Exercise sheet 11
Theoretical Physics 5 : SS 2023

26.06.2023

Write your name and your tutor's name on every page you hand in. Please staple said pages together.

Exercise 0.

How much time did you take to complete this homework sheet?

Exercise 1. (100 points):
Dirac particle in a spherical potential well

Consider the Dirac equation,

$$\left[\boldsymbol{\alpha} \cdot \boldsymbol{p} c + \beta m_0 c^2\right] \psi(\boldsymbol{r}) = [E - V(r)]\, \psi(\boldsymbol{r}), \tag{1}$$

in a spherical potential well,

$$V(r) = \begin{cases} -V_0 & \text{for} \quad r \leq R, \\ 0 & \text{for} \quad r > R. \end{cases} \tag{2}$$

a) (*25 p.*) Show that

$$\boldsymbol{\alpha} \cdot \boldsymbol{p} = -i(\boldsymbol{\alpha} \cdot \hat{\boldsymbol{e}}_r) \left(\hbar \frac{\partial}{\partial r} + \frac{\hbar}{r} - \frac{\beta}{r} K \right), \tag{3}$$

with $K = \beta\,(\boldsymbol{\Sigma} \cdot \boldsymbol{L} + \hbar)$ and $\hat{\boldsymbol{e}}_r = \boldsymbol{r}/r$.

Hint: use $\boldsymbol{\nabla} = \hat{\boldsymbol{e}}_r(\hat{\boldsymbol{e}}_r \cdot \boldsymbol{\nabla}) - \hat{\boldsymbol{e}}_r \times (\hat{\boldsymbol{e}} \times \boldsymbol{\nabla})$.

b) (*25 p.*) Use the ansatz

$$\psi(\boldsymbol{r}) = \begin{pmatrix} g(r) \phi_{j l_A m}(\theta, \phi) \\ i f(r) \phi_{j l_B m}(\theta, \phi) \end{pmatrix}, \tag{4}$$

where $\phi_{jl_{A/B}m}$ are the eigenstates of the $\boldsymbol{\sigma} \cdot \boldsymbol{L}$ operator,

$$
\begin{aligned}
(\boldsymbol{\sigma} \cdot \boldsymbol{L})\, \phi_{jl_Am} &= -\hbar(\kappa + 1)\phi_{jl_Am}, \\
(\boldsymbol{\sigma} \cdot \boldsymbol{L})\, \phi_{jl_Bm} &= \hbar(\kappa - 1)\phi_{jl_Bm},
\end{aligned}
$$

to find the differential equations for $G(r)$ and $F(r)$. They are related to $g(r)$ and $f(r)$ as

$$
f(r) = \frac{F(r)}{r}, \qquad g(r) = \frac{G(r)}{r}.
$$

c) *(25 p.)* For

$$
k^2 = \frac{1}{(\hbar c)^2}\left[(E + V_0)^2 - m_0^2 c^4\right] > 0
$$

the general solution is given by

$$
\begin{aligned}
G(r) &= r\left[a_1 j_{l_A}(kr) + a_2 y_{l_A}(kr)\right], \\
F(r) &= \frac{\kappa}{|\kappa|}\frac{\hbar c k r}{E + V_0 + m_0 c^2}\left[a_1 j_{l_B}(kr) + a_2 y_{l_B}(kr)\right],
\end{aligned}
$$

where j_l and y_l are the spherical Bessel functions of the first and second kind. For

$$
\tilde{k}^2 = -\frac{1}{(\hbar c)^2}\left[(E + V_0)^2 - m_0^2 c^4\right] > 0
$$

the general solution is given by:

$$
\begin{aligned}
G(r) &= r\sqrt{\frac{2\tilde{k}r}{\pi}}\left[b_1 K_{l_A+1/2}(\tilde{k}r) + b_2 I_{l_A+1/2}(\tilde{k}r)\right], \\
F(r) &= \frac{\hbar c \tilde{k} r}{E + V_0 + m_0 c^2}\sqrt{\frac{2\tilde{k}r}{\pi}}\left[-b_1 K_{l_B+1/2}(\tilde{k}r) + b_2 I_{l_B+1/2}(\tilde{k}r)\right],
\end{aligned}
$$

where $K_{l+1/2}$ and $I_{l+1/2}$ are the modified Bessel functions. Furthermore,

$$
l_A = \begin{cases} j + \frac{1}{2} & \text{for} \quad \kappa = j + \frac{1}{2}, \\ j - \frac{1}{2} & \text{for} \quad \kappa = -(j + \frac{1}{2}), \end{cases}
$$

and

$$
l_B = \begin{cases} j - \frac{1}{2} & \text{for} \quad \kappa = j + \frac{1}{2}, \\ j + \frac{1}{2} & \text{for} \quad \kappa = -(j + \frac{1}{2}). \end{cases}
$$

Determine the transcendental equation that gives the energy spectrum of bound states for which $E - m_0 c^2 > -V_0$ and $-m_0 c^2 < E < m_0 c^2$. You do not need to solve this equation.

Hint: your solution must be normalizable. Consider limits of the special functions (Bessel, etc.) – do they exclude certain solutions?

d) (*25 p.*) By exploiting the continuity condition at $r = R$, derive the following relation for s-states ($l = 0$), which correspond to $j = \frac{1}{2}$ and $\kappa = -1$,

$$\frac{kR\sin(kR)}{\sin(kR) - kR\cos(kR)} = \frac{k}{\tilde{k}_0} \frac{e^{-\tilde{k}_0 R}}{e^{-\tilde{k}_0 R}\left(1 + \frac{1}{\tilde{k}_0 R}\right)} \frac{E + m_0 c^2}{E + V_0 + m_0 c^2}, \tag{5}$$

where

$$\tilde{k}_0 = \frac{1}{\hbar c}\sqrt{m_0^2 c^4 - E^2}.$$

If you did not solve part c), take the above equation as your point of departure.

Then, rewrite the above equation into

$$1 = \tan\left(\frac{R}{\hbar c}\sqrt{(E + V_0)^2 - m_0^2 c^4}\right)\sqrt{\frac{E + V_0 + m_0 c^2}{E + V_0 - m_0 c^2}} \times \left\{\frac{\hbar c}{R}\left[\frac{1}{E + m_0 c^2} - \frac{1}{E + V_0 + m_0 c^2}\right] - \sqrt{\frac{m_0 c^2 - E}{m_0 c^2 + E}}\right\}. \tag{6}$$

These equations relate the energy eigenvalues of the s-states to the properties of the spherical potential well. It is rather remarkable and nice that we can (still) find analytic solutions!

Hint: If necessary, you can use the following Rayleigh formulas for (spherical) Bessel functions:

$$j_n(x) = (-1)^n x^n \left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(\frac{\sin(x)}{x}\right),$$

$$y_n(x) = -(-1)^n x^n \left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(\frac{\cos(x)}{x}\right),$$

$$i_n(x) = x^n \left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(\frac{\sinh(x)}{x}\right),$$

$$k_n(x) = (-1)^n x^n \left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(\frac{e^{-x}}{x}\right),$$

with

$$i_n(x) = \sqrt{\frac{\pi}{2x}} I_{n+1/2}(x) \qquad \text{and} \qquad k_n(x) = \sqrt{\frac{\pi}{2x}} K_{n+1/2}(x). \tag{7}$$