Exercise sheet 13
Theoretical Physics 5 : SS 2023

10.07.2023

Write your name and your tutor's name on every page you hand in. Please staple said pages together.

Exercise 0.

How much time did you take to complete this homework sheet?

Exercise 1. (40 points): Gauge transformations and Quantum Electrodynamics

Consider the Lagrangian for the charged Klein-Gordon Field

$$\mathcal{L}_{\mathrm{KG}} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - \mu^2 \phi^\dagger \phi \tag{1}$$

a) (*2.5 p.*) Verify that this Lagrangian is invariant under a global transformation $\phi(x) \to e^{i\alpha}\phi(x)$ where $\alpha \in \mathbb{R}$.

b) (*5 p.*) Verify that this Lagrangian is **not** invariant under U(1) gauge transformation of the form $\phi(x) \to \phi'(x) = e^{i\alpha(x)}\phi(x)$, where now α is a function of x.

c) (*5 p.*) Replace the derivatives with the covariant one $D_\mu = \partial_\mu + ieA_\mu$ and check whether now the Lagrangian is invariant under a local U(1) symmetry.

d) (*2.5 p.*) Write down the interaction Lagrangian and identify the terms (i.e. what do the terms represent?).

Now consider the QED Lagrangian,

$$\mathcal{L}_{\mathrm{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi, \tag{2}$$

where $\not{D} = \gamma_\mu D^\mu$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

e) (*5 p.*) Show that the QED Lagrangian is invariant under the global transformation

$$\begin{aligned}\psi(x) &\to e^{ie\Lambda}\psi(x),\\ A_\mu(x) &\to A_\mu(x).\end{aligned}$$

We call such a transformation global because $\Lambda \in \mathbb{R}$ is not a function of x.

f) (*5 p.*) Actually, we can subsume the global transformation in a local gauge transformation, where now $\Lambda = \Lambda(x)$. Show that the QED Lagrangian is invariant under

$$\begin{aligned}\psi(x) &\to e^{ie\Lambda(x)}\psi(x),\\ A^\mu(x) &\to A^\mu(x) - \partial_x^\mu \Lambda(x).\end{aligned}$$

Remark: Filling in these exponentials 'by hand' works fine for QED, but generalizes poorly to, say, Quantum Chromodynamics (the theory of the strong interaction). In such a theory the equivalent bosons to the photon (A_μ) are the gluons. These gluon fields, however, are now 3×3 matrices! You are now going to do the same gauge-transformation calculation in a manner that generalizes easily to, for example, QCD.

g) (*5 p.*) First, show that

$$-\frac{i}{e}\left[D^\mu, D^\nu\right] = F^{\mu\nu}, \tag{3}$$

where the covariant derivatives in the commutator act on *everything* to the right of them. Consider using a test function (letting the commutator act on some function $f(x)$ and showing this is equivalent to multiplying the function f by $F^{\mu\nu}$).

Remark: this definition for the field strength generalizes to non-Abelian gauge theories, i.e. where instead of the U(1) symmetry we have, for example, an SU(N) symmetry. One example is Quantum Chromodynamics, whose gauge group is SU(3). Accordingly, there are three color charges, in contrast to the single electric charge.

h) (*5 p.*) In an elegant bit of notation, let us define

$$e^{ie\Lambda(x)} =: U(x) \in \mathrm{U}(1), \tag{4}$$

where $U^\dagger(x)U(x) = 1$. You should convince yourself that

$$\psi(x) \to U(x)\psi(x), \tag{5}$$

$$A^\mu(x) \to U(x)\left(A^\mu(x) - \frac{i}{e}\partial_x^\mu\right)U^\dagger(x), \tag{6}$$

where the final derivate acts only on $U^\dagger$, is completely equivalent to our earlier gauge transformation.

For the following, do not write U as an exponential (that defeats the point of this exercise) – keep it general and at most use its Hermiticity.

First, show that

$$D^\mu(x) \to U(x) D^\mu(x) U^\dagger(x), \tag{7}$$

where, again, here D^μ acts on everything to the right of it (not just $U^\dagger$). It is helpful to use a test function.

Using this result, show that

$$F^{\mu\nu} = -\frac{i}{e}\left[D^\mu, D^\nu\right] \to U F^{\mu\nu} U^\dagger = F^{\mu\nu}. \tag{8}$$

Finally, show that the QED Lagrangian is gauge invariant.

Remark: Notice how in this notation $U(x)$ can easily be upgraded to a matrix! If you continue with gauge theories, you will learn to appreciate this fact.

i) *(5 p.)* Of course, a gauge transformation should not have observable consequences. However, this does not mean that the requirement that a theory be gauge invariant cannot have observable consequences. Quite the contrary! Gauge invariance actually explains why photons are massless.

Show that if we were to give the photon field, A_μ, a mass term,

$$\frac{1}{2} m_\gamma^2 A_\mu A^\mu, \tag{9}$$

the QED Lagrangian would not be gauge-invariant anymore under a gauge transformation.

Remark: All our fundamental theories (QED, QCD, General Relativity) are gauge theories. Their corresponding gauge bosons (photons, gluons, gravitons) are massless. The exception are the W and Z bosons of the weak interaction. They start out massless, but obtain a dynamically-generated mass via spontaneous symmetry breaking. For the electroweak theory this effect is better known as the Higgs mechanism. But, even the gauge symmetry of QED can be spontaneously broken. For example, you might have heard of the Meissner effect: in a superconducting material magnetic fields cannot enter a superconducting region. This is because of a broken symmetry, which causes the photons get a Meissner mass (the inverse of what is called the London penetration depth), which leads to an exponential decrease of the magnetic field inside of a superconductor.

Exercise 2. (20 points): The Hamiltonian of the electromagnetic field

Using the normal mode expansion of the photon field

$$A^\mu(\boldsymbol{x},t) \quad = \quad \sum_{\boldsymbol{k}} \sum_{\lambda=0}^{3} \left(\frac{\hbar\, c^2}{2\,\omega_k\, L^3}\right)^{1/2} \left[a(\boldsymbol{k})\,\epsilon^\mu(\boldsymbol{k},\lambda)\, e^{-i\,k\cdot x} + a^\dagger(\boldsymbol{k})\,\epsilon^{\mu *}(\boldsymbol{k},\lambda)\, e^{i\,k\cdot x}\right]$$

and the commutator relations

$$\begin{aligned}
&[a(\boldsymbol{k},\lambda), a^\dagger(\boldsymbol{k}',\lambda')] = \xi_\lambda\, \delta_{\lambda,\lambda'}\, \delta_{\boldsymbol{k},\boldsymbol{k}'}, \quad \xi_0 = -1,\ \xi_{i=1,2,3} = +1 \\
&[a(\boldsymbol{k},\lambda), a(\boldsymbol{k}',\lambda')] = 0 \\
&[a^\dagger(\boldsymbol{k},\lambda), a^\dagger(\boldsymbol{k}',\lambda')] = 0
\end{aligned}$$

show that the Hamiltonian

$$H = \int \mathrm{d}^3x\, N\left((\Pi^\nu(x)\dot{A}_\nu(x) - \mathcal{L}(x))\right), \tag{10}$$

with $\mathcal{L} = -\frac{1}{4}\, F^{\mu\nu}F_{\mu\nu}$ where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ takes the following form

$$H = \sum_{\boldsymbol{k}} \hbar\,\omega_k \left(-a^\dagger(\boldsymbol{k},0)\, a(\boldsymbol{k},0) + \sum_{i=1}^{3} a^\dagger(\boldsymbol{k},i)\, a(\boldsymbol{k},i)\right) \tag{11}$$

Exercise 3. (40 points): More gauge transformations

Let $|\Psi_T\rangle$ be a state which contains only transverse photons. Show that replacing $|\Psi_T\rangle$ by $|\Psi_T'\rangle$

$$\begin{aligned}
|\Psi_T'\rangle &= \left\{1 + \alpha\left[a^\dagger(\boldsymbol{k},3) - a^\dagger(\boldsymbol{k},0)\right]\right\}|\Psi_T\rangle\,, \\
&\alpha \in \mathbb{C},
\end{aligned}$$

corresponds to a gauge transformation:

$$\langle\Psi_T'|A^\mu(x)|\Psi_T'\rangle = \langle\Psi_T|A^\mu(x) + \partial^\mu\Lambda|\Psi_T\rangle\,, \tag{12}$$

where Λ is given by

$$\Lambda(x) = \left(\frac{2\hbar\, c^2}{\omega_k^3\, L^3}\right)^{1/2} \mathrm{Re}\left(i\alpha e^{-ik\cdot x}\right).$$

Hint: In Eq. 12 try writing the new states in terms of the old ones, and using the mode expansion of the photon field. The expectation values should simplify (beware of the sign of the longitudinal part). Then, use the explicit representations for the polarization vectors (lecture notes) to wrangle part of the expression into a derivative.