Exercise sheet 12
Theoretical Physics 5 : SS 2023

03.07.2023

Write your name and your tutor's name on every page you hand in. Please staple said pages together.

Exercise 0.

How much time did you take to complete this homework sheet?

Exercise 1. (40 points): Ground-state Coulomb Dirac wave functions

In the lecture notes you have looked at the hydrogen atom using Dirac theory. There, you expressed the radial Dirac Coulomb wave functions $F(\rho)$ and $G(\rho)$ as a power series,

$$F(\rho) = \sqrt{k_2} e^{-\rho/2} \sum_{m=0}^{n'} a_m \rho^{m+\gamma} \tag{1}$$

$$G(\rho) = \sqrt{k_1} e^{-\rho/2} \sum_{m=0}^{n'} b_m \rho^{m+\gamma}, \tag{2}$$

where $n' = n - (j + 1/2)$, $\rho = 2\sqrt{k_1 k_2} r$ and $k_{1,2} = \frac{1}{\hbar c}(\pm E + m_0 c^2)$.

Consider the $1s_{1/2}$ state ($n = 1$, $j = 1/2$).

a) *(30 p.)* Show that from the normalization condition (the integral is over r, not ρ!)

$$\int_0^\infty \mathrm{d}r \left[F^2 + G^2\right] = 1 \tag{3}$$

it follows that

$$F(\rho) = \left(\frac{m_0c^2}{\hbar c}\right)^{\frac{1}{2}} \sqrt{\frac{Z\alpha(1-\gamma)}{\Gamma(2\gamma+1)}} \rho^\gamma e^{-\rho/2}, \tag{4}$$

$$G(\rho) = -\left(\frac{m_0c^2}{\hbar c}\right)^{\frac{1}{2}} \sqrt{\frac{Z\alpha(1+\gamma)}{\Gamma(2\gamma+1)}} \rho^\gamma e^{-\rho/2}. \tag{5}$$

b (*10 p.*) When can we expect the equation for the energy spectrum,

$$E_{nj} = \frac{m_0c^2}{\sqrt{1+\left(\frac{Z\alpha}{n-(j+1/2)+\gamma}\right)^2}}, \tag{6}$$

to break down? What happens, physically, in this limit? No calculations are necessary.

Hint: Take a look at the expression for γ in the lecture notes.

Exercise 2. (60 points): Dirac field anticommutators

Using the normal mode expansion of the Dirac field

$$\psi(\boldsymbol{x},t) = \sum_{\boldsymbol{p}} \sum_{s_z} \left(\frac{m_0c^2}{E_pV}\right)^{1/2} \left[b(\boldsymbol{p},s_z)\, u(\boldsymbol{p},s_z) e^{-\frac{i}{\hbar}p\cdot x} + d^\dagger(\boldsymbol{p},s_z)\, v(\boldsymbol{p},s_z) e^{\frac{i}{\hbar}p\cdot x}\right]$$

$$\bar{\psi}(\boldsymbol{x},t) = \sum_{\boldsymbol{p}} \sum_{s_z} \left(\frac{m_0c^2}{E_pV}\right)^{1/2} \left[b^\dagger(\boldsymbol{p},s_z)\, \bar{u}(\boldsymbol{p},s_z) e^{\frac{i}{\hbar}p\cdot x} + d(\boldsymbol{p},s_z)\, \bar{v}(\boldsymbol{p},s_z) e^{-\frac{i}{\hbar}p\cdot x}\right], \tag{7}$$

and the equal-time creation and annihilation operators anticommutation relations

$$\begin{aligned}
\{b(\boldsymbol{p},s_z), b^\dagger(\boldsymbol{p}',s_z')\} &= \delta_{\boldsymbol{p},\boldsymbol{p}'}\delta_{s_z,s_z'}, \\
\{d(\boldsymbol{p},s_z), d^\dagger(\boldsymbol{p}',s_z')\} &= \delta_{\boldsymbol{p},\boldsymbol{p}'}\delta_{s_z,s_z'}, \\
\{b(\boldsymbol{p},s_z), b(\boldsymbol{p}',s_z')\} &= 0, \qquad \{d(\boldsymbol{p},s_z), d(\boldsymbol{p}',s_z')\} = 0, \\
\{b(\boldsymbol{p},s_z), d(\boldsymbol{p}',s_z')\} &= 0, \qquad \{d(\boldsymbol{p},s_z), b(\boldsymbol{p}',s_z')\} = 0, \\
\{b(\boldsymbol{p},s_z), d^\dagger(\boldsymbol{p}',s_z')\} &= 0, \qquad \{d(\boldsymbol{p},s_z), b^\dagger(\boldsymbol{p}',s_z')\} = 0;
\end{aligned}$$

a (*20 p.*) Express

$$H = c\int \mathrm{d}^3x\, N\left(\bar{\psi}(-i\hbar\gamma^i\partial_i + m_0c)\psi\right), \tag{8}$$

in terms of creation and annihilation operators.

b) (*20 p.*) Express the momentum operator,

$$\boldsymbol{P} = -i\hbar \int \mathrm{d}^3x \, N\left(\psi^\dagger \boldsymbol{\nabla}\psi\right) \tag{9}$$

in terms of creation and annihilation operators.

c) (*20 p.*) Show that

$$[H, b^\dagger(\boldsymbol{p}, s_z) b(\boldsymbol{p}, s_z)] = 0. \tag{10}$$