Exercise sheet 1
Theoretical Physics 3 : QM WS2023/24

25.10.2023

Exercise 1. (25 points)

Consider a particle described at time $t = 0$ by the following wave function

$$\Psi(x,0) = \begin{cases} A\,(\frac{x}{a})^2 & \text{for} \quad 0 \leq x \leq a \\ A\,(\frac{b-x}{b-a})^2 & \text{for} \quad a \leq x \leq b \\ 0 & \text{otherwise,} \end{cases}$$

where a, b and A are real constants.

a) Determine A such that the wave function is normalized to 1.

b) Make a plot of $\Psi(x,0)$ as a function of x.

c) Where is the highest probability to find the particle at $t = 0$?

d) What is the probability to find the particle in the range $-\infty < x \leq a$ (left side of a)? What are these probabilities in the special cases $b = a$ and $b = 2a$?

e) What is the expectation value of x?

Exercice 2. (25 points)

Consider the following wave function:

$$\Psi(x,t) = \begin{cases} (\frac{a}{2L})^{1/2} e^{-i\omega t} & |x| < L \\ 0 & \text{otherwise,} \end{cases}$$

where a, L and ω are real constants.

a) Normalize the wave function to 1.

b) Compute the expectation values of x and x^2 to obtain the variance σ^2.

c) What is the probability of finding the system outside the region defined by $\pm\sigma$ around $\langle x \rangle$?.

Exercise 3. (25 points)

A particle with mass m is in the state

$$\Psi(x,t) = A\, e^{-a(\frac{m}{\hbar} x^2 + it)},$$

with real and positive constants A and a.

a) Determine A such that the wave function is normalized to 1.

b) Which potential $V(x)$ should one choose for $\Psi(x,t)$ to satisfy the SchrÃ¶dinger equation?

c) Compute the expectation values of x and x^2, as well as the quantities

$$\left\langle \frac{\hbar}{i}\frac{\partial}{\partial x} \right\rangle := \int \mathrm{d}x\, \Psi^* \left[\frac{\hbar}{i}\frac{\partial}{\partial x}\Psi\right], \qquad \left\langle \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2 \right\rangle := \int \mathrm{d}x\, \Psi^* \left[\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2 \Psi\right].$$

d) Compute the variance σ^2 of x and $\frac{\hbar}{i}\frac{\partial}{\partial x}$. How large is the product of the standard deviation of these two quantities?

Exercise 4. (25 points)

In this exercise we will study few of the properties of the gaussian wave functions in one dimension. We consider a general gaussian wave function given by

$$\Psi(x) = \Psi_0 e^{-Ax^2+Bx},$$

where A, B are complex numbers with $\mathrm{Re}[A] > 0$. After normalizing the wave function derive the following expectation values:

a) $\langle x \rangle = \frac{\mathrm{Re}[B]}{2\mathrm{Re}[A]}$,

b) $\sigma^2 = \frac{1}{4\mathrm{Re}[A]}$.

Hint: Decompose the wave function in a real and an imaginary part.
Mathematical hint: $\int_{-\infty}^{\infty} e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} e^{b^2/4a}$.