETWEEN x & $x+dx$

PROB. TO FIND PARTICLE BETWEEN $-\infty$ & $+\infty$ $= 1$

$$1 = \int_{-\infty}^{+\infty} dx\, |\Psi(x,t)|^2.$$

IF Ψ IS SOLUTION OF SCHRÖDINGER EQ.

$A\Psi$ IS ALSO SOLUTION OF SCHRÖDINGER EQ.

- NON-NORMALIZABLE SOLUTION $\int_{-\infty}^{+\infty} dx \; |\Psi(x,t)|^2 \neq 1$

  ↳ DOES NOT DESCRIBE A PHYSICAL STATE

- NORMALIZABLE SOLUTION $\int_{-\infty}^{+\infty} dx \; |\Psi(x,t)|^2 = 1.$

SQUARE, INTEGRABLE SOLUTION

↳ DESCRIBES PHYSICAL STATES, PARTICLES

→ NORMALIZATION IS TIME-INDEPENDENT

$$\int_{-\infty}^{+\infty} dx \; |\Psi(x,t)|^2 = 1$$

$$\frac{d}{dt} \int_{-\infty}^{+\infty} dx \; |\Psi(x,t)|^2 = 0$$

$$\int_{-\infty}^{+\infty} dx \; \frac{\partial}{\partial t} |\Psi(x,t)|^2 = 0.$$

PROOF THAT A SOLUTION OF SCHRÖDINGER EQ. PRESERVES THE NORMALIZATION

$$\hookrightarrow \frac{\partial}{\partial t}|\Psi|^2 = \frac{\partial}{\partial t}\left(\Psi^* \Psi\right)$$

$$= \left(\frac{\partial \Psi^*}{\partial t}\right)\Psi + \Psi^*\left(\frac{\partial \Psi}{\partial t}\right)$$

$$\hookrightarrow \quad i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi$$

POTENTIAL V IS REAL

$$-i\hbar \frac{\partial \Psi^*}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + V\Psi^*$$

$$\hookrightarrow \quad \Psi^*\left(\frac{\partial \Psi}{\partial t}\right) = \frac{i\hbar}{2m}\Psi^*\frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar}V|\Psi|^2$$

$$\left(\frac{\partial \Psi^*}{\partial t}\right)\Psi = -\frac{i\hbar}{2m}\frac{\partial^2 \Psi^*}{\partial x^2}\Psi + \frac{i}{\hbar}V|\Psi|^2$$

$+$

$$\frac{\partial}{\partial t}|\Psi|^2 = \frac{i\hbar}{2m}\left(\Psi^*\frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 \Psi^*}{\partial x^2}\Psi\right)$$

$$= \frac{i\hbar}{2m}\frac{\partial}{\partial x}\left(\Psi^*\frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right)$$

$$\hookrightarrow \int_{-\infty}^{+\infty} dx \; \frac{\partial}{\partial t} |\Psi(x,t)|^2$$

$$= \frac{i\hbar}{2m} \int_{-\infty}^{+\infty} dx \; \frac{\partial}{\partial x} \left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x} \Psi \right)$$

$$\downarrow \qquad \int_a^b dx \; \frac{\partial F(x)}{\partial x} = F(b) - F(a)$$

$$= \frac{i\hbar}{2m} \left[\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x} \Psi \right]_{-\infty}^{+\infty}$$

$\downarrow$ FOR A NORMALIZABLE WAVE FUNCTION

$$\Psi(x = +\infty, t) = \Psi(x = -\infty, t) = 0.$$

$$= 0$$

■ QED

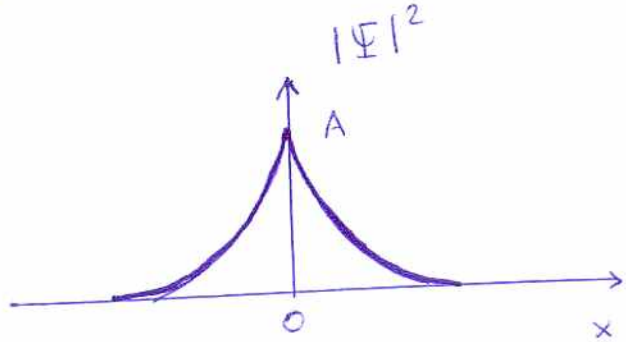
$\therefore$ IF $\Psi(x,t)$ IS NORMALIZED AT $t = 0$

IT STAYS NORMALIZED AT ALL TIMES.

→ EXAMPLE : PROBLEM 1.5

$$\Psi(x,t) = A\, e^{-\lambda|x|}\, e^{-i\omega t}$$

$$|\Psi(x,t)|^2 = A^2\, e^{-2\lambda|x|}$$



- NORMALIZATION

$$\int_{-\infty}^{+\infty} dx\, |\Psi(x,t)|^2 = 1$$

$$\Downarrow$$

$$A^2 \int_{-\infty}^{+\infty} dx\, e^{-2\lambda|x|} = 1.$$

$$2A^2 \underbrace{\int_{0}^{+\infty} dx\, e^{-2\lambda x}}_{\left[-\frac{1}{2\lambda} e^{-2\lambda x}\right]_0^{+\infty}} = 1$$

$$\frac{2A^2}{2\lambda} = 1 \implies \underline{\underline{A = \sqrt{\lambda}}}$$

- $\langle x \rangle = 0$
- $\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$

$$= \langle x^2 \rangle = A^2 \int_{-\infty}^{+\infty} dx\, x^2 e^{-2\lambda|x|}$$

$$= 2A^2 \int_0^{+\infty} dx\, x^2 e^{-2\lambda x}$$

MATH HELP

$$\int_0^{+\infty} dx \; x^2 \, e^{-2\lambda x}$$

$$= \frac{1}{4} \frac{d^2}{d\lambda^2} \int_0^{+\infty} dx \; e^{-2\lambda x}$$

$$= \frac{1}{4} \frac{d^2}{d\lambda^2} \left(\frac{1}{2\lambda} \right)$$

$$= \frac{1}{4} \cdot \frac{2}{2\lambda^3}$$

$$\sigma^2 = 2A^2 \cdot \frac{1}{4\lambda^3} = \frac{1}{2\lambda^2}$$

$$\sigma = \frac{1}{\sqrt{2}\lambda}$$

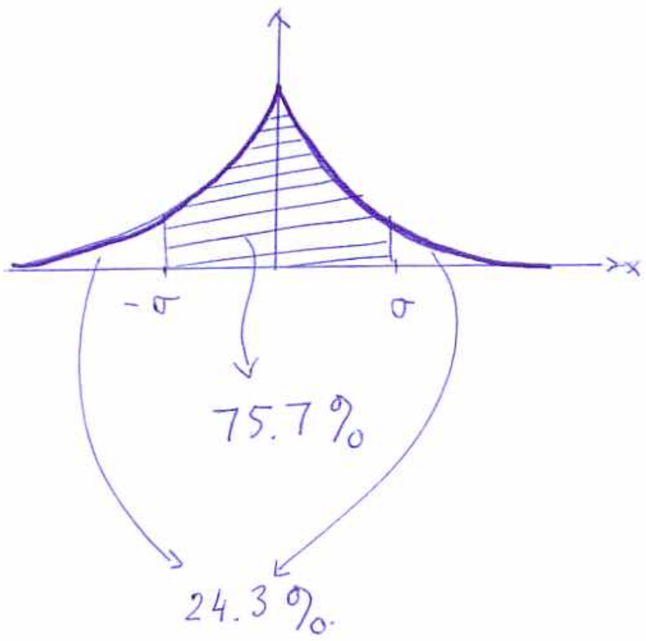
- $P\left(x \in [-\sigma, \sigma]\right)$

$$= \int_{-\sigma}^{\sigma} dx \; |\Psi(x,t)|^2$$

$$= 2A^2 \int_0^{\sigma} dx \; e^{-2\lambda x}$$

$$= -\frac{2A^2}{2\lambda} \, e^{-2\lambda x} \Big|_0^{\sigma}$$

$$= 1 - \underbrace{e^{-\sqrt{2}}}_{0.243} = \underline{0.757} \quad (75.7\,\%)$$



1.5 MOMENTUM

→ EXPECTATION VALUE

$\Psi(x,t)$ DESCRIBES STATE OF SYSTEM

IMAGINE AN ENSEMBLE OF SYSTEMS

↳ AVERAGE POSITION $\langle x \rangle$ OVER THIS ENSEMBLE

$$\langle x \rangle = \int_{-\infty}^{+\infty} dx \; x \; |\Psi(x,t)|^2$$

↳ TIME EVOLUTION OF $\langle x \rangle$

$$\frac{d}{dt}\langle x \rangle = \int_{-\infty}^{+\infty} dx \; x \; \frac{\partial}{\partial t} |\Psi(x,t)|^2$$

SEE P. 1.14

$$= \frac{i\hbar}{2m} \int_{-\infty}^{+\infty} dx \; x \; \frac{\partial}{\partial x} \left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x} \Psi \right)$$

↓ INTEGRATION BY PARTS

$$= -\frac{i\hbar}{2m} \int dx \left(\Psi^* \frac{\partial \Psi}{\partial x} - \underbrace{\frac{\partial \Psi^*}{\partial x} \Psi \Big)}_{\text{INTEGRATION BY PARTS}} + \cancel{\text{BOUNDARY TERM @ } -\infty \text{ AND } +\infty}$$

0

$\Psi(x = \pm\infty, t) = 0$

$$= -\frac{i\hbar}{m} \int dx \; \Psi^* \frac{\partial \Psi}{\partial x}$$

$\langle v \rangle$ EXPECTATION VALUE OF VELOCITY

$$\langle v \rangle = \frac{d}{dt} \langle x \rangle$$

↳ $\langle p \rangle$ EXPECTATION VALUE OF MOMENTUM

$$\langle p \rangle = m \langle v \rangle = -i\hbar \int dx \, \Psi^* \frac{\partial \Psi}{\partial x}$$

↳ 'OPERATORS'

$$\langle x \rangle = \int dx \, \Psi^* \underset{\substack{\uparrow \\ \text{POSITION 'OPERATOR'}}}{x} \Psi$$

$$\langle p \rangle = \int dx \, \Psi^* \underset{\substack{\uparrow \\ \text{MOMENTUM 'OPERATOR'}}}{\left(-i\hbar \frac{\partial}{\partial x} \right)} \Psi$$

$\langle T \rangle$ KINETIC ENERGY

$$T = \frac{p^2}{2m} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}$$

$$\langle T \rangle = \int dx \, \Psi^* \underset{\substack{\uparrow \\ \text{KINETIC ENERGY 'OPERATOR'}}}{\left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right)} \Psi$$

$$\langle V \rangle = \int dx \, \Psi^* \, V \, \Psi$$

↳ POTENTIAL ENERGY 'OPERATOR'

SCHRÖDINGER EQ

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi + V\Psi = i\hbar\frac{\partial \Psi}{\partial t}$$

MULTIPLY ON LEFT BY Ψ^* AND $\int dx$

$$\underbrace{\int dx\ \Psi^*\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\Psi}_{\langle T \rangle} + \underbrace{\int dx\ \Psi^* V \Psi}_{\langle V \rangle} = \underbrace{\int dx\ \Psi^*\left(i\hbar\frac{\partial}{\partial t}\right)\Psi}_{\langle E \rangle}$$

EXPECTATION VALUE OF TOTAL ENERGY

TIME EVOLUTION OF $\langle p \rangle$

$$\frac{d}{dt}\langle p \rangle = \frac{d}{dt}\int dx\ \Psi^*\left(-i\hbar\frac{\partial \Psi}{\partial x}\right)$$

$$= -i\hbar\int dx\left(\frac{\partial \Psi^*}{\partial t}\cdot\frac{\partial \Psi}{\partial x} + \Psi^*\frac{\partial}{\partial x}\cdot\frac{\partial \Psi}{\partial t}\right)$$

- $$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi$$

$$-i\hbar\frac{\partial \Psi^*}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + V\Psi^*$$

$$\frac{d}{dt}\langle p\rangle = \int dx \left(-\frac{\hbar^2}{2m}\frac{\partial^2\Psi^*}{\partial x^2}\cdot\frac{\partial\Psi}{\partial x} + V\Psi^*\frac{\partial\Psi}{\partial x} + \frac{\hbar^2}{2m}\Psi^*\frac{\partial}{\partial x}\frac{\partial^2\Psi}{\partial x^2} - \Psi^*\frac{\partial}{\partial x}(V\Psi)\right)$$

$\downarrow$ INTEGRATION BY PARTS TWICE ON FIRST TERM

$$\int_{-\infty}^{+\infty} dx\, \frac{\partial^2\Psi^*}{\partial x^2}\cdot\frac{\partial\Psi}{\partial x} = -\int_{-\infty}^{+\infty} dx\, \frac{\partial\Psi^*}{\partial x}\frac{\partial^2\Psi}{\partial x^2}$$

$$= +\int_{-\infty}^{+\infty} dx\, \Psi^*\frac{\partial^3\Psi}{\partial x^3}$$

$$\frac{d}{dt}\langle p\rangle = \int dx \left(\cancel{-\frac{\hbar^2}{2m}\Psi^*\frac{\partial^3\Psi}{\partial x^3}} + \cancel{V\Psi^*\frac{\partial\Psi}{\partial x}} + \cancel{\frac{\hbar^2}{2m}\Psi^*\frac{\partial^3\Psi}{\partial x^3}} - \Psi^*\frac{\partial V}{\partial x}\Psi - \cancel{V\Psi^*\frac{\partial\Psi}{\partial x}}\right)$$

$$\boxed{\frac{d}{dt}\langle p\rangle = \int dx\, \Psi^*\left(-\frac{\partial V}{\partial x}\right)\Psi = \left\langle -\frac{\partial V}{\partial x}\right\rangle}$$

EHRENFEST's THEOREM

$\hookrightarrow$ EXPECTATION VALUES OBEY CLASSICAL LAWS (e.g. NEWTON's LAWS)

1.6 UNCERTAINTY PRINCIPLE

→ WAVE FUNCTION Ψ

WAVELENGTH λ

$\updownarrow$

PARTICLE MOMENTUM p

PARTICLE - WAVE DUALITY IN Q.M.

DE BROGLIE FORMULA :

$$\boxed{p = \frac{h}{\lambda} = \frac{2\pi\hbar}{\lambda}}$$

$\hbar = \frac{h}{2\pi}$

→ HEISENBERG's UNCERTAINTY PRINCIPLE

THE MORE PRECISE WE KNOW POSITION,
THE LESS PRECISE WE KNOW ITS MOMENTUM
AND VICE VERSA

$$\boxed{\sigma_x \cdot \sigma_p \geqslant \frac{\hbar}{2}}$$

SOMETIMES ONE ALSO DENOTES $\sigma_x = \Delta x$

$\sigma_p = \Delta p$

$$\Delta x \cdot \Delta p \geqslant \frac{\hbar}{2}$$