Practice exam
Theoretical Physics 5 : SS 2023

17.07.2023

Exercise 1. (25 points) :
A real scalar field in 1+1 dimensions

Consider the following Lagrangian for a real scalar field ϕ in 1+1 dimensions (one spatial and one temporal)

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{\lambda}{4}\left(\phi^2 - v^2\right)^2,$$

a) (*10 p.*) Construct the corresponding Hamiltonian and find the condition on the classical field configurations $\phi_0(x)$ that minimize the energy.

   Hint: The configurations that minimize the energy minimize the potential.

b) (*5 p.*) Derive the equations of motions for the field ϕ.

c) (*10 p.*) The static solution that interpolates between two vacuum states

$$\phi_0(x) = v\tanh\left(\sqrt{\frac{\lambda}{2}}vx\right)$$

   is called the kink solution. Show that the kink is indeed a valid solution of the equations of motion.

   Hint: Recall that

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

Exercise 2. (25 points): A helium atom

Helium is composed of two electrons bound by the electromagnetic force to a nucleus containing two protons and either one or two neutrons, depending on the isotope.

The Hamiltonian of the two-electron system in the Helium atom with the approximation of an infinitely heavy nucleus is given by

$$H = \left[-\frac{\hbar^2}{2m}\nabla_1^2 - \alpha\frac{2e^2}{r_1} \right] + \left[-\frac{\hbar^2}{2m}\nabla_2^2 - \alpha\frac{2e^2}{r_2} \right] + \alpha\frac{e^2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|} \tag{1}$$

a) (*5 p.*) Under the assumption that the interaction between the electrons is a small perturbation, we can factorize the wave function into a product of wave functions of separate electrons. The total wave function of the fermion system should be antisymmetric and is given by the product of a coordinate and a spin wave functions. If both electrons are in the 1s state, each with a known coordinate-space wave function ψ_{1s}, then the total coordinate wave function must be symmetric. Write down the spin wave function for this state.

b) (*10 p.*) If the two electrons are in different states, we can construct symmetric and antisymmetric combinations of coordinate wave functions. Let us assume that one electron is in the 1s state and the other in the 2s state, with known coordinate-space wave functions ψ_{1s} and ψ_{2s}. Write down all possible total wave functions (so the spin and coordinate space parts).

c) (*10 p.*) The total energy of the system, for the states considered in (c), will consist of the energies of the individual electrons, $E_{1s,2s}$, and two other terms. Give expressions for these two other terms. In both, you may leave the integrals unevaluated. Moreover, you do not need to fill in an explicit expression for the wave functions. What do both terms represent, physically?

Exercise 3. (25 points): The Hermiticity of the Lagrangian

The, by now hopefully familiar, Dirac Lagrangian is given by

$$\mathcal{L}_D = \bar{\psi}(i\not{\partial} - m)\psi. \tag{2}$$

By extremizing the action one obtains the Dirac equation,

$$(i\not{\partial} - m)\psi = 0. \tag{3}$$

Throughout the course, we've mentioned that the Lagrangian should be a Hermitian scalar. Actually, this was somewhat imprecise langauge – the *action* should be a Hermitian scalar, and usually this implies that the Lagrangian is too.

a) (*5 p.*) Show that $\mathcal{L}_D$ is not Hermitian.

Should we panic? No! If you think back hard enough to your classical mechanics classes, you might recall that two Lagrangians are equivalent if they differ by a total derivative. After all, then the action one gets from either Lagrangian will be the same. We can rewrite the Dirac Lagrangian into a Hermitian form,

$$\tilde{\mathcal{L}}_D = \bar{\psi}\left[\frac{1}{2}i\left(\overrightarrow{\not{\partial}} - \overleftarrow{\not{\partial}}\right) - m\right]\psi, \tag{4}$$

where the left and right derivates mean

$$A\overrightarrow{\partial}_\mu B := A(\partial_\mu B), \quad A\overleftarrow{\partial}_\mu B = (\partial_\mu A)B \implies \bar{\psi}\overleftarrow{\not{\partial}}\psi = (\partial_\mu\bar{\psi})\gamma^\mu\psi.$$

b) (*5 p.*) Show that $\tilde{\mathcal{L}}_D$ is Hermitian.

c) (*5 p.*) Let us denote the actions by S_D and $\tilde{S}_D$. Show that $S_D - \tilde{S}_D = 0$.

d) (*10 p.*) Derive the equations of motion as well as the conjugate momentum from both $\mathcal{L}_D$ and $\tilde{\mathcal{L}}_D$. The equations of motion have observable consequences – they should be the same for both Lagrangians.

Exercise 4. (25 points): Hyperfine splitting in hydrogen and the 21 cm line

The interaction between an electron and a magnetic field is given by

$$H_{\text{int}} = -\boldsymbol{\mu} \cdot \boldsymbol{B}, \tag{5}$$

where $\boldsymbol{\mu}$ is the magnetic moment,

$$\boldsymbol{\mu} = \frac{e}{2m}\frac{\hbar}{c}\boldsymbol{\sigma}, \tag{6}$$

of the electron.

a) (*10 p.*) Express H_{int} in its normal mode expansion.

   Hint: Use

$$\boldsymbol{A} = \sum_{\boldsymbol{k},\sigma} N_k \left\{ a_{\boldsymbol{k},\sigma} \boldsymbol{\epsilon}_{\boldsymbol{k},\sigma} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + a^\dagger_{\boldsymbol{k},\sigma} \boldsymbol{\epsilon}^*_{\boldsymbol{k},\sigma} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \right\}, \tag{7}$$

   where $\boldsymbol{\epsilon}_{\boldsymbol{k},\sigma}$ is the photon polarization vector and that $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$.

- (*15 p.*) The 1S-state with $f = 1$ has a slightly higher energy than the 1S-state with $f = 0$. In the transition between the initial state

$$|\psi_i\rangle = |1S\rangle\, |\uparrow\rangle_e\, |\uparrow\rangle_p\,, \tag{8}$$

  and the final state (with total electron + proton spin equal to zero)

$$|\psi_f\rangle = |1S\rangle\, \frac{1}{\sqrt{2}} \left\{ |\uparrow\rangle_e\, |\downarrow\rangle_p - |\downarrow\rangle_e\, |\uparrow\rangle_p \right\}, \tag{9}$$

  a photon with a wavelength of $\lambda \approx 21$ cm is emitted.

  Derive the lifetime of this transition and estimate its numerical value in years (an order of magnitude is enough). You may use that, numerically,

$$\alpha = \frac{e^2}{4\pi\hbar c} \approx \frac{1}{137}, \quad \hbar c \approx 1.97{\times}10^{-7}\ \text{eVm}, \quad mc^2 \approx 0.5{\times}10^6\ \text{eV}, \quad c \approx 3{\times}10^8\ \text{m/s}.$$

  Hint: Calculate the matrix element

$$\langle f|\, \hat{H}_{int}\, |i\rangle \tag{10}$$

  in dipole approximation and apply Fermi's golden rule.

Formula sheet:

Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Dirac matrices:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}.$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}1;$$

Euler-Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \phi_r} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} = 0$$