3) DIRAC PARTICLE IN CENTRAL POTENTIAL

- STATIONARY STATES IN CENTRAL POTENTIAL $V(r)$

e.g. H-ATOM $V(r) = -\frac{\alpha}{r}\hbar c$

$$\alpha \simeq \frac{1}{137} = \frac{e^2}{4\pi\varepsilon_0 \hbar c}$$

$$\left(c\bar{\alpha}.\hat{\bar{p}} + \beta m_0 c^2 + V(r) \right) \Psi = i\hbar \frac{\partial}{\partial t}\Psi$$

$$\hat{\bar{p}} = -i\hbar \bar{\nabla}$$

$\downarrow$ FOR STATIONARY SOLUTION $\Psi(\bar{r},t) = e^{-\frac{i}{\hbar}Et}\psi(\bar{r})$

(V DOES NOT DEPEND ON t)

$$\left(c\bar{\alpha}.\hat{\bar{p}} + \beta m_0 c^2 + V(r) \right) \psi(\bar{r}) = E\psi(\bar{r})$$

$$\underline{\underline{H_D\,\psi(\bar{r}) = E\psi(\bar{r})}}$$

DIRAC HAMILTONIAN

$$\boxed{H_D = c\bar{\alpha}.\hat{\bar{p}} + \beta m_0 c^2 + V(r)}$$

$$\bar{\alpha} = \begin{pmatrix} 0 & \bar{\sigma} \\ \bar{\sigma} & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

- ORBITAL ANGULAR MOMENTUM

$$\hat{\bar{L}} = \bar{r} \times \hat{\bar{p}}$$

↳ FOR SCHRÖDINGER EQ. (SPINLESS PARTICLE IN CENTRAL POTENTIAL)

↓

$\bar{L}^2, L_z$ COMMUTE WITH H_S

↳ SCHRÖDINGER

⇓

ℓ^2, ℓ_z ARE GOOD QUANTUM NUMBERS FOR SCHRÖDINGER THEORY

↳ IN DIRAC THEORY

$$[H_D, L_i]$$

$$= \varepsilon_{ijk} [H_D, r_j \hat{\bar{p}}_k] \qquad [V(r), L_i] = 0$$

$$= \varepsilon_{ijk} [c\alpha_\ell \hat{\bar{p}}_\ell, r_j \hat{\bar{p}}_k]$$

$$= c\,\varepsilon_{ijk}\, \alpha_\ell \underbrace{[\hat{\bar{p}}_\ell, r_j]}_{-i\hbar\delta_{j\ell}} \hat{\bar{p}}_k \qquad \text{AS } [\hat{\bar{p}}_\ell, \hat{\bar{p}}_k] = 0$$

$$= -i\hbar c\, \varepsilon_{ijk}\, \alpha_j \hat{\bar{p}}_k$$

$$= -i\hbar c\, (\bar{\alpha} \times \hat{\bar{p}})_i \neq 0.$$

∴ $\bar{L}$ DOES NOT COMMUTE WITH H_D

↳ ℓ_z IS NOT GOOD QUANTUM NUMBER

- **SPIN** (INTRINSIC ANGULAR MOMENTUM)

$$\bar{S} = \frac{\hbar}{2}\bar{\Sigma} = \frac{\hbar}{2}\begin{pmatrix} \bar{\sigma} & 0 \\ 0 & \bar{\sigma} \end{pmatrix}$$

$$[H_D, S_i] = \frac{\hbar}{2}\left[c\,\bar{\alpha}.\hat{\bar{p}}, \bar{\Sigma}\right]$$

$$= \frac{\hbar c}{2}\left\{\begin{pmatrix} 0 & \bar{\sigma}.\hat{\bar{p}} \\ \bar{\sigma}.\hat{\bar{p}} & 0 \end{pmatrix}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} - \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}\begin{pmatrix} 0 & \bar{\sigma}.\hat{\bar{p}} \\ \bar{\sigma}.\hat{\bar{p}} & 0 \end{pmatrix}\right\}$$

$$= \frac{\hbar c}{2}\begin{pmatrix} 0 & [\bar{\sigma}.\hat{\bar{p}}, \sigma_i] \\ [\bar{\sigma}.\hat{\bar{p}}, \sigma_i] & 0 \end{pmatrix}$$

$$\downarrow \qquad [\sigma_k, \sigma_l] = 2i\,\varepsilon_{klm}\,\sigma_m$$

$$= \frac{\hbar c}{2}\begin{pmatrix} 0 & 2i\,\varepsilon_{kij}\,\hat{p}_k\sigma_j \\ 2i\,\varepsilon_{kij}\,\hat{p}_k\sigma_j & 0 \end{pmatrix}$$

$$= i\hbar c\,\varepsilon_{ijk}\,\hat{p}_k\begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}$$

$$= i\hbar c\,\varepsilon_{ijk}\,\hat{p}_k\,\alpha_j$$

$$= i\hbar c\left(\bar{\alpha} \times \hat{\bar{p}}\right)_i \neq 0$$

$\therefore$ $\bar{S}$ DOES <u>NOT</u> COMMUTE WITH H_D

$\hookrightarrow$ S_z IS NOT GOOD QUANTUM NUMBER.

- TOTAL ANGULAR MOMENTUM

$$\bar{J} = \bar{L} + \bar{S}$$

$$[H_D, J_i] = [H_D, L_i] + [H_D, S_i]$$

$$= 0$$

$$\Downarrow$$

ONLY TOTAL ANGULAR MOMENTUM IS CONSERVED BY DIRAC EQ. (FOR PARTICLE IN CENTRAL POTENTIAL)

IN DIRAC EQ. THERE IS A COUPLING BETWEEN SPIN $\bar{S}$ AND ORBITAL ANGULAR MOMENTUM $\bar{L}$
$\Rightarrow$ SPIN-ORBIT INTERACTION

$$[H_D, J^2] = 0$$

$$[H_D, J_i] = 0$$

$$[J^2, J_i] = 0$$

BUT $[J_i, J_j] = i\hbar\varepsilon_{ijk} J_k$

$\therefore$ H_D, J^2, J_z COMMUTE SIMULTANEOUSLY

SOLUTIONS OF H_D HAVE GOOD QUANTUM NUMBERS j, j_z

$$\begin{aligned} J^2 \psi &= \hbar^2 j(j+1)\psi \\ J_z \psi &= \hbar j_z \psi \end{aligned}$$

$\rightsquigarrow$ $\underline{\underline{j = \ell \pm \frac{1}{2}}}$ FOR GIVEN VALUE OF j, 2 SOLUTIONS

↳ WE NEED ANOTHER CONSERVED QUANTITY (GOOD QUANTUM NUMBER) TO DISTINGUISH BOTH CASES.

- CONSIDER OPERATOR $K \equiv \beta(\bar{\Sigma}.\bar{L} + \hbar)$

WILL SHOW THAT $\underline{\underline{[H_D, K] = 0}}$

$\hookrightarrow$ $\bar{\Sigma}.\bar{L}$ SPIN-ORBIT OPERATOR

CAN BE WRITTEN EQUIVALENTLY AS
$$\bar{\Sigma}.\bar{L} = \bar{\Sigma}.(\bar{J} - \tfrac{\hbar}{2}\bar{\Sigma})$$
$$= \bar{\Sigma}.\bar{J} - \tfrac{\hbar}{2}\underbrace{\Sigma^2}_{\Sigma^2 = 3\mathbb{1}}$$
$$= \bar{\Sigma}.\bar{J} - \tfrac{3\hbar}{2}$$

$$K = \beta(\bar{\Sigma}.\bar{L} + \hbar)$$
$$= \beta\left(\bar{\Sigma}.\bar{J} - \tfrac{\hbar}{2}\right)$$

$\hookrightarrow$ PROOF $[H_D, K] = 0$

$$[H_D, K]$$
$$= [H_D, \beta]\left(\bar{\Sigma}.\bar{J} - \tfrac{\hbar}{2}\right) + \beta\left[H_D, \bar{\Sigma}.\bar{J} - \tfrac{\hbar}{2}\right]$$
$$= c[\bar{\alpha}.\hat{P}, \beta]\left(\bar{\Sigma}.\bar{J} - \tfrac{\hbar}{2}\right) + \beta[H_D, \bar{\Sigma}.\bar{J}]$$

$\downarrow$ AS $\alpha_i\beta = -\beta\alpha_i$ $\qquad\qquad$ $\downarrow$ AS $[H_D, J_i] = 0$

$$= -2c\beta(\bar{\alpha}.\hat{P})\left(\bar{\Sigma}.\bar{J} - \tfrac{\hbar}{2}\right) + \beta[H_D, \Sigma_i]J_i$$

$\downarrow$ $[H_D, \Sigma_i] = 2ic(\bar{\alpha}\times\hat{P})_i$

$$= -2c\beta\begin{pmatrix} 0 & \bar{\sigma}.\hat{P} \\ +\bar{\sigma}.\hat{P} & 0 \end{pmatrix}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} J_i$$

$$+ \hbar c\,\beta(\bar{\alpha}.\hat{P}) + 2ic\,\beta(\bar{\alpha}\times\hat{P}).\bar{J}$$

$$(\bar{\sigma}.\hat{\bar{p}})(\bar{\sigma}.\bar{J}) = \hat{\bar{p}}.\bar{J} + i\bar{\sigma}.(\hat{\bar{p}}\times\bar{J})$$

$$\downarrow$$

$$(\bar{\sigma}.\hat{\bar{p}})\,\sigma_i = \hat{p}_i + i\,\varepsilon_{kij}\,\hat{p}_k\,\sigma_j$$

$$[H_D, K]$$

$$= -2c\,\beta\begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}\hat{\bar{p}}.\bar{J} \cancel{- 2c\,\beta\, i\,\varepsilon_{ijk}\begin{pmatrix}0 & \sigma_j\hat{p}_k\\ \sigma_j\hat{p}_k & 0\end{pmatrix}J_i}$$

$$+ c\hbar\,\beta\,\hat{\bar{p}}.\begin{pmatrix}0 & \bar{\sigma}\\ \bar{\sigma} & 0\end{pmatrix} \cancel{+ 2c\,\beta\, i\,\varepsilon_{ijk}\begin{pmatrix}0 & \sigma_j\hat{p}_k\\ \sigma_j\hat{p}_k & 0\end{pmatrix}J_i}$$

$$= -2c\,\beta\begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}\hat{\bar{p}}.\bar{J}$$

$$+ 2c\,\beta\begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}\hat{\bar{p}}.\begin{pmatrix}\frac{\hbar}{2}\bar{\sigma} & 0\\ 0 & \frac{\hbar}{2}\bar{\sigma}\end{pmatrix}$$

$$= -2c\,\beta\underbrace{\begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}}_{\gamma_5}\cdot\hat{\bar{p}}.\underbrace{\left(\bar{J} - \frac{\hbar}{2}\bar{\Sigma}\right)}_{\bar{L}}$$

$$\downarrow \qquad \hat{\bar{p}}.\bar{L} = 0$$

$$\overset{\text{IV}}{\overset{\circ}{=}} 0$$

$\hookrightarrow$ PROOF $[K, J_i] = 0$

$$[K, J_i]$$

$$= \left[\beta\left(\bar{\Sigma}.\bar{J} - \frac{\hbar}{2}\right), J_i\right]$$

$$= [\beta, J_i]\left(\bar{\Sigma}.\bar{J} - \frac{\hbar}{2}\right) + \beta\left[\bar{\Sigma}.\bar{J} - \frac{\hbar}{2}, J_i\right]$$

$$= \frac{\hbar}{2}\underbrace{[\beta, \Sigma_i]}_{0}\left(\bar{\Sigma}.\bar{J} - \frac{\hbar}{2}\right) + \beta\left[\bar{\Sigma}.\bar{J}, J_i\right]$$

$$= \beta\,\Sigma_j \underbrace{[J_j, J_i]}_{i\hbar\,\varepsilon_{jik} J_k} + \beta \underbrace{[\Sigma_j, J_i]}_{\frac{\hbar}{2}[\Sigma_j, \Sigma_i]} J_j$$

$$= \frac{\hbar}{2} 2i\, \varepsilon_{jik} \Sigma_k$$

$$= i\hbar\,\varepsilon_{jik}\,\beta\,\Sigma_j J_k + i\hbar\,\varepsilon_{jik}\,\beta\,\Sigma_k J_j$$

$$\overset{!}{=} 0$$

$\therefore$ $\boxed{H_D, J^2, J_z, K \text{ FORM A SET OF SIMULTANEOUSLY COMMUTING OPERATORS}}$

$\Downarrow$

HAVE COMMON EIGENSTATES

- EIGENVALUES OF $K = \beta(\bar{\Sigma}.\bar{L} + \hbar)$

↳ DEFINE EIGENVALUE OF K AS $-\hbar\kappa$

$$K\Psi = -\hbar\kappa\Psi$$

↳ DETERMINE K^2

$$K^2 = \beta(\bar{\Sigma}.\bar{L} + \hbar)\,\beta(\bar{\Sigma}.\bar{L} + \hbar)$$

$$\downarrow \quad \beta\Sigma_i = \Sigma_i\beta$$

$$= (\Sigma_i L_i)(\Sigma_j L_j) + 2\hbar(\bar{\Sigma}.\bar{L}) + \hbar^2$$

$$\downarrow \quad \Sigma_i\Sigma_j = \delta_{ij} + i\,\varepsilon_{ijk}\Sigma_k$$

$$= L^2 + i\,\varepsilon_{ijk}L_iL_j\Sigma_k + 2\hbar(\bar{\Sigma}.\bar{L}) + \hbar^2$$

$$= L^2 + \frac{i}{2}\varepsilon_{ijk}\underbrace{(L_iL_j - L_jL_i)}_{i\hbar\,\varepsilon_{ijl}L_l}\Sigma_k + 2\hbar(\bar{\Sigma}.\bar{L}) + \hbar^2$$

$$\downarrow \quad \varepsilon_{ijk}\varepsilon_{ijl} = 2\delta_{kl}$$

$$= L^2 - \hbar(\bar{\Sigma}.\bar{L}) + 2\hbar(\bar{\Sigma}.\bar{L}) + \hbar^2$$

$$= L^2 + 2(\bar{S}.\bar{L}) + \hbar^2$$

$$= (\bar{L} + \bar{S})^2 - \bar{S}^2 + \hbar^2 \qquad \bar{S}^2 = \frac{\hbar^2}{4}\Sigma^2 = \frac{3\hbar^2}{4}\mathbb{1}$$

$$= J^2 + \frac{\hbar^2}{4}\mathbb{1}$$

$$K^2 = J^2 + \frac{\hbar^2}{4}\mathbb{1}$$

FOR EIGENSTATE OF J

$$J^2 \psi = \hbar^2 j(j+1)\psi$$

$$\downarrow$$

$$K^2 \psi = \left(J^2 + \frac{\hbar^2}{4}\right)\psi$$

$$= \hbar^2 \left(j(j+1) + \frac{1}{4}\right)\psi$$

$$= \hbar^2 \left(j + \frac{1}{2}\right)^2 \psi$$

$$= \hbar^2 \kappa^2 \psi$$

$$\boxed{|\kappa| = j + \frac{1}{2}}$$

$$\kappa = \pm\left(j + \frac{1}{2}\right)$$

↳ SIGN OF κ

$$K\psi = \mp \hbar \left(j + \frac{1}{2}\right)\psi$$

$$\beta \begin{pmatrix} \bar{\sigma}.\bar{L} + \hbar & 0 \\ 0 & \bar{\sigma}.\bar{L} + \hbar \end{pmatrix} \psi = \mp \hbar \left(j + \frac{1}{2}\right)\psi$$

$$\downarrow \quad \psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$

WITH ψ_A, ψ_B : 2×1 PAULI SPINORS

$$\begin{pmatrix} \bar{\sigma}.\bar{L} + \hbar & 0 \\ 0 & -\bar{\sigma}.\bar{L} - \hbar \end{pmatrix} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \mp \hbar \left(j + \frac{1}{2}\right) \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$

$$\begin{cases} (\bar{\sigma}.\bar{L})\,\psi_A = \mp\hbar\left(j+\frac{1}{2}\pm 1\right)\psi_A \\ -(\bar{\sigma}.\bar{L})\,\psi_B = \mp\hbar\left(j+\frac{1}{2}\mp 1\right)\psi_B \end{cases}$$

FOR $\underset{\sim\sim\sim}{K = +(j+\frac{1}{2})}$ $\begin{cases} (\bar{\sigma}.\bar{L})\,\psi_A = -\hbar\left(j+\frac{3}{2}\right)\psi_A \\ (\bar{\sigma}.\bar{L})\,\psi_B = \hbar\left(j-\frac{1}{2}\right)\psi_B \end{cases}$

$\underset{\sim\sim\sim}{K = -(j+\frac{1}{2})}$ $\begin{cases} (\bar{\sigma}.\bar{L})\,\psi_A = \hbar\left(j-\frac{1}{2}\right)\psi_A \\ (\bar{\sigma}.\bar{L})\,\psi_B = -\hbar\left(j+\frac{3}{2}\right)\psi_B \end{cases}$

ψ_A, ψ_B ARE EIGENSTATES OF $\bar{\sigma}.\bar{L}$
BUT WITH DIFFERENT EIGENVALUES

$\hookrightarrow$ $L^2 = J^2 - \hbar\,\bar{\Sigma}.\bar{L} - \frac{3}{4}\hbar^2$

$$L^2 \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \hbar^2\left(j(j+1)-\frac{3}{4}\right)\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} - \hbar \begin{pmatrix} \bar{\sigma}.\bar{L}\ \psi_A \\ \bar{\sigma}.\bar{L}\ \psi_B \end{pmatrix}$$

FOR $\underset{\sim\sim\sim}{K = +(j+\frac{1}{2})}$

$$L^2 \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \hbar^2\left[j(j+1)-\frac{3}{4}\right]\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} + \hbar^2 \begin{pmatrix} (j+\frac{3}{2})\,\psi_A \\ -(j-\frac{1}{2})\,\psi_B \end{pmatrix}$$

$$= \hbar^2 \begin{pmatrix} \left(j^2+2j+\frac{3}{4}\right)\psi_A \\ \left(j^2-\frac{1}{4}\right)\psi_B \end{pmatrix}$$

DENOTE $\underline{\underline{\ell_{\pm} \equiv j \pm \frac{1}{2}}}$

$$\ell_+(\ell_+ + 1) = j^2 + 2j + \frac{3}{4}$$
$$\ell_-(\ell_- + 1) = j^2 - \frac{1}{4}$$

$$L^2 \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \hbar^2 \begin{pmatrix} \ell_+(\ell_+ + 1)\,\psi_A \\ \ell_-(\ell_- + 1)\,\psi_B \end{pmatrix}$$

ψ_A, ψ_B ARE EIGENSTATES OF L^2

BUT WITH <u>DIFFERENT</u> EIGENVALUES.

↳ DIRAC SPINOR <u>NOT</u> EIGENSTATE OF L^2

FOR $\underset{\sim\sim\sim}{K = -(j + \frac{1}{2})}$

$$L^2 \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \hbar^2 \begin{pmatrix} \ell_-(\ell_- + 1)\,\psi_A \\ \ell_+(\ell_+ + 1)\,\psi_B \end{pmatrix}$$

$\therefore$

$$K = +(j + \tfrac{1}{2}) > 0 \;\Rightarrow\; \ell_A = \ell_+ ,\; \ell_B = \ell_-$$

$$K = -(j + \tfrac{1}{2}) < 0 \;\Rightarrow\; \ell_A = \ell_- ,\; \ell_B = \ell_+$$

<u>NOTE</u> SIGN OF K INDICATES WHETHER $\underline{L}$ & $\bar{\sigma}$ ARE // OR ANTI-//

(OPPOSITE FOR UPPER COMPONENT ψ_A

& LOWER COMPONENT ψ_B)

$$\begin{cases} (\bar{\sigma}.\underline{L})\,\psi_A = -\hbar\,(K + 1)\,\psi_A \\ (\bar{\sigma}.\underline{L})\,\psi_B = +\hbar\,(K - 1)\,\psi_B \end{cases}$$

- SOLUTION OF DIRAC EQ. IN CENTRAL POTENTIAL.

$\hookrightarrow \quad H_D \psi = E\psi \qquad \psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$

$$\begin{pmatrix} 0 & c\,\bar{\sigma}.\hat{\bar{p}} \\ c\,\bar{\sigma}.\hat{\bar{p}} & 0 \end{pmatrix} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \begin{pmatrix} (E - V(r) - m_0 c^2)\,\psi_A \\ (E - V(r) + m_0 c^2)\,\psi_B \end{pmatrix}$$

$$\begin{cases} c(\bar{\sigma}.\hat{\bar{p}})\,\psi_B = (E - V(r) - m_0 c^2)\,\psi_A \\ c(\bar{\sigma}.\hat{\bar{p}})\,\psi_A = (E - V(r) + m_0 c^2)\,\psi_B \end{cases}$$

DENOTE $\boxed{\psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \begin{pmatrix} g(r)\,\varphi_{j l_A m} \\ i f(r)\,\varphi_{j l_B m} \end{pmatrix}}$

ψ HAS GOOD j, m, K

$$\begin{cases} J^2 \psi = \hbar^2 j(j+1)\psi \\ J_z \psi = \hbar m\, \psi \\ K\psi = -\hbar K \psi \quad \text{WITH } K = \pm (j + \frac{1}{2}) \end{cases}$$

FOR $K > 0 \longrightarrow l_A = j + \frac{1}{2}$

$l_B = j - \frac{1}{2}$

$K < 0 \longrightarrow l_A = j - \frac{1}{2}$

$l_B = j + \frac{1}{2}$

φ_{jlm} DEPENDS ONLY ON ANGLES

$\hookrightarrow$ EIGENSTATES OF J^2, J_z, L^2

PAULI SPINOR $\downarrow$

$$\boxed{\varphi_{jlm}(\bar{e}_r) = \sum_{m_l} \sum_{m_s} \langle l\, m_l, \tfrac{1}{2} m_s | j\, m \rangle Y_{l m_l}(\bar{e}_r)\, \chi_{m_s}}$$

$$\hookrightarrow \quad \bar{\sigma}.\hat{\bar{p}} = \underbrace{(\bar{\sigma}.\bar{e}_r)(\bar{\sigma}.\bar{e}_r)}_{1}(\bar{\sigma}.\bar{p}) \qquad \bar{e}_r = \frac{\bar{r}}{r}$$

$$= (\bar{\sigma}.\bar{e}_r)\left\{ \bar{e}_r.\hat{\bar{p}} + i\,\bar{\sigma}.(\bar{e}_r \times \hat{\bar{p}}) \right\}$$

$$\downarrow \qquad \bar{e}_r.\hat{\bar{p}} = -i\hbar\,\bar{e}_r.\bar{\nabla}$$

$$= -i\hbar\,\frac{\partial}{\partial r}$$

$$= (\bar{\sigma}.\bar{e}_r)\left\{ -i\hbar\frac{\partial}{\partial r} + \frac{i}{r}\,\bar{\sigma}.\underbrace{(\bar{r}\times\hat{\bar{p}})}_{\bar{L}} \right\}$$

$$\bar{\sigma}.\hat{\bar{p}} = (\bar{\sigma}.\bar{e}_r)\left\{ -i\hbar\frac{\partial}{\partial r} + \frac{i}{r}\,\bar{\sigma}.\bar{L} \right\}$$

NOTE $\bar{\sigma}.\bar{e}_r$ & $\bar{\sigma}.\bar{L}$ ONLY ACT ON ANGULAR PARTS OF WAVE FUNCTION.

$$\begin{cases} c(\bar{\sigma}.\hat{\bar{p}})\,\psi_B = (E - V(r) - m_0c^2)\,\psi_A \\ c(\bar{\sigma}.\hat{\bar{p}})\,\psi_A = (E - V(r) - m_0c^2)\,\psi_B \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} c(\bar{\sigma}.\bar{e}_r)\left\{ \hbar f'(r) - \hbar\frac{f(r)}{r}(K-1) \right\} \varphi_{j l_B m} \\ = (E - V(r) - m_0c^2)\, g(r)\, \varphi_{j l_A m} \\ c(\bar{\sigma}.\bar{e}_r)\left\{ -i\hbar\, g'(r) - i\hbar\frac{g(r)}{r}(K+1) \right\} \varphi_{j l_A m} \\ = (E - V(r) + m_0c^2)\, i f(r)\, \varphi_{j l_B m} \end{cases}$$

$\hookrightarrow \quad (\bar{\sigma}\cdot\bar{e}_r)\ \varphi_{jlm}$

$\rightsquigarrow \quad [\bar{\sigma}\cdot\bar{e}_r\ ,\ J_i]$

$$= \frac{1}{r}\left[\bar{\sigma}\cdot\bar{r}\ ,\ (\bar{r}\times\hat{\bar{p}})_i + \frac{\hbar}{2}\sigma_i\right]$$

$$= \frac{1}{r}\Big\{\varepsilon_{ijk}\ \sigma_l\ \underbrace{[r_l\ ,\ r_j\hat{\bar{p}}_k]}_{r_j\underbrace{[r_l,\hat{\bar{p}}_k]}_{i\hbar\delta_{kl}}} + \frac{\hbar}{2} r_l\ \underbrace{[\sigma_l\ ,\ \sigma_i]}_{2i\,\varepsilon_{lij}\sigma_j}\Big\}$$

$$= \frac{i\hbar}{r}\left\{\varepsilon_{ijk}\ r_j\sigma_k + \varepsilon_{ijl}\ \sigma_j r_l\right\}$$

$$= 0$$

$\therefore\quad (\bar{\sigma}\cdot\bar{e}_r)\ \varphi_{jlm}$ IS EIGENSTATE OF $\bar{J}^2, J_z$ WITH QUANTUM NUMBERS j, m

$\rightsquigarrow$ UNDER SPATIAL INVERSION

$\bar{r}\rightarrow-\bar{r}$

$\bar{\sigma}\rightarrow\bar{\sigma}$

$(\bar{\sigma}\cdot\bar{e}_r)\longrightarrow-(\bar{\sigma}\cdot\bar{e}_r)$ TRANSFORMS AS PSEUDOSCALAR

φ_{jlm} : PARITY $(-1)^l$

$(\bar{\sigma}\cdot\bar{e}_r)$ CHANGES PARITY OF STATE

$$(\bar{\sigma}\cdot\bar{e}_r)\underbrace{\varphi_{jl_Am}}_{\text{PARITY }(-1)^{l_A}} = C\ \underbrace{\varphi_{jl_Bm}}_{(-1)^{l_B}=-(-1)^{l_A}} \quad \text{AS } \underline{l_B = l_A \mp 1}$$

$$(\bar{\sigma}\cdot\bar{e}_r)^2 = 1 \Rightarrow C^2 = 1 \Rightarrow C = \pm 1$$

$$\underline{\underline{(\bar{\sigma}\cdot\bar{e}_r)\,\varphi_{j\ell_A m} = -\,\varphi_{j\ell_B m}}}$$

PHASE CONVENTION

CHECK FOR $\ell_A = 0$

$$(\bar{\sigma}\cdot\bar{e}_r) = \sin\theta\cos\phi\,\sigma_x + \sin\theta\sin\phi\,\sigma_y + \cos\theta\,\sigma_z$$

$$\downarrow \qquad S_\pm = \frac{\hbar}{2}(\sigma_x \pm i\sigma_y)$$

$$\hbar\sigma_x = S_+ + S_-$$

$$\hbar\sigma_y = -i(S_+ - S_-)$$

$$\hbar(\bar{\sigma}\cdot\bar{e}_r) = \sin\theta\, e^{-i\phi} S_+ + \sin\theta\, e^{+i\phi} S_- + 2\cos\theta\, S_z$$

$$Y_{1,1} = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$\downarrow \qquad Y_{1,-1} = +\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\phi}$$

$$Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$\hbar(\bar{\sigma}\cdot\bar{e}_r) = 2\sqrt{\frac{2\pi}{3}}\,Y_{1,-1}\,S_+ - 2\sqrt{\frac{2\pi}{3}}\,Y_{1,+1}\,S_- + 2\sqrt{\frac{4\pi}{3}}\,Y_{1,0}\,S_z$$

SPECIAL CASE $\ell_A = 0$, $\ell_B = 1$, $j = +\frac{1}{2}$, $m = +\frac{1}{2}$

$$\varphi_{\frac{1}{2}0\frac{1}{2}}(\bar{e}_r) = Y_{00}(\bar{e}_r)\begin{pmatrix}1\\0\end{pmatrix} \rightarrow \chi_{+\frac{1}{2}}$$

$$\varphi_{\frac{1}{2}1\frac{1}{2}}(\bar{e}_r) = \sum_{m_\ell m_s} \langle 1\, m_\ell, \tfrac{1}{2}\, m_s \,|\, \tfrac{1}{2} + \tfrac{1}{2}\rangle\, Y_{1 m_\ell}(\bar{e}_r)\,\chi_{m_s}$$

$$\varphi_{\frac{1}{2}1\frac{1}{2}}(\bar{e}_r) = \langle 1\,0, \tfrac{1}{2} + \tfrac{1}{2} \mid \tfrac{1}{2} + \tfrac{1}{2} \rangle Y_{10}(\bar{e}_r)\, \chi_{+\frac{1}{2}}$$

$$+ \langle 1 +1, \tfrac{1}{2} - \tfrac{1}{2} \mid \tfrac{1}{2} + \tfrac{1}{2} \rangle Y_{1+1}(\bar{e}_r)\, \chi_{-\frac{1}{2}}$$

$$= -\sqrt{\tfrac{1}{3}}\, Y_{10}(\bar{e}_r) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sqrt{\tfrac{2}{3}}\, Y_{11}(\bar{e}_r) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\therefore \hbar\, (\bar{\sigma}\cdot\bar{e}_r)\, \varphi_{\frac{1}{2}0\frac{1}{2}}(\bar{e}_r)$$

$$= -2\sqrt{\frac{2\pi}{3}}\, \frac{1}{\sqrt{4\pi}}\, Y_{1,+1}(\bar{e}_r)\, \underbrace{S_- \begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\hbar \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

$$+ 2\sqrt{\frac{4\pi}{3}}\, \frac{1}{\sqrt{4\pi}}\, Y_{1,0}(\bar{e}_r)\, \underbrace{S_z \begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\frac{\hbar}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

$$= \hbar \left\{ -\sqrt{\tfrac{2}{3}}\, Y_{11}(\bar{e}_r) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sqrt{\tfrac{1}{3}}\, Y_{10}(\bar{e}_r) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

$$= -\hbar\, \varphi_{\frac{1}{2}1\frac{1}{2}}(\bar{e}_r)$$

$$\therefore \quad \underline{\underline{(\bar{\sigma}\cdot\bar{e}_r)\, \varphi_{\frac{1}{2}0\frac{1}{2}}(\bar{e}_r) = -\varphi_{\frac{1}{2}1\frac{1}{2}}(\bar{e}_r)}}$$

↳ COUPLED RADIAL EQUATIONS FOR f, g

$$\begin{cases} -\hbar c \left\{ f' - \frac{f}{r}(K-1) \right\} = \left(E - V(r) - m_0 c^2\right) g \\ -\hbar c \left\{ -g' - \frac{g}{r}(K+1) \right\} = \left(E - V(r) + m_0 c^2\right) f \end{cases}$$

$$\Big\downarrow \qquad g(r) \equiv \frac{G(r)}{r}, \quad f(r) \equiv \frac{F(r)}{r}$$

$$\boxed{\begin{cases} \dfrac{\partial F}{\partial r} - K\dfrac{F}{r} = \dfrac{1}{\hbar c}\left(-E + V(r) + m_0 c^2\right) G \\[2ex] \dfrac{\partial G}{\partial r} + K\dfrac{G}{r} = \dfrac{1}{\hbar c}\left(E - V(r) + m_0 c^2\right) F \end{cases}}$$

↳ TOTAL SOLUTION

$$\psi = \begin{pmatrix} \dfrac{G(r)}{r}\, \varphi_{j \ell_A m}(\bar{e}_r) \\[2ex] i\,\dfrac{F(r)}{r}\, \varphi_{j \ell_B m}(\bar{e}_r) \end{pmatrix}$$

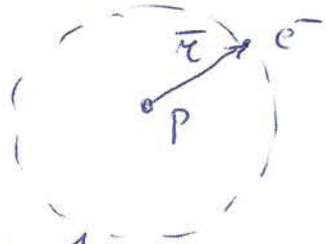
FOR $K = +\left(j + \frac{1}{2}\right) \qquad \ell_A = j + \frac{1}{2}, \; \ell_B = j - \frac{1}{2}$

$K = -\left(j + \frac{1}{2}\right) \qquad \ell_A = j - \frac{1}{2}, \; \ell_B = j + \frac{1}{2}$

HYDROGEN ATOM IN DIRAC THEORY

↳ BOUND STATES IN

COULOMB POTENTIAL OF e^- ORBITING AROUND PROTON



$$\boxed{V(r) = -\frac{Z\alpha \hbar c}{r}}$$

$$\alpha \equiv \frac{e^2}{4\pi\varepsilon_0 \hbar c} \simeq \frac{1}{137}$$

↳ FINE-STRUCTURE CONSTANT

FOR PROTON $Z = +1$
He^+ $Z = +2$, ...

↳ INTRODUCE

$$k_1 \equiv \frac{1}{\hbar c}\left(E + m_0 c^2\right)$$

$$k_2 \equiv \frac{1}{\hbar c}\left(-E + m_0 c^2\right)$$

FOR BOUND STATES $E < m_0 c^2 \implies k_2 > 0$

$$\begin{cases} \dfrac{\partial F}{\partial r} - K\dfrac{F}{r} = \left(k_2 - \dfrac{Z\alpha}{r}\right) G \\[2ex] \dfrac{\partial G}{\partial r} + K\dfrac{G}{r} = \left(k_1 + \dfrac{Z\alpha}{r}\right) F \end{cases}$$

DIMENSIONLESS VARIABLE

$$\rho \equiv 2\sqrt{k_1 k_2}\, r = \frac{2}{\hbar c}\left(m_0^2 c^4 - E^2\right)^{1/2} r$$

$$\begin{cases} \dfrac{\partial F}{\partial \rho} - K\dfrac{F}{\rho} = \left(\dfrac{1}{2}\sqrt{\dfrac{k_2}{k_1}} - \dfrac{Z\alpha}{\rho}\right) G \\[2ex] \dfrac{\partial G}{\partial \rho} + K\dfrac{G}{\rho} = \left(\dfrac{1}{2}\sqrt{\dfrac{k_1}{k_2}} + \dfrac{Z\alpha}{\rho}\right) F \end{cases}$$

↳ ASYMPTOTIC FORM $\rho \to \infty$

$$\frac{\partial F}{\partial \rho} \simeq \frac{1}{2}\sqrt{\frac{k_2}{k_1}}\, G \qquad\qquad \frac{\partial^2 F}{\partial \rho^2} = \frac{1}{4} F$$

$$\Longrightarrow$$

$$\frac{\partial G}{\partial \rho} \simeq \frac{1}{2}\sqrt{\frac{k_1}{k_2}}\, F \qquad\qquad \frac{\partial^2 G}{\partial \rho^2} = \frac{1}{4} G$$

$$\therefore \quad F(\rho) \underset{\rho \to \infty}{\sim} e^{-\rho/2}$$

$$G(\rho) \underset{\rho \to \infty}{\sim} e^{-\rho/2}$$

↳ SOLUTION IN POWER SERIES FORM

$$F(\rho) = \sqrt{k_2}\, e^{-\rho/2} \sum_{m=0}^{\infty} a_m \rho^{m+\gamma}$$

$$G(\rho) = \sqrt{k_1}\, e^{-\rho/2} \sum_{m=0}^{\infty} b_m \rho^{m+\gamma}$$

BEHAVIOR FOR $\rho \to 0$ $\qquad F(\rho) \sim \rho^{\gamma}$

$$G(\rho) \sim \rho^{\gamma}$$

↳ RECURSION RELATIONS

$$\frac{\partial F}{\partial \rho} = -\frac{1}{2} F + \sqrt{k_2}\, e^{-\rho/2} \sum_{m=0}^{\infty} a_m (m+\gamma) \rho^{m+\gamma-1}$$

$$\frac{\partial G}{\partial \rho} = -\frac{1}{2} G + \sqrt{k_1}\, e^{-\rho/2} \sum_{m=0}^{\infty} b_m (m+\gamma) \rho^{m+\gamma-1}$$

$$\hookrightarrow \frac{\partial F}{\partial \rho} - K\frac{F}{\rho} = \sqrt{k_2}\, e^{-\rho/2} \sum_{m=0}^{\infty} \left\{ -\frac{1}{2} a_m \rho^{m+\gamma} - K a_m \rho^{m+\gamma-1} + a_m (m+\gamma) \rho^{m+\gamma-1} \right\}$$

$\downarrow$ IN 1° TERM : INTRODUCE $\underline{a_{-1} = 0}$

& EXTEND SUM $-1 \to +\infty$

$$= \sqrt{k_2}\, e^{-\rho/2} \left\{ -\sum_{m=-1}^{\infty} \frac{1}{2} a_m \rho^{m+\gamma} + \sum_{m=0}^{\infty} a_m \rho^{m+\gamma-1} (-K + m + \gamma) \right\}$$

$\downarrow$

$m' = m - 1$

$$= \sqrt{k_2}\, e^{-\rho/2} \sum_{m=-1}^{\infty} \rho^{m+\gamma} \left\{ -\frac{1}{2} a_m + a_{m+1} (m+1+\gamma-K) \right\}$$

$$\hookrightarrow \left(\frac{1}{2}\sqrt{\frac{k_2}{k_1}} - \frac{Z\alpha}{\rho} \right) G = \sqrt{k_2}\, e^{-\rho/2} \sum_{m=-1}^{\infty} \left\{ \frac{1}{2} b_m - Z\alpha \frac{k_1}{\sqrt{k_1 k_2}} b_{m+1} \right\} \rho^{m+\gamma}$$

WITH $\underline{b_{-1} = 0}$

∴ EQUATE BOTH SIDES

$$\boxed{-\frac{1}{2} a_m + (m+1+\gamma-K)\, a_{m+1} = \frac{1}{2} b_m - \frac{Z\alpha\,(E + m_0 c^2)}{\hbar c \sqrt{k_1 k_2}}\, b_{m+1}}$$

ANALOGOUSLY $a \leftrightarrow b$, $K \leftrightarrow -K$, $Z\alpha \to -Z\alpha$, $k_1 \leftrightarrow k_2$

$$\boxed{-\frac{1}{2} b_m + (m+1+\gamma+K)\, b_{m+1} = \frac{1}{2} a_m + \frac{Z\alpha\,(-E + m_0 c^2)}{\hbar c \sqrt{k_1 k_2}}\, a_{m+1}}$$

↳ TERM FOR $m = -1$

$$a_{-1} = b_{-1} = 0$$

$$\begin{cases} (\gamma - K)\, a_0 = - \dfrac{Z\alpha\,(E + m_0 c^2)}{\hbar c \sqrt{k_1 k_2}}\, b_0 \\ (\gamma + K)\, b_0 = + \dfrac{Z\alpha\,(-E + m_0 c^2)}{\hbar c \sqrt{k_1 k_2}}\, a_0 \end{cases}$$

$$\Downarrow$$

$$(\gamma^2 - K^2) = -(Z\alpha)^2 \frac{m_0^2 c^4 - E^2}{(\hbar c)^2 k_1 k_2}$$

$$= -(Z\alpha)^2$$

CONDITION TO HAVE A REGULAR SOLUTION AT $r = 0$

$$\gamma = \sqrt{K^2 - (Z\alpha)^2}$$

$$\boxed{\gamma = \sqrt{(j + \tfrac{1}{2})^2 - (Z\alpha)^2}}$$

↳ ADD EQUATIONS & $m \to m-1$

$$\left(m + \gamma + \frac{Z\alpha E}{\hbar c\sqrt{k_1 k_2}}\right)(a_m + b_m)$$

$$= (a_{m-1} + b_{m-1}) + \left(K + \frac{Z\alpha m_0 c^2}{\hbar c\sqrt{k_1 k_2}}\right)(a_m - b_m)$$

↳ SUBTRACT EQUATIONS & $m \to m-1$

$$\left(m + \gamma - \frac{Z\alpha E}{\hbar c\sqrt{k_1 k_2}}\right)(a_m - b_m)$$

$$= \left(K - \frac{Z\alpha m_0 c^2}{\hbar c\sqrt{k_1 k_2}}\right)(a_m + b_m)$$

$$\therefore \quad (a_m - b_m) = \frac{\left(K - \frac{Z\alpha m_0 c^2}{\hbar c\sqrt{k_1 k_2}}\right)}{\left(m + \gamma - \frac{Z\alpha E}{\hbar c\sqrt{k_1 k_2}}\right)}(a_m + b_m)$$

↳ PLUG INTO 1° EQ.

$$(a_m + b_m)\left[(m+\gamma)^2 - \frac{(Z\alpha)^2 E^2}{(\hbar c)^2 k_1 k_2} - K^2 + \frac{(Z\alpha)^2 m_0^2 c^4}{(\hbar c)^2 k_1 k_2}\right]$$

$$= \left(m + \gamma - \frac{Z\alpha E}{\hbar c\sqrt{k_1 k_2}}\right)(a_{m-1} + b_{m-1})$$

$$\Updownarrow$$

$$(a_m + b_m)\left[m^2 + 2m\gamma + \gamma^2 + (Z\alpha)^2 - K^2\right]$$

$$= \left(m + \gamma - \frac{Z\alpha E}{\hbar c\sqrt{k_1 k_2}}\right)(a_{m-1} + b_{m-1})$$

$$\Downarrow \qquad \gamma^2 = K^2 - (Z\alpha)^2$$

$$\left(a_m + b_m\right) = \frac{m - \left(\frac{Z\alpha E}{\hbar c\sqrt{k_1 k_2}} - \gamma\right)}{m\,(2\gamma + m)} \left(a_{m-1} + b_{m-1}\right)$$

$$= \frac{\left[1 - \left(\frac{Z\alpha E}{\hbar c\sqrt{k_1 k_2}} - \gamma\right)\right] \cdots \left[m - \left(\frac{Z\alpha E}{\hbar c\sqrt{k_1 k_2}} - \gamma\right)\right]}{m!\,(2\gamma + 1) \cdots (2\gamma + m)} \left(a_0 + b_0\right)$$

$$\left(a_m - b_m\right) = \frac{\left(K - \frac{Z\alpha\, m_0 c^2}{\hbar c\sqrt{k_1 k_2}}\right)}{\left[m - \left(\frac{Z\alpha E}{\hbar c\sqrt{k_1 k_2}} - \gamma\right)\right]} \left(a_m + b_m\right)$$

$$\downarrow \quad \left(a_0 + b_0\right) = \frac{\left(\gamma - \frac{Z\alpha E}{\hbar c\sqrt{k_1 k_2}}\right)}{\left(K - \frac{Z\alpha\, m_0 c^2}{\hbar c\sqrt{k_1 k_2}}\right)} \left(a_0 - b_0\right)$$

$$\left(a_m - b_m\right) = (-1)^m \frac{\left[\frac{Z\alpha E}{\hbar c\sqrt{k_1 k_2}} - \gamma\right]\left[\left(\frac{Z\alpha E}{\hbar c\sqrt{k_1 k_2}} - \gamma\right) - 1\right] \cdots \left[\left(\frac{Z\alpha E}{\hbar c\sqrt{k_1 k_2}} - \gamma\right) - m + 1\right]}{m!\,(2\gamma + 1) \cdots (2\gamma + m)}$$

$$\cdot \left(a_0 - b_0\right).$$

SERIES CAN BE EXPRESSED THROUGH CONFLUENT HYPERGEOMETRIC FUNCTION

$${}_1F_1(a, b; z) \equiv \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!}$$

$$= 1 + \frac{a}{b} z + \frac{a(a+1)}{b(b+1)} \frac{z^2}{2!} + \cdots$$

POCHHAMMER SYMBOL $(a)_k = \dfrac{\Gamma(a+k)}{\Gamma(a)}$

↳ FOR NORMALIZABLE SOLUTIONS

SERIES MUST TERMINATE FOR $m = n' \geq 0$

i.e. $a_{n'+1} = b_{n'+1} = 0$

$$\Downarrow$$

$$\underline{\underline{a_{n'} = -b_{n'}}}$$

$$(a_{n'} - b_{n'})\left(n' + \gamma - \frac{Z\alpha E}{\hbar c\sqrt{k_1 k_2}}\right)$$

$$= (a_{n'} + b_{n'})\left(K - \frac{Z\alpha\, m_0 c^2}{\hbar c\sqrt{k_1 k_2}}\right)$$

$$= 0$$

$$\Downarrow$$

$$\boxed{n' = \frac{Z\alpha E}{\hbar c\sqrt{k_1 k_2}} - \gamma} \geq 0$$

INTEGER
$n' = 0, 1, \ldots$

$$\left(\frac{m_0^2 c^4}{E^2} - 1\right)^{1/2} = \frac{Z\alpha}{(n' + \gamma)}$$

$$E = \frac{m_0 c^2}{\left[1 + \frac{(Z\alpha)^2}{(n'+\gamma)^2}\right]^{1/2}}$$

$$\Downarrow \qquad \gamma = \left[(j + \tfrac{1}{2})^2 - (Z\alpha)^2\right]^{1/2}$$

$$\boxed{E = \frac{m_0 c^2}{\left[1 + \frac{(Z\alpha)^2}{\left(n' + \sqrt{(j+\frac{1}{2})^2 - (Z\alpha)^2}\right)^2}\right]^{1/2}}}$$

$n' = 0, 1, \ldots$

↳ PERTURBATIVE EXPANSION IN $(Z\alpha)$

$$(Z\alpha) \ll 1$$

FOR $Z=1$, $\alpha = \frac{1}{137}$: $Z\alpha \simeq 10^{-2} \ll 1$

$$E \simeq m_0 c^2 \left[1 + \frac{(Z\alpha)^2}{\left(n' + j + \frac{1}{2} - \frac{Z^2\alpha^2}{2(j+\frac{1}{2})} \right)^2} \right]^{-1/2}$$

$$\simeq m_0 c^2 \left[1 + \frac{(Z\alpha)^2}{(n'+j+\frac{1}{2})^2 \left(1 - \frac{(Z\alpha)^2}{(j+\frac{1}{2})(n'+j+\frac{1}{2})} \right)} \right]^{-1/2}$$

$$\simeq m_0 c^2 \left[1 + \frac{(Z\alpha)^2}{(n'+j+\frac{1}{2})^2} + \frac{(Z\alpha)^4}{(j+\frac{1}{2})(n'+j+\frac{1}{2})^3} \right]^{-1/2}$$

$$\downarrow \qquad (1 + ax + bx^2)^{-1/2} \underset{x \ll 1}{\simeq} 1 - \frac{a}{2}x + \frac{1}{2}x^2\left(\frac{3}{4}a^2 - b\right)$$

$$E \simeq m_0 c^2 \left[1 - \frac{1}{2}\frac{(Z\alpha)^2}{(n'+j+\frac{1}{2})^2} + \frac{1}{2}\frac{(Z\alpha)^4}{(n'+j+\frac{1}{2})^4}\left(\frac{3}{4} - \frac{n'+j+\frac{1}{2}}{j+\frac{1}{2}} \right) \right]$$

INTRODUCE PRINCIPAL QUANTUM NUMBER

$$\underline{\underline{n \equiv n' + j + \tfrac{1}{2}}}$$

$n = 1, 2, \ldots$

$n' = 0,\ j = \tfrac{1}{2} \Rightarrow n = 1$

BINDING ENERGY E_{nj}

$$\boxed{E_{nj} \equiv E - m_0 c^2 \simeq -\frac{1}{2}(m_0 c^2)\frac{(Z\alpha)^2}{n^2} \cdot \Big[\underset{\substack{\uparrow \\ \text{BOHR THEORY}}}{1} + \underbrace{\frac{(Z\alpha)^2}{n^2}\Big(-\frac{3}{4} + \frac{n}{j+\frac{1}{2}}\Big)}_{\text{FINE STRUCTURE}} \Big]}$$

IN DIRAC THEORY BINDING ENERGY DEPENDS ON $\underline{\underline{n \text{ AND } j}}$

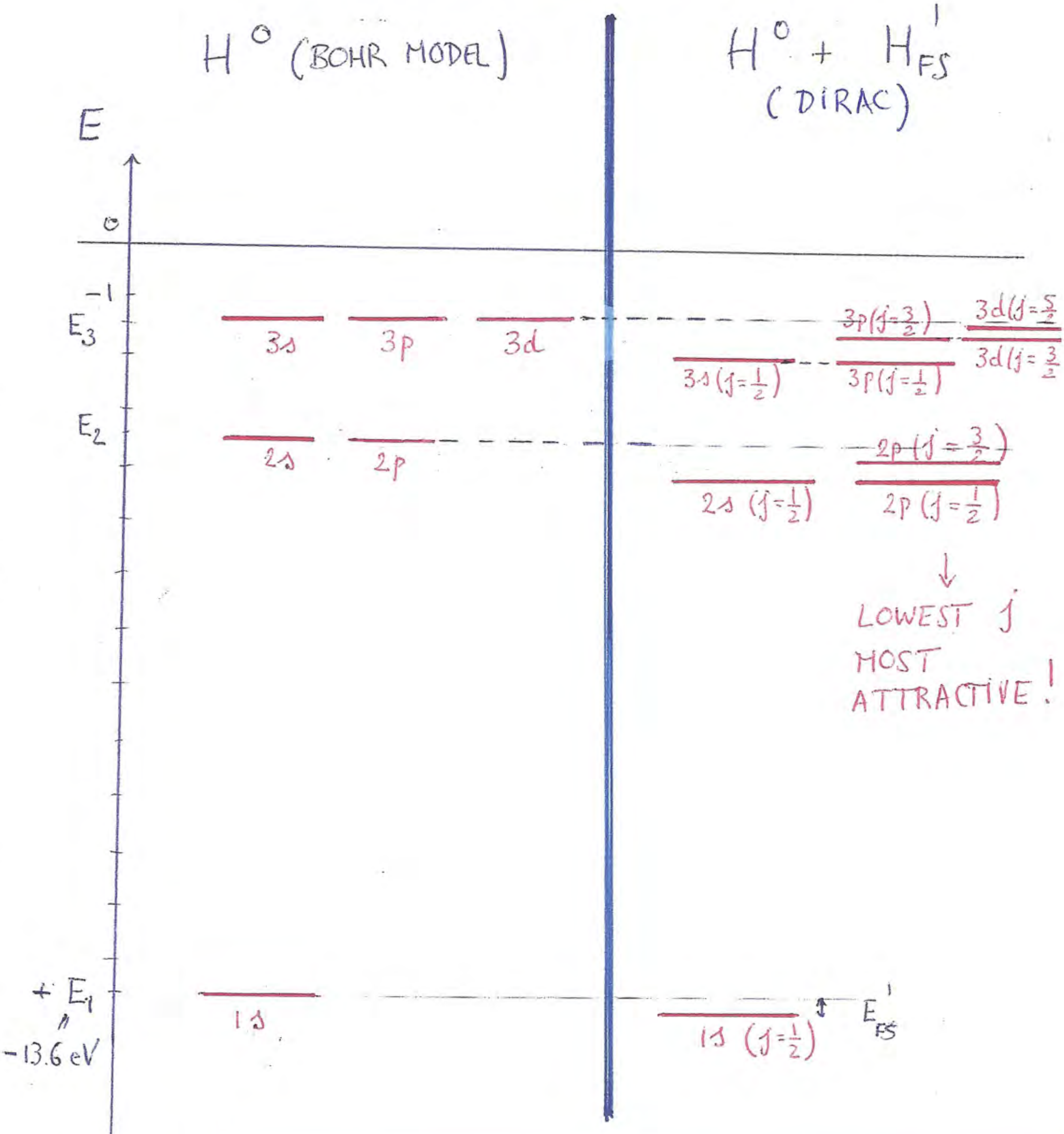
FOR H $(Z = 1)$

$$E_{nj} = -\frac{13.6\ eV}{n^2}\Big[1 + \frac{\alpha^2}{n^2}\Big(-\frac{3}{4} + \frac{n}{j+\frac{1}{2}}\Big)\Big]$$

$$= E_n + E'_{FS} \qquad \text{WITH } E_n = -\frac{13.6\ eV}{n^2}$$

- $n = 1$, $j = 1/2$: $\quad E_{1\frac{1}{2}} = -13.6\ eV\Big[1 + \frac{\alpha^2}{4}\Big] < -13.6\ eV$
- $n = 2$, $j = 1/2$: $\quad E_{2\frac{1}{2}} = -\frac{13.6\ eV}{4}\Big[1 + \frac{\alpha^2}{4}\cdot\frac{5}{4}\Big]$
- $n = 2$, $j = 3/2$: $\quad E_{2\frac{3}{2}} = -\frac{13.6\ eV}{4}\Big[1 + \frac{\alpha^2}{4}\cdot\frac{1}{4}\Big]$

FINE - STRUCTURE OF H



TOTAL E

$$\boxed{E = E_n + E'_{FS} = -\frac{13.6\,eV}{n^2}\left[1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right)\right]}$$

DEPENDS ON n & j