Exercise sheet 10
Theoretical Physics 5 : SS 2023

19.06.2023

Write your name and your tutor's name on every page you hand in. Please staple said pages together.

Exercise 0.

How much time did you take to complete this homework sheet?

Exercise 1. (25 points): Gamma matrices form a basis

Show that the most general 4×4 matrix can be written as a linear combination (with complex coefficients) of

$$1,\, \gamma^\mu,\, \sigma^{\mu\nu},\, \gamma^\mu\gamma_5,\, \gamma_5,$$

where 1 is the identity matrix and $\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right]$. If you write code (you certainly don't need to), be sure to include the code and the code's result in your answer, and explain what the code does.

Hint: The standard inner product of two matrices is given by $\mathrm{tr}\left(A^\dagger B\right)$. Show that all matrices are linearly independent and then count the number of elements that you have.

Exercise 2. (40 points) : Dirac bilinears

The point of this exercise is for you to do the calculation yourself at least once and understand what you are doing. Do not blindly copy.

Since a spinor gains a minus sign after rotating over 2π, physical quantities must be bilinears in the fields ψ. In this way physical quantities turn into themselves after a rotation over 2π. These bilinears have the general form $\bar{\psi}\Gamma\psi$. As you've seen in the previous exercise, there are 16 independent bilinears related to 16 complex 4×4 matrices:

- $\Gamma_S = 1$ (scalar),
- $\Gamma_P = \gamma_5$ (pseudoscalar),
- $\Gamma_V^\mu = \gamma^\mu$ (vector),
- $\Gamma_A^\mu = \gamma^\mu \gamma_5$ (axial vector),
- $\Gamma_T^{\mu\nu} = \sigma^{\mu\nu}$ (tensor).

Without referring to any explicit representation for the Γ matrices,

a) (*10 p.*) show that $\Gamma_i^2 = \pm 1$ for $i = S, P, V, A, T$ ($\Gamma_i^2 = (\Gamma_i^\mu)^2$ or $(\Gamma^{\mu\nu})^2$ without a sum over the Lorentz indices – you're taking the square of each of the 16 matrices, not contracting them!).

b) (*15 p.*) show that $\operatorname{tr} \Gamma_i = 0$ for $i \neq S$,

c) (*15 p.*) using the Lorentz transformation of the Dirac spinor $\psi'(x') = S(a)\psi(x)$ with $x'^\mu = a^\mu_\nu x^\nu$, check if the bilinears transform according as they are named, i.e. a vector bilinear transforms like a vector, $\bar{\psi}'\psi' = \bar{\psi}\psi$, $\bar{\psi}'\gamma_5\psi' = \det(a)\bar{\psi}\gamma_5\psi$, $\bar{\psi}'\gamma^\mu\psi' = a^\mu{}_\nu\bar{\psi}\gamma^\nu\psi$, $\bar{\psi}'\gamma^\mu\gamma_5\psi' = \det(a)a^\mu{}_\nu\bar{\psi}\gamma^\mu\gamma_5\psi$, $\bar{\psi}'(x')\sigma^{\mu\nu}\psi'(x') = a^\mu_\kappa a^\nu_\lambda \bar{\psi}(x)\sigma^{\kappa\lambda}\psi(x)$. Distinguish between proper and improper (orthochronous) Lorentz transformations.

Exercise 3. (35 points): Discrete symmetries

Consider the Lagrangian for a Dirac fermion and a boson,

$$\mathcal{L} = \bar{\Psi}\left(i\gamma_\mu\partial^\mu - m\right)\Psi + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}M^2\varphi^2 + ih\varphi\bar{\Psi}\gamma_5\Psi,$$

where $h \in \mathbb{R}$ and $\bar{\Psi} = \Psi^\dagger\gamma^0$.

a) (*10 p.*) Is φ a Hermitian field?

Hint: the action must be a Hermitian Lorentz scalar. Remember that some Lagrangians are equivalent up to certain factor.

b) (*10 p.*) Determine how φ transforms under parity and under time reversal for these two transformations to both be symmetries of the theory (i.e. leave the action invariant).

Hint: to not mess up signs, remember that a parity transformation and a space inversion each belong to a specific subgroup of Lorentz transformation.

Now consider,

$$\mathcal{L} = \bar{\Psi}\left(i\gamma_\mu\partial^\mu - m\right)\Psi + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}M^2\varphi^2 + g\varphi\bar{\Psi}\Psi,$$

where $g \in \mathbb{R}$.

c) (*15 p.*) Determine how φ must transform for parity and time reversal to be a symmetry of the theory.