Exercise sheet 9
Theoretical Physics 5 : SS 2023

12.06.2023

Write your name and your tutor's name on every page you hand in. Please staple said pages together.

Exercise 0.

How much time did you take to complete this homework sheet?

Exercise 1. (50 points): Lorentz invariance and time- and space-like four-vectors

a) (*7.5 p.*) Consider a four vector p with $p^2 > 0$, i.e. $(p^0)^2 - \boldsymbol{p}^2 > 0$. Such a four vector is called time like. Show that one can find a Lorentz transformation (a boost, in this case) $p \to p'$ such that the three-momentum $\boldsymbol{p}'$ vanishes in the new frame.

   Hint: The use of rotations is allowed. So, feel free to choose your coordinate system such that $\boldsymbol{p}$ points along one of the axes.

b) (*7.5 p.*) Show that it is not possible for $p^2 < 0$ (space-like four vector) to find a Lorentz transformation such that the three momentum vanishes in the new frame. Show that instead one can find a Lorentz transformation such that the *energy* vanishes in the new frame.

c) (*7.5 p.*) Consider two four-vectors p_1 and p_2 of massive particles with masses m_1, $m_2 \neq 0$, i.e. $p_i^0 = \sqrt{\boldsymbol{p}_i^2 + m_i^2}$, $i = 1, 2$. Show that $p = p_1 + p_2$ satisfies $p^2 > 0$. Do not use that one may choose a frame $\boldsymbol{p}_1 + \boldsymbol{p}_2 = \boldsymbol{0}$, because this would require $p^2 > 0$.

   Hint: Make use of the result of (a), i.e. for any four vector k where you know that $k^2 > 0$ holds you can choose a frame with $\boldsymbol{k} = \boldsymbol{0}$.

d) (*7.5 p.*) Consider two four vectors q_1 and q_2 of massive particles with the same mass $m \neq 0$. Show that $q = q_1 - q_2$ satisfies $q^2 \leq 0$.

e) (*10 p.*) Show that the following quantities are Lorentz invariant, i.e. do not change when a system is rotated or boosted. The new system is denoted by a prime:

- The measure of the four-dimensional energy-momentum integral

$$\mathrm{d}^4 p = \mathrm{d}^4 p', \tag{1}$$

- the four-dimensional delta function for energy and momentum,

$$\delta^{(4)}(p - q) = \delta^{(4)}(p' - q'), \tag{2}$$

- the sign of the energy if the corresponding four vector is time like: if $p^0 > 0$ and $p^2 > 0$ then $p'^0 > 0$.

f) (*10 p.*) Show in two ways for a three momentum $\boldsymbol{p}$ and the corresponding energy $E_p = \sqrt{\boldsymbol{p}^2 + m^2}$ that

$$\frac{\mathrm{d}^3 p}{2E_p} = \frac{\mathrm{d}^3 p'}{2E'_p} \tag{3}$$

holds for boosts and rotations. First by a direct calculation and then by relating

$$\mathrm{d}^3 p \int \mathrm{d}p^0 \, \delta(p^2 - m^2)\theta(p^0) \tag{4}$$

to Eq. (3).

Exercise 2. (50 points): Rotations around the x-axis

a) (*40 p.*) A finite rotation can be expressed as a series of N successive infinitesimal rotations where $N \to \infty$. For a rotation around the x-axis, this can be written as

$$x'^\mu = \lim_{N\to\infty} \left[\left(1 + \frac{\varphi}{N} J_x \right) N \right]^\mu{}_\nu x^\nu,$$

where J_x is the generator of an infinitesimal rotation around the x-axis. Show that the finite rotation is given by

$$x'^\mu = \left[1 + J_x^2 + \cos\varphi(-J_x^2) + \sin\varphi J_x \right]^\mu{}_\nu x^\nu$$

and calculate the corresponding rotation matrix.

b) (*10 p.*) Show that

$$(\Lambda^{\mu}{}_{\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\varphi & -\sin\varphi \\ 0 & 0 & \sin\varphi & \cos\varphi \end{pmatrix} \tag{5}$$

is a Lorentz transformation, i.e. that it satisfies

$$\Lambda^{\mu}{}_{\rho} g_{\mu\nu} \Lambda^{\nu}{}_{\sigma} = g_{\rho\sigma}. \tag{6}$$