Exercise sheet 8
Theoretical Physics 5 : SS 2023

05.06.2023

Write your name and your tutor's name on every page you hand in. Please staple said pages together.

Exercise 0.

How much time did you take to complete this homework sheet?

Exercise 1. (60 points): Dirac matrix gymnastics

Without using an explicit representation for the Dirac matrices, show that:

a) *(5 p.)* $\gamma_\mu \gamma^\mu = 4$;

b) *(5 p.)* $\mathrm{tr}[\gamma^\mu \gamma^\nu] = 4 g^{\mu\nu}$;

c) *(10 p.)* $\mathrm{tr}[\not{a}\,\not{b}\,\not{c}\,\not{d}] = 4\left(a_\mu b^\mu c_\nu d^\nu - a_\mu c^\mu b_\nu d^\nu + a_\mu d^\mu b_\nu c^\nu\right)$, where $\not{a} := \gamma^\mu a_\mu$;

d) *(5 p.)* $\gamma_5^2 = 1$, with $\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$;

e) *(5 p.)* $\{\gamma_5, \gamma^\mu\} = 0$;

f) *(5 p.)* $\mathrm{tr}[\gamma_5] = 0$;

g) *(10 p.)* $\mathrm{tr}[\gamma^\mu \gamma^\nu \gamma_5] = 0$;

h) *(10 p.)* $\mathrm{tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_5] = 4i\,\varepsilon^{\mu\nu\rho\sigma}$, where $\varepsilon_{0123} = +1$;

i) *(5 p.)* $\mathrm{tr}[\gamma^{\mu_1} \cdots \gamma^{\mu_n}] = 0$ if n is odd.

Exercise 2. (40 points): Weyl representation

In the Dirac representation, Dirac matrices have the form

$$\gamma_D^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\gamma}_D = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix},$$

while in the so-called *Weyl representation*, they have the form

$$\gamma_W^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix},$$

with $\sigma^\mu = (1, \boldsymbol{\sigma})$ and $\bar{\sigma}^\mu = (1, -\boldsymbol{\sigma})$. The Dirac representation is useful for applications at low energy, while the Weyl representation is useful when chirality is important.

a) (*12.5 p.*) Write down a unitary matrix S connecting both representations $\gamma_D^\mu = S\gamma_W^\mu S^{-1}$.

b) (*7.5 p.*) Write the γ_5 matrix in both representations.

c) (*20 p.*) In the Weyl representation the four-component Dirac spinor can be written in terms of two two-component Weyl spinors. That is, $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$. Show that the two-component bispinors ψ_L and ψ_R are independent for massless particles. The left and right handed states are eigenstates of the chirality operator γ_5. What are their corresponding eigenvalues?

   Hint: Try writing down the Dirac equation and using the explicit form of the Dirac matrices in the Weyl representation.